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Example 8. *Find the critical point(s), and look at the graphs of the functions near those points, of the following:*

- $f(x, y) = x^2 + y^2$

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One way to test what kind of behavior is occurring is to take values of many points near the critical point. If, for example, they are all less than the value at the critical point, then the critical point is probably a local maximum.

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Theorem 9. *Let $f(x, y)$ be a function in two variables and let (a, b) be a critical point, so that $f_x(a, b) = 0$ and $f_y(a, b) = 0$.*

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Theorem 9. *Let $f(x, y)$ be a function in two variables and let (a, b) be a critical point, so that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let D be the quantity $f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$. If $D < 0$ then (a, b) is a saddle point.*

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