Problem 1. (11.10 #7) Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. Also find the associated radius of convergence.

$$f(x) = \sin \pi x$$

Solution.

$$f^{(n)}(x) = \begin{cases} \pi^n \sin \pi x & \text{if } n = 4k \\ \pi^n \cos \pi x & \text{if } n = 4k+1 \\ \pi^n - \sin \pi x & \text{if } n = 4k+2 \\ \pi^n - \cos \pi x & \text{if } n = 4k+3 \end{cases}$$

where k is an integer greater than or equal to 0. Thus

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 4k \\ \pi^n & \text{if } n = 4k+1 \\ 0 & \text{if } n = 4k+2 \\ -\pi^n & \text{if } n = 4k+3 \end{cases}$$

Thus only the odd terms in our Maclaurin series have nonzero coefficients.

$$c_n = \begin{cases} 0 & \text{if } n = 4k \\ \frac{\pi^n}{n!} & \text{if } n = 4k+1 \\ 0 & \text{if } n = 4k+2 \\ \frac{-\pi^n}{n!} & \text{if } n = 4k+3 \end{cases}$$

Hence, since we only consider the odd terms of our series, (and we can write any odd number as $n = 2k+1$), our series is

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1} x^{2n+1}}{(2n+1)!}$$

If we just used the Maclaurin series for $\sin x$ we would get the same result by replacing x with πx. Using the ratio test, we see that the radius of convergence is $R = \infty$. □

Problem 2. (11.10 #16) Find the Taylor series for $f(x)$ centered at the given value of a.

$$f(x) = 1/x \qquad a = -3$$

Solution.

$$f^{(n)}(x) = -n!x^{-n-1}$$

$$f^{(n)}(-3) = -\frac{n!}{3^{n+1}}$$

$$c_n = \frac{-1}{3^{n+1}}$$

$$f(x) = \frac{1}{x} = \sum_{n=0}^{\infty} \frac{-(x-3)^n}{3^{n+1}}$$

This series is geometric with radius of convergence $R = 3$. □

Problem 3. (11.10 #26) Use the binomial series to expand the function as a power series. State the radius of convergence.

$$\frac{1}{(1+x)^4}$$

Solution.

$$\frac{1}{(1+x)^4} = \sum_{n=0}^{\infty} \binom{-4}{n} x^n$$

Since this is a binomial series, the radius of convergence is $R = 1$. □

Problem 4. (11.10 #33) Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

$$f(x) = x \cos\left(\frac{1}{2}x^2\right)$$

Solution.

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$x \cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2^{2n}(2n)!}$$

Since the first series has infinite radius of convergence, so does this one. □