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Theorem 2. *The inverse of the* 2×2 *matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is* $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$.

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Note that the number $ad - bc$ which appears everywhere in this formula has its own special name; it is the *determinant* of the matrix.

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Example 5. *Solve the matrix equation* $\begin{bmatrix} 3 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

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With these techniques (and calculators in hand) we can move on to swiftly solving problems with more variables.

Example 6. *The average yield on A-bonds is 6%, on B-bonds is 7% and on C-bonds is 10%. Because of a hedging scheme, you must invest twice as much money in* A *bonds as* C *bonds. Find the amounts to invest for the following desired outcomes:*

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- *\$25K invested with an annual return of \$1.8K.*
- *\$30K invested with an annual return of \$2.2K.*
- *\$40K invested with an annual return of \$2.9 K.*

Linear inequalities

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Definition 7. *A linear inequality in two variables is an equation of the form* $ax + by \geq c$ *(or* $\leq c$ *instead of* $\geq c$*), where* a, b, c *are numbers and* x *and* y *are variables or in many variables is one of the form* $\begin{bmatrix} a & b & c & \cdots \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \ldots \end{bmatrix} \geq p$

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Linear inequalities arise natually as *constraints*. For example, $x \geq 0$ is a linear inequality which means that we only want to consider positive or zero values of x, *(or $\leq p$ instead of $\geq p$) where $\begin{bmatrix} a & b & c & \cdots \end{bmatrix}$ is a row vector of numbers, $\begin{bmatrix} x \\ y \\ z \\ \ldots \end{bmatrix}$ is a column vector of variables, and p is some number.*

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Because they are easier to visualize, we will focus on linear inequalities in two variables. The set of points which satisfies a linear inequality in two variables forms a *half-space*. We check this in examples before talking about the general case. Note that we will be using the funny convention (which will make sense later) of shading in the points which do *not* satisfy the inequality.

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This procedure works because, as we willl see when we develop derivatives of functions of many variables, linear functions in any number of variables have constant derivatives,

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