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Final Exam Practice Problems - Solutions

Problem 1. Prove that if $d = ax + by$ where $a, b, d \in \mathbb{Z}^+$ $x, y \in \mathbb{Z}$, $d|a$, and $d|b$, then $d = \gcd(a, b)$.

Proof. If $d|a$ and $d|b$, then $d|\gcd(a, b)$ since d is a common divisor of a and b. Since $d = ax + by$, then $\gcd(a, b)|d$ (since $\gcd(a, b)|a$ and b, and so by the 2 out of 3 rule, $\gcd(a, b)|d$.) Since $d > 0$, then $d = \gcd(a, b)$. □

Problem 2. For each of the following statements, either prove it, or provide a counterexample. Let $n \in \mathbb{Z}^+$.

Statement A: If $12|n^2$, then $12|n$.
Statement B: If $14|n^2$, then $14|n$.

Statement A: False
Counterexample: Let $n = 6$, then $12|36 = 6^2$, but 12 doesn't divide 6.
Statement B: True

Proof. Suppose $14|n^2$, then since $2|n^2$, $2|n$ (by the fact that 2 is prime). Thus $n = 2m$ for some $m \in \mathbb{Z}$.

Note that $7|n^2$ as well, thus $7|4m^2$. By primality of 7, $7|4$ or $7|m^2$. Hence, $7|m^2$. Again by primality of 7, $7|m$. Hence, $m = 7k$ for some $k \in \mathbb{Z}$.

Therefore,

$$n = 2m = 2(7k) = 14k$$

and hence $14|n$. □

Problem 3. (I just think this one is fun!)

Let $f : \mathbb{R} \to \mathbb{R}$ where

$$f(ab) = a \cdot f(b) + b \cdot f(a) \;\; \forall a, b \in \mathbb{R}$$

(a) What is $f(1)$?
(b) What is $f(0)$?
(c) If $n \in \mathbb{Z}^+$, $a \in \mathbb{R}$, prove that $f(a^n) = na^{n-1}f(a)$.

(a) Let $a = 1$ and $b = 1$, then

$$f(1) = f(ab) = a \cdot f(b) + b \cdot f(a) = f(1) + f(1) = 2 \cdot f(1)$$

Hence, $f(1) = 0$.

(b) Let $a = 0$ and $b = 0$, then

$$f(0) = f(ab) = a \cdot f(b) + b \cdot f(a) = 0 \cdot f(0) + 0 \cdot f(0) = 0$$

Hence, $f(0) = 0$.

(c) We can prove this inductively on n.

Proof. Base Case: $n = 1$

$$1 \times a^0 \times f(a) = f(a)$$

Inductive Step: Assume $f(a^k) = ka^{k-1}f(a)$. We want to show $f(a^{k+1}) = (k+1)a^k f(a)$.

$$\begin{aligned} f(a^{k+1}) &= f(a^k \cdot a) = a^k \cdot f(a) + a \cdot f(a^k) \\ &= a^k \cdot f(a) + a \cdot (ka^{k-1}) \cdot f(a) \\ &= a^k \cdot f(a) + ka^k \cdot f(a) \\ &= (k+1) \cdot a^k \cdot f(a) \end{aligned}$$

□

Problem 4. Let $A, B \subseteq \mathbb{N}$ with $1 < |A| < |B|$. If there are $262,144$ relations from A to B, determine all the possibilities for $|A|$ and $|B|$.

Note that 262144 must be a power of 2 or this question doesn't make sense. In fact

$$262144 = 2^1 8$$

Thus $|A|\,|B| = 18$. Since $|A| < |B|$, then the possibilities are

$$\begin{aligned} |A| &= 1 & |B| &= 18 \\ |A| &= 2 & |B| &= 9 \\ |A| &= 3 & |B| &= 6 \end{aligned}$$

Problem 5. Let S be a set of seven positive integers the maximum of which is at most 24. Prove that the sums of the elements in all the nonempty subsets of S cannot be distinct.

Proof. This one has a little trick to it. How many subsets A are there of S of cardinality 5 or less? There are

$$2^7 - 1 - 1 - \binom{7}{1} = 119$$

nonempty subsets A of S. (Think pigeons)

What is the maximum sum of elements of A?

$$24 + 23 + 22 + 21 + 20 = 110$$

What is the minimum sum? 1

Thus there are 110 possibilities for the sum of the elements of A. But there are 119 such sets. Thus at least 2 of them must have the same sum. □

Problem 6. Prove or disprove:

Let $A, B, C \subseteq \mathcal{U}$.

$$A - C = B - C \Rightarrow A = B$$

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4\}$, $C = \{3, 4, 5\}$

Both $A - C$ and $B - C$ are the set $\{1, 2\}$, but $A \neq B$.

