Name________________

Final Exam Practice Problems

Problem 1. Prove that if $d = ax + by$ where $a, b, d \in \mathbb{Z}^+$ $x, y \in \mathbb{Z}$, $d|a$, and $d|b$, then $d = \gcd(a, b)$.

Problem 2. For each of the following statements, either prove it, or provide a counterexample. Let $n \in \mathbb{Z}^+$.
Statement A: If $12|n^2$, then $12|n$.
Statement B: If $14|n^2$, then $14|n$.

Problem 3. (I just think this one is fun!)
Let $f : \mathbb{R} \to \mathbb{R}$ where

$$f(ab) = a \cdot f(b) + b \cdot f(a) \;\; \forall a, b \in \mathbb{R}$$

(a) What is $f(1)$?
(b) What is $f(0)$?
(c) If $n \in \mathbb{Z}^+$, $a \in \mathbb{R}$, prove that $f(a^n) = na^{n-1}f(a)$.

Problem 4. Let $A, B \subseteq \mathbb{N}$ with $1 < |A| < |B|$. If there are $262,144$ relations from A to B, determine all the possibilities for $|A|$ and $|B|$.

Problem 5. Let S be a set of seven positive integers the maximum of which is at most 24. Prove that the sums of the elements in all the nonempty subsets of S cannot be distinct.

Problem 6. Prove or disprove:
Let $A, B, C \subseteq \mathcal{U}$.

$$A - C = B - C \Rightarrow A = B$$

Problem 7. Let $A = \{1, 2, 3, \ldots, 15\}$.
a) How many subsets of A contain all of the odd integers in A?
b) How many subsets fo A contain exactly three odd integers?
c) How many eight-element subsets of A contain exactly three odd integers?

Problem 8. Prove that

$$(A \cap B) \cup C = A \cap (B \cup C) \Leftrightarrow C \subseteq A$$

Problem 9. How many permutations of the letters $A, B, C, \ldots, Z$ either start with a D or end with an R?

Problem 10. For all $x \in \mathbb{R}$,

$$|x| = \sqrt{x^2} = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

a) Prove that $|x+y|^2 \leq (|x|+|y|)^2$. (As a corollary, we get that $|x+y| \leq |x|+|y|$)
b) Prove that if $n \in \mathbb{Z}^+$, $n \geq 2$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$, then

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Problem 11. Prove that for all $n \in \mathbb{Z}^+$, $n > 3$,

$$2^n < n!$$

Problem 12. Determine the smallest perfect cube that is divisible by 7!. (My punctuation may look odd, but I mean 7 factorial, not 7).