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Theorem 1. *To find the area between the graph of $f(x)$ and that of $g(x)$ between $x = a$ and $x = b$ we evaluate $\int_a^b |f(x) - g(x)| dx$.*

Example 2. *Find the area of the region between the functions* $f(x) = x$ *and* $g(x) = x^2$ *and the vertical lines at* $x = -3$ *and* $x = -1$.

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Example 4. *Find the area between the curves* $f(x) = \frac{2}{x}$ *and* $g(x) = x - 1$ *between* $x = 1$ *and* $x = 3$.

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Theorem 5. *The average value of the function $f(x)$ over the interval from a to b is*

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

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Before further examples, we justify the average value theorem. We can use the techniques from last lecture.

We can approximate the average value by taking a large, finite set of values and averaging them:

$$\frac{f(x_0) + f(x_1) + \cdots + f(x_{N-1})}{N}.$$

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