

Quiz #4 10/30/09

Name: _____

Directions: Make sure to read each problem carefully. To receive full credit, you must show all of your work.

Problem 1. (2 points each part) Let $p(x, y)$ be the open statement $|x| \leq |y|$ in the universe of real numbers. Which of these statements are true, and which are false?

a) $\forall y \exists x [p(x, y)]$

True - certainly given any number, there exists a number that is less than it

b) $\forall x \exists y [p(x, y)]$

True - certainly given any number, there exists a number that is larger than it

c) $\exists y \forall x [p(x, y)]$

False - there doesn't exist a number such that all numbers are less than that number

Problem 2. (6 points) Recall that an odd number is defined to be a number of the form $2k + 1$ where k is an integer. Using this definition, prove that if the product of two integers is odd, then both of the integers themselves must be odd.

Proof. (Contrapositive) The contrapositive of the original statement, is that if one of two integers is even, their product is even.

Suppose one of n and m is even. Without loss of generality, let's assume n is even (the proof is the same if m is even). Thus $n = 2k$ for some integer k . Hence,

$$nm = n(2k) = 2(kn)$$

and therefore nm is even. □

Problem 3. (3 points) For the universe of all integers, let

$$\begin{aligned} p(x) &: x \text{ is divisible by } 4 \\ q(x) &: x \text{ is even} \\ r(x) &: x \text{ is a perfect square} \end{aligned}$$

Write the statement

If x is even and x is a perfect square, then x is divisible by 4

in symbolic form.

$$\forall x[(q(x) \wedge r(x)) \rightarrow p(x)]$$

Problem 4. (Bonus - 3 points) Prove that the above statement is true.

Proof. (Using the Rule of Universal Generalization) Suppose that x is even and a perfect square. Then $x = 2k$ for some integer k and $x = i^2$ for some integer i . Thus,

$$2k = i^2$$

and therefore $\frac{i^2}{2}$ is an integer. Since 2 is prime, then 2 divides i . Hence, $i = 2j$ for some integer j , and thus

$$x = i^2 = (2j)^2 = 4(j^2)$$

Therefore x is divisible by 4. □