Practice Midterm

Problem 1. In how many ways can the symbols $\diamondsuit, \clubsuit, \spadesuit, \blacksquare, \bigstar, \bigstar, \bigstar, \bigstar$ be arranged so that no $\bigstar$ is adjacent to another $\bigstar$?

We can arrange the $\diamondsuit, \clubsuit, \spadesuit, \blacksquare$ in 4! ways. There are 5 places to put the $\bigstar$ and 4 $\bigstar$s (i.e. there are $\binom{5}{4}$ choices for how to distribute the $\bigstar$s).

$$4! \times \binom{5}{4} = 5!$$

Problem 2. (a) How many arrangements are there of the letters RIEMANNIAN?

(b) In how many of the arrangements in part (a) are the R and M adjacent (i.e next to each other)?

(c) In how many of the arrangements in part (a) are all the vowels adjacent?

(a) There are 2 Is, 2 As, and 3 Ns. Thus there are

$$\frac{10!}{3!2!2!}$$

arrangements

(b) Consider the R and M as one letter (though there are 2 ways to represent it - RM and MR). Thus there are

$$2 \times \frac{9!}{3!2!2!}$$

arrangements with R and M adjacent.

(c) Consider the vowels all as one character, there are 6 characters to consider, and 3 of them are the same, so there are $\frac{6!}{3!}$ arrangements. There are $\frac{5!}{2!2!}$ arrangements of just the vowels, so there are

$$\frac{6!5!}{3!2!2!}$$

total arrangements.

Problem 3. A student is to answer 7 out of 10 questions on an examination. In how many ways can he make his selection if

(a) there are no restrictions?

(b) he must answer the first two questions?

(c) he must answer at least four of the first six questions?

(a) $\binom{10}{7}$

(b) $\binom{8}{5}$

(c) $\binom{6}{4}\binom{4}{3} + \binom{6}{5}\binom{4}{2} + \binom{6}{6}\binom{4}{1}$

Problem 4. Determine the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 53$$

where

(a) $x_i \geq 0$, $1 \leq i \leq 4$.

(b) $x_i \geq -2$, $1 \leq i \leq 4$.

(a) $\binom{56}{53}$
(b) $\binom{64}{61}$

Problem 5. In how many ways can Lisa toss 100 (identical) dice so that at least three of each type of face will be showing?

There are 6 different faces, and we get at least 3 of each (18 total), so we want the number of solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 82$$

which is

$$\binom{87}{82}$$

Problem 6. Determine all of the truth values assignments of p and q that would make the following statement false:

$$q \leftrightarrow (-p \vee -q)$$

(you don't need a truth table, but you could determine this by constructing one ... in fact, you should try it out just so that you make sure that you remember how to do it).

There are two ways this statement can be false. Either q is false and $(-p \vee -q)$ is true, or q is true and $(-p \vee -q)$ is false. The first scenario works if q is false and p is either true or false. The second scenario works if q is true and p is true. That is, the only way the statement is true is if p is false and q is true.

p	q	$-p$	$-q$	$-p \vee -q$	$q \leftrightarrow (-p \vee -q)$
0	0	1	1	1	0
0	1	1	0	1	1
1	0	0	1	1	0
1	1	0	0	0	0

Problem 7. Simplify

$$(-p \vee q) \wedge (p \wedge (p \wedge q))$$

$(-p \vee q) \wedge (p \wedge (p \wedge q)) \Leftrightarrow (-p \vee q) \wedge ((p \wedge p) \wedge q)$	Associative Law
$(-p \vee q) \wedge ((p \wedge p) \wedge q) \Leftrightarrow (-p \vee q) \wedge (p \wedge q)$	Idempotent Law
$(-p \vee q) \wedge (p \wedge q) \Leftrightarrow ((-p \vee q) \wedge p) \wedge q$	Associative Law
$((-p \vee q) \wedge p) \wedge q \Leftrightarrow ((-p \wedge p) \vee (q \wedge p)) \wedge q$	Distributive Law
$((-p \wedge p) \vee (q \wedge p)) \wedge q \Leftrightarrow (F_0 \vee (q \wedge p)) \wedge q$	Inverse Law
$(F_0 \vee (q \wedge p)) \wedge q \Leftrightarrow (q \wedge p) \wedge q$	Identity Law
$(q \wedge p) \wedge q \Leftrightarrow (p \wedge q) \wedge q$	Commutative Law
$(p \wedge q) \wedge q \Leftrightarrow p \wedge (q \wedge q)$	Associative Law
$p \wedge (q \wedge q) \Leftrightarrow p \wedge q$	Idempotent Law

Problem 8. Verify the following logical implication

$$\begin{array}{l} p \wedge q \\ p \rightarrow (r \wedge q) \\ r \rightarrow (s \vee t) \\ -s \\ \hline \therefore t \end{array}$$

1.	$p \wedge q$	Premise
2.	p	Conjunctive Simplification
3.	$p \rightarrow (r \wedge q)$	Premise
4.	$r \wedge q$	Modus Ponens
5.	r	Conjunctive Simplification
6.	$r \rightarrow (s \vee t)$	Premise
7.	$s \vee t$	Modus Ponens
8.	$-s$	Premise
9.	$\therefore t$	Disjunctive Syllogism

Problem 9. Let $p(x)$, $q(x)$, and $r(x)$ denote the following open statements

$p(x)$: $x^2 - 8x + 15 = 0$
$q(x)$: x is odd
$r(x)$: $x > 0$

For the universe of all integers, determine the truth or falsity of each of the following statements.

a) $\forall x[p(x) \rightarrow q(x)]$
b) $\exists x[p(x) \rightarrow q(x)]$
c) $\forall x[(p(x) \vee q(x)) \rightarrow r(x)]$

a) True (3 and 5 are odd)
b) True (see above)
c) False: $x = -1$ is odd, but not greater than 0.

Problem 10. Let n be an integer. Prove that n is even if and only if $31n + 12$ is even.

Naively, to prove this we need to show

$$\forall n[p(n) \rightarrow q(n)] \text{ and } \forall n[q(n) \rightarrow p(n)]$$

We will show the first, but instead of $\forall n[q(n) \rightarrow p(n)]$, we shall show $\forall n[-p(n) \rightarrow -q(n)]$ which is equivalent.

Proof. Suppose that n is even. Then $n = 2k$ for some integer k.

$$31n + 12 = 31(2k) + 12 = 2(31k + 6)$$

Hence, $31n + 12$ is even.

Suppose that n is odd. Then $n = 2k + 1$ for some even integer k.

$$31n + 12 = 31(2k + 1) + 12 = 62k + 43 = 2(31k + 21) + 1$$

Hence, $31n + 12$ is odd. □