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Limits at finite points are conceptually tricky. Sometimes one has to pretend that the function does not exist at that point and use the values at nearby points to try to come up with a single possible value which fits with the rest.

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- $\lim_{h \to 0} \frac{3h+h^2}{h} = 3.$

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Intuitively, functions which are continuous everywhere are those which can be graphed without picking up the pencil.

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Continuity is a property enjoyed by almost all basic functions at almost all of their values.

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The Intermediate Value Theorem is clear from pictures illustrating it. It is also easy to think of concrete examples such as the function $f(x) = 1 - 3x$.

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- A great trick for trying to compute $\lim_{x\to c} \frac{f(x)}{g(x)}$ where both the numerator and denominator go to zero or infinity is to multiply both the numerator and

denominator by the same factor to get a finite limit.

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1. Substitute the limiting value into the function.

2. (Optional) Make a table of values of the function near the limiting value (which for limits at positive or negative infinity means substituting large positive or negative numbers).

3. (Optional) Graph the function near the limiting value.

4. If the result of substitution is a finite number (including zero) you're done - the limit is that value. We may also deduce this when the graph indicates the function is continuous, or by plugging in values. We must use words to explain the deduction with these methods.

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Example: $\lim_{h\to 0}\frac{h^2-3}{h+1} = \frac{0-3}{0+1} = -3.$

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Example: $\lim_{s\to\infty} \frac{45}{s^2+s}$ substitutes to $\frac{45}{\infty}$, so the limit is zero since 45 divided by a large number approaches zero.

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Example: $\lim_{x\to 3} \frac{x^2+4}{x^2-9} = +\infty$ since the denominator goes to zero while the numerator approaches 13, and 13 divided by a really small number is a really big number.

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Example: $\lim_{x\to+\infty} x^2+2x$ is $+\infty$ since $x^2+2x>x$ so the function gets larger as x does.

8. If the resulting expression is zero over zero, then sometimes one can find a common factor, divide, and then get a well-behaved limit. Here the table and graphing methods can point us to the right answer, which is then best justified by the division method (though the table method is acceptable).

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Example: $\lim_{x\to 3} \frac{x^2-x-6}{x-3}$ looks like $\frac{0}{0}$ so we check and find that $x^2 - x - 6 = (x - 3)(x + 2)$ so that the limit is the same as $\lim_{x\to 3}(x + 2) = 5$.

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Example: $\lim_{s\to+\infty}\frac{2s^2+1}{s^3-3s}$ looks like $\frac{\infty}{\infty}$. After trying a few things, we multiply both the numerator and denominator by $\frac{1}{s^2}$. We get $\lim_{s\to\infty}\frac{2+\frac{1}{s^2}}{s-\frac{3}{s}}$ which when we plug in looks like $\frac{2}{\infty}$, so the limit is zero.

Using limits to understand rules for derivatives

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For one example, to take the derivative of $f(x) + g(x)$ we consider

$$\lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h},$$

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