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In the last lecture, we saw the theoretical idea behind the method of Lagrange multipliers, namely that an optima for a function constrained to some curve will happen at a points where the level set for the function is tangent to the curve.

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**Example 4.** *Find the optimal dimensions for a fish tank if it is to hold fifty thousand cubic centimeters of water, is supposed to have a total surface area of five thousand square centimeters, and costs one dollar per square centimeter for the base and fifty cents per square* *centimeter for the sides.*