Problem 7. Let $A = \{1, 2, 3, \ldots, 15\}$.

a) How many subsets of A contain all of the odd integers in A?

b) How many subsets of A contain exactly three odd integers?

c) How many eight-element subsets of A contain exactly three odd integers?

a) Since we are only considering subsets of A containing all of the odd integers, then we only care about which even integers are in the set. (In fact there is a bijection from the set of subsets of A containing all the odd integers to the set of subsets of the even integers between 2 and 14). There are 7 even integers in A, and hence 2^7 subsets containing all of the odd integers.

b) We should choose the odd integers first, and then the even ones. (or vice versa)

$$\binom{8}{3} \times 2^7 \text{ subsets}$$

c) If there are 3 odd integers, there are 5 even integers, so there are

$$\binom{8}{3} \times \binom{7}{5} \text{ subsets}$$

Problem 8. Prove that

$$(A \cap B) \cup C = A \cap (B \cup C) \Leftrightarrow C \subseteq A$$

Proof. "$\Rightarrow$" Suppose that $(A \cap B) \cup C = A \cap (B \cup C)$.

$$C \subseteq (A \cap B) \cup C = A \cap (B \cup C) \subseteq A$$

Hence, $x \in A$ and thus $C \subseteq A$.

"$\Leftarrow$" Suppose that $C \subseteq A$. Then,

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C) = A \cap (B \cup C)$$

as desired. □

Problem 9. How many permutations of the letters $A, B, C, \ldots, Z$ either start with a D or end with an R?

On this one we want to be careful not to double count.

There are 25! permutations that start with a D, and 25! permutations that end with an R. (Note: if we just add these 2 numbers, we'll be double counting those that start with D and end with R). There are 24! that start with a D and end with an R. Thus, there are

$$2 \times 25! - 24!$$

permutations that start with a D or end with an R.

Problem 10. For all $x \in \mathbb{R}$,

$$|x| = \sqrt{x^2} = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

a) Prove that $|x+y|^2 \leq (|x|+|y|)^2$. (As a corollary, we get that $|x+y| \leq |x|+|y|$)
b) Prove that if $n \in \mathbb{Z}^+$, $n \geq 2$ and $x_1, x_2, \ldots, x_n \in \mathbb{R}$, then

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Proof. (a)

$$|x+y|^2 = (x+y)^2 = x^2 + 2xy + y^2 = |x|^2 + 2xy + |y|^2 \leq |x|^2 + 2|x||y| = (|x|+|y|)^2$$

(b) This is to be proven by induction.
Base Case: $n = 1$, then

$$|x_1| \leq |x_1|$$

Inductive Step: Suppose that $k \geq 1$, and

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

for all $1 \leq n \leq k$. We want to show this result is true for $k+1$.

$$|x_1 + x_2 + \cdots + x_k + x_{k+1}| \leq |x_1 + x_2 + \cdots + x_k| + |x_{k+1}|$$

by the $n = 2$ case. By the $n = k$ case we get that

$$|x_1 + x_2 + \cdots + x_k| + |x_{k+1}| \leq |x_1| + |x_2| + \cdots + |x_k| + |x_{k+1}|$$

and the result follows. □

Problem 11. Prove that for all $n \in \mathbb{Z}^+$, $n > 3$,

$$2^n < n!$$

Proof. Base Case: $n = 4$ (why? because we said above $n > 3$)

$$2^4 = 16 < 24 = 4!$$

Inductive Step: Suppose it is true for $n = k \geq 4$, i.e. $2^k < k!$. Then

$$2^{k+1} = 2^k \times 2 < k! \times 2 < k! \times (k+1) = (k+1)!$$

□

Problem 12. Determine the smallest perfect cube that is divisible by 7!. (My punctuation may look odd, but I mean 7 factorial, not 7).

Note that

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$$

The exponents of the primes in the prime decomposition of a perfect cube have to be multiples of 3. Hence, the smallest such perfect cube will have only the primes listed in 7!, and the exponents in the prime decomposition will be multiples of 3. Thus the smallest such perfect cube is

$$2^6 \cdot 3^3 \cdot 5^3 \cdot 7^3$$