

Midterm Extra Credit - Solutions

Problem 1. (6 points) Verify the following argument:

$$[(\neg p \vee \neg q) \wedge (r \rightarrow q) \wedge p] \rightarrow \neg r$$

1.	$(\neg p \vee \neg q)$	premise
2.	$p \rightarrow \neg q$	substitution
3.	p	premise
4.	$\neg q$	Modus Ponens
5.	$r \rightarrow q$	premise
6.	$\therefore \neg r$	Modus Tollens

Problem 2. (6 points) Find a counterexample:

For a given open statement $p(x, y)$, the statement

$$\exists x \forall y [p(x, y)]$$

is always equivalent to the statement

$$\forall y \exists x [p(x, y)]$$

If $p(x, y)$ is the statement " $x + y = 17$ ", then these statements are not equivalent. In particular, $\exists x \forall y [p(x, y)]$ is false (there is no magic number such that when it has any other number added to it, we get 17). However, $\forall y \exists x [p(x, y)]$ is true, as for each y , there exists a particular x (namely $17 - y$) such that $x + y = 17$.

Problem 3. A new government is formed in a small country of 27 citizens. There are 7 public officials: President, Prime Minister, Czar, Prince, and 3 Senators.

a. (6 points) How many ways can the offices be assigned from the 27 citizens?

b. (6 points) If Bob and Sue are each guaranteed that they will be one of the offices besides Senator, how many ways can the offices be assigned?

a.

$$P(27, 4) \times \binom{23}{3}$$

b.

$$P(4, 2) \times P(25, 2) \times \binom{23}{3}$$

Problem 4. (10 points) Simplify

$$\neg[\neg[(p \wedge q) \vee r] \wedge \neg q]$$

1. $\neg[\neg[(p \wedge q) \vee r] \wedge \neg q] \Leftrightarrow [(p \wedge q) \vee r] \vee q$	DeMorgan's Laws
2. $[(p \wedge q) \vee r] \vee q \Leftrightarrow (p \wedge q) \vee (r \vee q)$	Associative Law
3. $(p \wedge q) \vee (r \vee q) \Leftrightarrow (p \wedge q) \vee (q \vee r)$	Commutative Law
4. $(p \wedge q) \vee (q \vee r) \Leftrightarrow [(p \wedge q) \vee q] \vee r$	Associative Law
5. $[(p \wedge q) \vee q] \vee r \Leftrightarrow q \vee r$	Absorption Law

Problem 5. (6 points) Suppose a license plate contains 3 letters followed by 3 numbers. If

- 1) there is no repetition among the letters,
 - 2) no repetition among the numbers,
 - 3) the 3-digit number must be divisible by 2
 - 4) and the 3rd letter must be a vowel (that is A,E,I,O, or U)
- how many license plates can be formed?

$$(5 \times 25 \times 24) \times (5 \times 9 \times 8)$$

Problem 6. (8 points) Using truth tables, show that

$$p \vee q \Leftrightarrow [(p \wedge \neg q) \vee (\neg p \wedge q)]$$

and

$$p \vee q \Leftrightarrow \neg(p \leftrightarrow q)$$

Problem 7. (8 points) Suppose that John has 15 cupcakes and 17 doughnuts to distribute among 4 children. If each child gets at least one cupcake or one doughnut, how many ways can John distribute the cupcakes and doughnuts among the 4 children?

The most elegant solution to this problem uses a generalization of the formula that counts the cardinality of the triple union. (See Inclusion-Exclusion Principle) The way we should think about this problem is in an opposite sort of way. We should count the ways that at least one child gets nothing and then subtract that from the number of ways to distribute the cupcakes and doughnuts among the children with no restrictions.

$$\# \text{ of ways to distribute doughnuts and cupcakes} = \binom{18}{15} \times \binom{20}{17}$$

The hard part is counting the ones of these in which at least one child gets nothing. The possibilities are that exactly one child could get nothing, exactly two could get nothing, or exactly three could get nothing. We can't have the scenario that all 4 get nothing (because where would the pastries go?). Where does this inclusion-exclusion principle start? Well, let's give the children names. I'll call them Child 1, Child 2, Child 3, and Child 4. Let A_1 be the set of distribution of doughnuts and cupcakes where Child 1 gets nothing. Define A_2 , A_3 , and A_4 analogously. We want to compute

$$|A_1 \cup A_2 \cup A_3 \cup A_4|$$

Of course this isn't just the sum of the cardinalities of the individual sets, because we'd end up double and triple counting some things. By the inclusion-exclusion principle we know that

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{i=1}^4 |A_i| - \sum_{i,j:1 \leq i < j \leq 4} |A_i \cap A_j| + \sum_{i,j,k:1 \leq i < j < k \leq 4} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

A few things to say before we compute this: Notice that $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$ because we can't have all 4 kids get nothing. The notation $\sum_{i,j:1 \leq i < j \leq 4}$ means to sum over pairs i, j where $i < j$ and i and j are between 1 and 4. In particular

$$\sum_{i,j:1 \leq i < j \leq 4} |A_i \cap A_j| = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|$$

Now let's start computing.

$$|A_i| = \binom{17}{15} \times \binom{19}{17} \quad \forall i$$

(this calculation is similar to the overall one, except we are distributing to 3 children instead of 4).

$$|A_i \cap A_j| = \binom{16}{15} \times \binom{18}{17} \quad \forall 1 \leq i < j \leq 4$$

(this calculation is similar to the overall one, except we are distributing to 2 children instead of 4).

$$|A_i \cap A_j \cap A_k| = 1 \quad \forall 1 \leq i < j < k \leq 4$$

(this calculation is easy because we are giving everything to one child). If we put this all together we get

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = (4) \times \binom{17}{15} \times \binom{19}{17} - \binom{4}{2} \times \binom{16}{15} \times \binom{18}{17} + \binom{4}{3} \times 1 - 0 = 91300$$

$$\text{Answer to original problem} = \binom{18}{15} \times \binom{20}{17} - |A_1 \cup A_2 \cup A_3 \cup A_4| = \boxed{838940}$$

Problem 8. Define a Gaussian integer to be a number of the form $a + bi$, where $a, b \in \mathbb{Z}$, and $i = \sqrt{-1}$. Define the conjugate of a Gaussian integer $a + bi$ (denoted $\overline{a + bi}$) to be

$$\overline{a + bi} = a - bi$$

a) (6 points) Prove that if $x = a + bi$ is a Gaussian integer with $a + b$ odd, then $x\bar{x}$ is an odd integer (i.e. a number of the form $c + 0i$).

b) (6 points) Prove that if $x = a + bi$ is a Gaussian integer with a and b odd, then $\frac{x\bar{x}}{2}$ is an integer.

(For both parts you may use the fact that if k is an integer, then so is k^2).

Proof. (a)

If $a + b$ is odd, then either a is odd and b is even, or a is even and b is odd (you can prove this for yourself). Without loss of generality assume that a is odd and b is even because the argument for the alternative is analogous.

$$x\bar{x} = a^2 + abi - abi + b^2i^2 = a^2 - b^2i^2 = a^2 + b^2$$

Since a is odd, a^2 is odd. Since b is even, b^2 is even. Thus $a^2 + b^2$ is odd.

(b)

If a and b are odd, then $x\bar{x} = a^2 + b^2$ is even. Hence it is divisible by 2, and $\frac{x\bar{x}}{2}$ is an integer. \square