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Sp 11
L7.1

a) quick review of FFTs
b) FFT "gotchas"
c) properties of FFT's

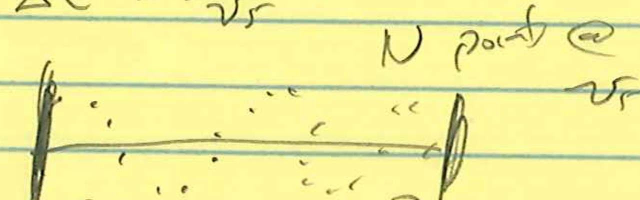
a) review

i) all real world data is discretely sampled. Data taken in equal time steps is called a "time series".
Data taken @ equal position steps can be substituted and analyzed similarly

ii) time series are sampled @ a constant rate ν_s (or @ constant spacing Δx)

iii) once the experimenter chose ν_s and N, the # of samples in a sample is set & given by

$$T_{sample} = N\Delta t = \frac{N}{\nu_s}$$



also lowest freq observable is:

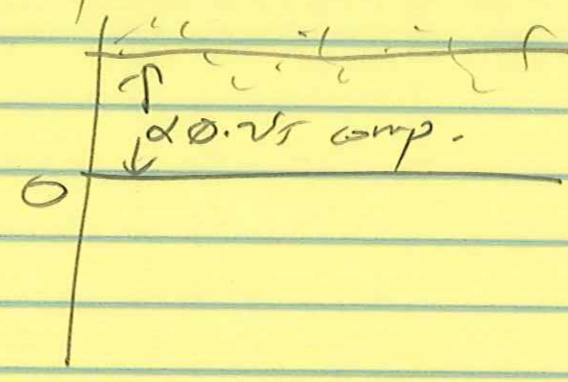
$$\Delta\nu = \frac{1}{T_{sam}} = \frac{\nu_s}{N}$$

iii) continued

Fourier analysis will then give coefficients for freqs

$$(0\nu_f, 1\Delta\nu, 2\Delta\nu, 3\Delta\nu \ldots \underbrace{\tfrac{N}{2}\Delta\nu}_{\text{the Nyquist}})$$

iv) $0\cdot\nu_f$ coeff. gives the "DC component" or offset



b.) FFT (fast-Fourier Transform) "guts"

i) So Fourier says:

$$y(\Delta t) = A_0 + A_1\cos(1\cdot 2\pi\cdot\Delta\nu\cdot\Delta t + \phi_1) + \ldots A_{\frac{N}{2}}\cos(\tfrac{N}{2}\cdot 2\pi\cdot\Delta\nu\cdot\Delta t + \phi_{\frac{N}{2}})$$

note Δt is a "time basis"

or in complex notation:

$$y(\Delta t) = C_0 + \sum_{j=1}^{\frac{N}{2}} \left[C_j e^{ij2\pi\Delta\nu\Delta t} + C_j^{*} e^{-ij2\pi\Delta\nu\Delta t} \right]$$

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this is critical to understanding F.C.'s

F.C.'s, (c_j's) belong to their respective freqs, e.g.

$\Rightarrow$ $c_2, c_2^* \longleftrightarrow e^{i 2 2\pi \Delta\nu \Delta t}$

ii) we use particular frequencies $j \cdot \Delta\nu = j \cdot \frac{\nu_s}{N}$ because together they form an "orthonormal set"

$$\sum_{cycle} \cos(2\pi \nu_a \Delta t + \phi_a) \cdot \cos(2\pi \nu_b \Delta t + \phi_b)$$

$$= \begin{cases} \phi & \text{if } \nu_a \neq \nu_b \\ \text{const} & \text{if } \nu_a = \nu_b \end{cases}$$

Only true for members of the orthonormal set and for summation over a complete cycle

{draw on board and ask for a volunteer to come up, ~~...~~ steps for checking orthonormality}

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c) new material: estimating FFTs + properties

iii) how do we mathematically estimate c_j's?

for the n^{th} c_j:

$$C(\nu_j) = \Delta t \sum_{\ell=0}^{N-1} \underbrace{x(\Delta t_\ell)}_{x \text{ @ time step}} \cdot \underbrace{e^{-i2\pi \frac{j}{N}\ell}}_{\text{ortho. basis fctn of freq. } \frac{j}{N}\Delta\nu \text{ evaluated @ pt. } \ell}$$

inner loop over time basis

our outer loop then steps through each successive basis fctn (@ our "Fourier frequencies")

$$\left[j = \frac{-N}{2}+1, \frac{-N}{2}+2, \cdots 0, 1, 2, \cdots \frac{N}{2} \right]$$

orthonormality ensures that each inner loop picks out only the same frequency of the signal components

iv) If we know c_j's, we can synthesize $x(t)$ from them

Inverse FFT $\nu \to t$

$$x_\ell(\Delta t) = \sum_{n=0}^{N-1} c_n e^{+i2\pi n \frac{\ell}{N}}$$

IFFT → pt. ℓ

(must now outer loop over all times)

