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Note that the dot product of two vectors is not another vector but a number.

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$$W = \begin{bmatrix} 1.2 & -26 \end{bmatrix} \begin{bmatrix} V \\ S \end{bmatrix}.$$

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Definition 4. *The product $M \cdot N$ of two matrices is the matrix whose entry in the ith row and jth column is the dot product of the ith row of M and the jth column of N, when this is defined.*

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Example 6. *The system of equations*

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can be written in matrix-vector notation as

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Example 7. *Suppose that the number of vampires and slayers in Sunnydale changes from one year to the next*

according to

$$\begin{bmatrix} V_{new} \\ S_{new} \end{bmatrix} = \begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix} \begin{bmatrix} V_o \\ S_o \end{bmatrix}.$$

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(Interpret this equation.) Then after ten year you would expect

$$\begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix} \begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1.2 & -26 \\ \frac{1}{100} & 1 \end{bmatrix} \begin{bmatrix} V_o \\ S_o \end{bmatrix}.$$

vampires and slayers to be on the prowl. Because matrix multiplication is associative, this can be written

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This example is a taste of population modeling using matrices. For a more complete story, take Math 342.

Solving systems of equations using inverses of matrices

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Matrices are collections of numbers which behave in some ways just like numbers themselves. We can add, subtract and multiply them, and there is a zero matrix. The similarities only go so far, though - only square matrices can both be added and multiplied, the multiplication is complicated and it depends on the order in which the matrices appear!

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Note that the number $ad - bc$ which appears everywhere in this formula has its own special name; it is the *determinant* of the matrix.

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One great advantage to matrix methods is in solving related systems.

Example 16. *Solve the matrix equation* $\begin{bmatrix} 3 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

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With these techniques (and calculators in hand) we can move on to swiftly solving problems with more variables.

Example 17. *The average yield on A-bonds is 6%, on B-bonds is 7% and on C-bonds is 10%. Because of a hedging scheme, you must invest twice as much money in* A *bonds as* C *bonds. Find the amounts to invest for the following desired outcomes:*

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