(Partial) Homework 9 Solutions (Evens)

Section 5.2: 3, 4, 5ab, 9, 16a-f, 18, 20 Section 5.5: 4, 5, 8 Section 5.6: 2, 3, 4, 6, 8, 9, 10c

Problem 1. (Problem #4 Section 5.2)

For this one, you need to remember how to count functions $f : A \to B$. There are $|B|^{|A|}$ of them. Hence, we are looking for a solution to $3^{|A|} = 2187$. This is $|A| = 7$.

Problem 2. (Problem #16a-f Section 5.2)

Recall that the notation $f(A)$ means the set of images of elements of the set A.

$$\begin{aligned}
&(\mathrm{a}) f(A) = \{4, 9\}\\
&(\mathrm{b}) f(A) = \{4, 9\}\\
&(\mathrm{c}) f(A) = [0, 9)\\
&(\mathrm{d}) f(A) = [0, 9)\\
&(\mathrm{e}) f(A) = [0, 49]\\
&(\mathrm{f}) f(A) = [9, 16) \cup [25, 36]
\end{aligned}$$

Problem 3. (Problem #18 Section 5.2)

Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2$ (in other words, $f = \{(x, x^2)|x \in \mathbb{R}\}$). Let $A_1 = \{0, 2\}$, and let $A_2 = \{0, -2\}$. Then $A_1 \cap A_2 = \{0\}$, and

$$f(A_1 \cap A_2) = \{0\}$$

However,

$$f(A_1) \cap f(A_2) = \{0, 4\}$$

since $(-2)^2 = 2^2 = 4$.

Problem 4. (Problem #20 Section 5.2)

First, we need to recall how to count injective functions. There are $P(|B|, |A|)$ functions $f : A \to B$ (if $|A|, |B| < \infty$). In this case we have that $P(|B|, 5) = 6720$. If no other way that guess and check, we see that

$$|B| = 8$$

Problem 5. (Problem #4 Section 5.5)

Let $S = \{3, 7, 11, 15, 19, \ldots, 95, 99, 103\}$. For this problem, the idea is essentially the same as Example 5.44. Basically, we need to count the pigeonholes, and make sure we have at least one more pigeon than pigeonholes.

In this case, the pigeons are the elements selected from S. The pigeonholes are the two (and one) element subsets of S given by

$$\{7, 103\}, \{11, 99\}, \ldots, \{51, 59\}, \{55\}, \{3\}$$

The 55 and 3 don't really fit with anything else. The number of sets listed above is 14 (if I've counted correctly). Thus if we have 15 elements of S, they all fit in one of these two (or one) element subsets of S. Since I have 15 elements of S, and only 14 two (or one) element subsets, then at least two belong to one of the subsets. Thus we are done, because if two elements live in any of the two element subsets, they sum to 110.

Problem 6. (Problem #8 Section 5.5)

a) Here the pigeons are the elements of S and the pigeonholes are the set of even integers and the set of odd integers. Since there are at least 3 pigeons, and only 2 pigeonholes, then at least two of the integers are even (or at least two are odd). In either case, you end up with two number that have an even sum.

b) Here again, the pigeons are the elements of S, and the pigeonholes are 1) (even, even) 2) (even, odd) 3) (odd, even) 4) (odd, odd). If we have any two ordered pairs in the same pigeonhole, then they sum to (even, even). Thus, we need $|S| \geq 5$.

c) Same idea, but this time there are 8 pigeonholes, so $|S| \geq 9$.

d) The general result is this:

Let $S \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$, where this cartestian product contains n copies of $\mathbb{Z}^+$. If $|S| \geq 2^n + 1$, then there exist ordered triples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ such that

$$x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n$$

are all even.

e) This is just an application of part (b). How do we ensure that the midpoint of $P_i(x_i, y_i)$ and $P_j(x_j, y_j)$ is a lattice point? The midpoint is "add the coordinates and divide by 2", hence we need the sums of both the coordinates to be even (as in part (b)). Thus, our answer is $n = 5$ as in part (b) since that is how many ordered pairs we need to ensure even sums of coordinates.

Problem 7. (Problem #2 Section 5.6)

a) If $x \in (-2, 7]$, then $x \neq -2$ and

$$g(x) = \frac{2x^2 - 8}{x + 2} = \frac{2(x^2 - 4)}{x + 2} = \frac{2(x + 2)(x - 2)}{x + 2} = 2(x - 2) = 2x - 4 = f(x)$$

b) Yes, the function g is not well defined on $[-7, 2)$.

Problem 8. (Problem #4 Section 5.6)

$$\begin{aligned} g \circ f(1) &= g(f(1)) = g(2) = 4 \\ g \circ f(2) &= 6 \\ g \circ f(3) &= 10 \\ g \circ f(4) &= 14 \end{aligned}$$

Hence, $g \circ f = \{(1, 4), (2, 6), (3, 10), (4, 14)\}$.

Problem 9. (Problem #6 Section 5.6)

We start with a computation

$$f \circ g(x) = f(cx + d) = a(cx + d) + b = acx + ad + b$$

and

$$g \circ f(x) = g(ax + b) = c(ax + b) + d = acx + cb + d$$

Thus, the relationship that must be satisfied is that

$$ad + b = cb + d \Leftrightarrow d(a-1) = b(c-1)$$

Problem 10. (Problem #8 Section 5.6)

First, recall that a function is onto if the range is the codomain of the function. Alternatively, every element of the codomain is the image of something in the domain. Alternatively, if $h : X \to Y$ is onto, then for every $y \in Y$, there exists some $x \in X$ such that $h(x) = y$.

Proof. (a) If $g \circ f : A \to C$ is onto, then for all $c \in C$, there exists some $a \in A$ such that

$$g \circ f(a) = c$$

This also ensures that there exists an element of B such that the image of that element is c. This element of B is just $f(a)$ (since $g(f(a)) = c$). Hence, g is onto.

(b) If $g \circ f : A \to C$ is one-to-one, then $g \circ f(a) = g \circ f(a') \Rightarrow a = a'$. (See the formulation of one-to-one on page 255 below the definition). Now suppose that $f(a) = f(a')$ (Aside: we want to show that a must be equal to a'). If this is the case then, certainly $g(f(a)) = g(f(a'))$. By the one-to-one property of $g \circ f$, we have that $a = a'$. ☐

Problem 11. (Problem #10c Section 5.6)

The function f is invertible. Why? f is one-to-one because if $a^3 = b^3$, then $a = b$ (as long as $a, b \in \mathbb{R}$). Also, it is onto because every element of $\mathbb{R}$ is a cube of some other element (i.e. for any $a \in \mathbb{R}$, a is the cube of $\sqrt[3]{a}$).

What is it's inverse? Well, we want to come up with a function that undoes what the original function did. Hence, we want the cube root function because

$$\sqrt[3]{x^3} = (\sqrt[3]{x})^3 = x$$

Note that the same idea doesn't work with the squaring function because squaring doesn't preserve sign.