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L2.5

c) FFT properties

Matlab Lecture 8.m FFT

[Linearity]: If $h(t) = \alpha f(t) + \beta g(t)$

[$\hat{h}$ = FFT of h] $\hat{h}(\omega) = \alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$

[Translation]: if $h(x) = f(x - x_0)$

then $$\hat{h}(k) = e^{-ikx_0} \hat{f}(k)$$

k = wave number, $\hat{h}(k) = FFT(h(x))$

[Differentiation] $h(t) = \frac{d}{dt} f(t) \Rightarrow$

$$\hat{h}(\omega) = i\omega \hat{f}(\omega)$$

further $\frac{d^n}{dt^n} f(t) = (i\omega)^n \hat{f}(\omega)$

derivatives become simple multipliers, easier to code!

see example

3a1
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~~L7.5~~
Sp Tx
L7.6

C) cont.

$\boxed{\text{Differentiation}}$ $h(t) = \frac{d}{dt} f(t)$

$$\Rightarrow \hat{h}(\omega) = i\omega \hat{f}(\omega)$$

further: $h(t) = \frac{d^n}{dt^n} f(t) \Rightarrow$

$$\hat{h}(\omega) = (i\omega)^n \cdot \hat{f}(\omega)$$

derivatives $\Rightarrow$ multiplication
(easier to code?)

(example in applied E&M)

say $\underbrace{\nabla \times \vec{E}}_{\text{curl } \vec{E}} = -\mu \frac{\partial}{\partial t} \vec{H}$

note

E = electric field in 3-D (e.g. $f(x,y,z,t)$)
H = magnetic field " " " "

one of Maxwell's eqn.

curl E looks like $\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$

$$= \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

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L7.7

hard to solve in computer
can instead let

$\hat{E}$ = FFT of $\vec{E}$ w.r.t. both space & time

$\hat{H}$ = FFT of $\vec{H}$

then can write (for example):

$\hat{x}$ component $-ik_y \hat{E}_z + ik_z \hat{E}_y = +i\omega\mu \hat{H}_x$

solve all these for Fourier k's & ω's

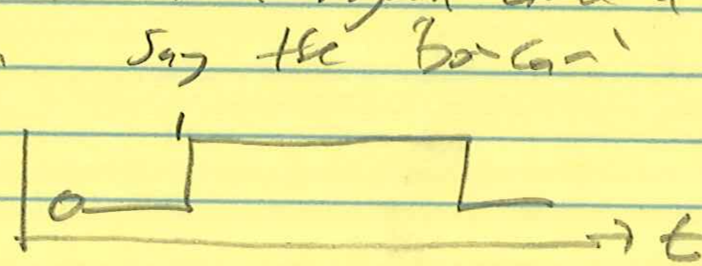
$\boxed{\text{Convolution}}$ if $h(t) = f(t) \otimes g(t)$

$$\equiv \int_{-\infty}^{\infty} f(t_0 - t)\, g(t)\, dt$$

then $\hat{h}(\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)$

$\Rightarrow$ $\boxed{\text{windowing}}$

FFT we have looked at are really the convolution of the real signal and a windowing fctn say the 'boxcar'



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7.7

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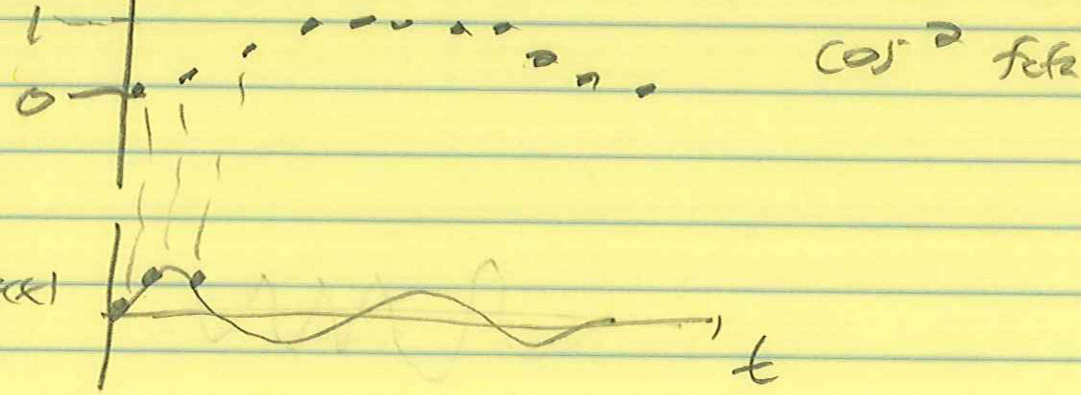
7.8

c) Cont'd

Can choose to first convolve data with a specially-chosen windowing fctn, eg



pt by pt mult of $W(t) \cdot x(t)$

then ⇒ FFT

ML Lecture 8 .. n J

finally - show LPF trick for aliasing