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Theorem 1. *To find the area between the graph of* $f(x)$ *and that of* $g(x)$ *between* $x = a$ *and* $x = b$ *we evaluate* $\int_a^b |f(x) - g(x)|dx$.

Example 2. *Find the area of the region between the functions $f(x) = x$ and $g(x) = x^2$ and the vertical lines at $x = -3$ and $x = -1$.*

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Example 4. *Find the area between the curves* $f(x) = \frac{2}{x}$ *and* $g(x) = x - 1$ *between* $x = 1$ *and* $x = 3$.

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We give and illustrate the answer first and then justify it in a few ways.

Theorem 5. *The average value of the function $f(x)$ over the interval from a to b is*

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

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Example 6. *Find the average value of the function* $f(x) = x\sqrt{x^2+1}$ *between* 0 *and* 1. *Does your answer make intuitive sense?*

Before further examples, we justify the average value theorem. We can use the techniques from last lecture.

We can approximate the average value by taking a large, finite set of values and averaging them:

$$\frac{f(x_0) + f(x_1) + \cdots + f(x_{N-1})}{N}.$$

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