MATH 241, LECTURE 9

1. The Derivative

The derivative of a function f at some point x is the slope of the tangent line to the graph of f at the point $(x, f(x))$. We can collect these numbers all together to make a new function, which we also call the derivative and which we name f'. We are about to learn many formal rules to find this f', but before we do that we should play a game to develop some intuition about the derivative.

Example 1. *The derivative game: given some graphs of derivative functions, sketch possible graphs for the original functions.*

2. Formula for instantaneous rate of change - the algebraic viewpoint

Last time we found what the derivative must be by looking at slopes of secant lines which were close to the tangent line. One point on the secant line was always $(x, f(x))$. The other we chose to be $(x+h, f(x+h))$, where we understood that h was meant to be small. The slope of this secant line would then be $\frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$.

Definition 2. *The derivative of $f(x)$ with respect to x is the function*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

We will talk about lim, which means the "limit", later. For now, we will adopt a working notion that we may compute this for smaller and smaller h; if all of those computations seem to approach a single number, for now we will call that the limit. We will often leave h in the equation "until the very end", at which point we "let it become very small - so small it is negligible".

Example 3. *Find the derivative of $f(x) = \frac{1}{x}$ at $x = 2$ first and then for any possible value of x.*

Example 4. *Find the derivative of $f(x) = -\frac{1}{2}x + 3$. Is the answer expected?*

Example 5. *Suppose price index, measuring the aggregate price for a large cross-section of household goods measured in thousands of dollars, has values $2 - \frac{1}{1+x}$ over two years. What is the rate of price increase over two years? What is the "instantaneous rate of inflation" at six months, one year, and two years?*

3. First rules of differentiation

It would be tedious to compute a difference quotient and take a limit every time one had to take a derivative, just as it would be tedious to add 57 to itself 13 times in order to compute 57×13. Our next class times will be devoted to learning efficient ways to compute derivatives. Just like doing multiplication, there are both some basic cases to memorize, and some rules to reduce complicated situations to simpler ones.

First, it is convenient to use a different name for the derivative.

Notation 6. *The derivative of $f(x)$ is sometimes denoted by $\frac{d}{dx}f(x)$.*

Next, we state a rule which relates for example the derivatives of x^3 and $-53x^3$.

Theorem 7. *If $g(x) = cf(x)$ then $\frac{d}{dx}g(x) = c\frac{d}{dx}$.*

Example 8. *If the derivative of* $f(x) = \sqrt{x}$ *is* $-\frac{1}{2\sqrt{x}}$, *what is the derivative of* $g(x) = 10\sqrt{x}$*?*

This rule can be deduced from the definition of the derivative, and some rules for limits. Our second rule, which may be similarly deduced, deals with addition of functions.

Theorem 9. $\frac{d}{dx}\{f(x) + g(x)\} = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

Example 10. *What is the derivative of* $3x^2 + \frac{1}{x}$*?*

Our next rule is one of the most famous, and one of which we have already seen many examples.

Theorem 11. $\frac{d}{dx}x^n = nx^{n-1}$.

This rule works even when n is a fraction, or negative, or zero. We may check it for all of the cases which we have seen so far, as well as the following.

Example 12.
- *Find the derivative of* x^{10000} *and of* $\frac{1}{x^3}$.
- *Find the derivative of* $2x^{\frac{7}{4}} - \sqrt{\frac{5}{x}}$.
- *Find the derivative of* $3x^{500} - 5.734 + x^{-500}$.