Constrained optimization and Lagrange multipliers

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What we see in these examples is that (in these cases) we can use a constraint equation to solve for one variable in terms of the other, substitute that expression into our function, and thus obtain a one-variable function to optimize.

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But what if I wanted to minimize the $f(x, y)$ from the example subject to the constraint $x^5y^7 - 3xy^2 = 2$? I could not just solve for one variable in terms of the other, so a new method is needed.

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This observation leads to a way to find optimum points because of the following theorem (which we will not be able to justify).

Theorem 3. *The slope of the level curve $f(x, y) = c$ for the function f at any point (x, y) is given by $m = -\frac{f_x}{f_y}$.*

Therefore, the level curve of $f(x, y)$ is tangent to the constraint curve $g(x, y) = c$ when $-\frac{f_x}{f_y} = -\frac{g_x}{g_y}$

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Theorem 4. *The maximum and minimum values for the function $f(x,y)$ subject to the constraint $g(x,y)=k$ occur at points (x,y) for which the following three equations hold:*

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We first apply this theorem to see that it gives the same results we found in our previous, simple, examples. pz