The definite integral

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Areas naturally correspond to aggregate amounts. If the bars on a bar graph of profit are of unit width, then the area of the bar graph represents total profit. We will see that for a continuous(ly modelled) revenue stream, the area under the marginal revenue curve represents total

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While we do not know how to find areas of regions with curved boundaries, we do know how to find areas of regions with straight boundaries, especially rectangles. We approximate the area of a curved region by trying to fill it (or cover it, with some "spillage") with rectangles. This method for finding curved areas, such as that inside a circle, was known to some Greek mathematicians a circle, was known to some Greek mathematicians but was lost until the Renaissance and the (re?)invention of integral calculus.

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We can see that ten rectangles gives a pretty good fit. The area of each rectangle is computed as base times height. Each rectangle has base of $\frac{1}{10} = 0.1$ and the ith rectangle has height equal to $f(0.1 \cdot i)$. Thus, the total area is

$$0.1(0.1)^2 + 0.1(0.2)^2 + 0.1(0.3)^2 + \cdots + 0.1(1.0)^2$$

$$
\begin{aligned}
0.1(0.1)^2 + 0.1(0.2)^2 + 0.1(0.3)^2 + \cdots + 0.1(1.0)^2 \\
= 0.1\left[(0.1)^2 + (0.2)^2 + \cdots + (0.9)^2 + (1.0)^2\right]
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$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_N)\Delta x$$
$$= \Delta x \left[f(x_1) + f(x_2) + \cdots + f(x_N) \right]$$

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Question: why would using any anti-derivative give the same answer when using the Fundamental Theorem?

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We will revisit the process of defining the definite integral as a sum and the Fundamental Theorem in our next lectures.