Embedded $S^2$ Self-Shrinkers with Rotational Symmetry

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Abstract

In this note, we use comparison arguments to show that an embedded $S^2$ self-shrinker in $\mathbb{R}^3$ with rotational symmetry must be mean-convex.

1 Introduction

In [5], Huisken showed that a mean-convex compact self-shrinker in $\mathbb{R}^3$ must be the standard sphere of radius 2 centered at the origin. In [1], Angenent constructed a compact self-shrinker with the topology type of a torus. In this note, we study compact embedded self-shrinkers in $\mathbb{R}^3$ with rotational symmetry. We use comparison arguments to show that an embedded $S^2$ self-shrinker in $\mathbb{R}^3$ with rotational symmetry must be mean-convex. An application of Huisken’s theorem gives us the following result.

Theorem 1. An embedded $S^2$ self-shrinker in $\mathbb{R}^3$ with rotational symmetry must be the standard sphere of radius 2 centered at the origin.

Huisken’s monotonicity formula [5] tells us that the blow-up at a type I singular point of mean curvature flow converges to a limiting surface which is a self-shrinker. Therefore, in order to understand the singularities of mean curvature flow, we need to understand the self-shrinking surfaces. It turns out that there is some rigidity for compact self-shrinkers. As mentioned above, Huisken [5] showed that a mean-convex compact self-shrinker must be the standard sphere. Also, the recent work of Colding and Minicozzi [2] shows that the only compact embedded $F$-stable self-shrinker is the standard sphere. On the other hand, Angenent’s torus self-shrinker [1] is an example of a compact, embedded self-shrinker in $\mathbb{R}^3$ that is not the standard sphere. It is unknown whether or not the standard sphere is the only embedded $S^2$ self-shrinker in $\mathbb{R}^3$. The result in this note tells us that the only embedded $S^2$ self-shrinker in $\mathbb{R}^3$ with rotational symmetry is the standard sphere.

Remark. In the independent work [7], Kleene and Möller showed that a complete, embedded self-shrinker in $\mathbb{R}^3$ with rotational symmetry must either be a plane, a standard cylinder, an embedded $S^1 \times S^1$, or the standard sphere.

2 Set-up

Let $M$ be an $n$-dimensional manifold, and let $F : M \times I \to \mathbb{R}^{n+1}$ be a family of embeddings. We say that $F$ is a solution of mean curvature flow if

$$ \frac{\partial F}{\partial t}(p,t) = \vec{H}(p,t), $$

where $\vec{H}(p,t)$ is the mean curvature of $M_t = F(M,t)$ at the point $F(p,t)$. If $g = g(t)$ is the metric induced on $M$ by the embedding $F(\cdot,t)$ and $\Delta_g$ is the Laplace-Beltrami operator, then the mean curvature is also given by $\Delta_g F$ and (1) becomes

$$ \frac{\partial F}{\partial t}(p,t) = \Delta_g F(p,t). $$
We say that a solution \( F : M \times (-\infty,0) \to \mathbb{R}^{n+1} \) of mean curvature flow is a self-shrinker if the submanifolds \( M_t \) satisfy \( M_t = \sqrt{-t} M_{-1} \). In this case, the embedding \( F(p) = F(p,-1) \) satisfies the equation

\[
\overrightarrow{H}(p) = -\frac{1}{2} F(p)^{\perp},
\]

where \( F(p)^{\perp} \) is the projection of \( F(p) \) into \( N_p M_{-1} \). Conversely, given a solution \( F : M \to \mathbb{R}^{n+1} \) of (3), if we define \( F(p,t) = \sqrt{-t} F(\phi(p,t)) \) where \( \phi : M \times (-\infty,0) \to M \) satisfies \( dF(\frac{\partial \phi}{\partial t}) = -\frac{1}{2t} F(\phi(x,t))^T \) and \( F^T \) is the projection of \( F \) into the tangent space, then \( F \) is a solution of mean curvature flow.

### 3 Basic Shape of Self-Shrinkers with Rotational Symmetry

Let \( M \) be a compact 2-dimensional manifold, and let \( F : M \to \mathbb{R}^3 \) be an embedding satisfying

\[
\overrightarrow{H}(p) = -\frac{1}{2} F(p)^{\perp}.
\]

We will refer to the image \( F(M) \) of the self-shrinker as \( M \). We assume that \( M \) has rotational symmetry. Then the evolution of \( M \) by mean curvature flow is rotationally symmetric about the same axis. Since \( M \) is a compact self-shrinker, its evolution by mean curvature flow shrinks to the origin, and it follows that the axis of rotation must be a line through the origin. We fix coordinates \((x,y,z)\) in \( \mathbb{R}^3 \), and assume that \( M \) has rotational symmetry about the \( z \)-axis. Since the distance between two disjoint compact surfaces evolving by mean curvature flow is non-decreasing (see Appendix A), a compact self-shrinker must be connected. Let \( C \) be the slice of \( M \) in the \((x,z)\)-plane. The rotational symmetry of \( M \) tells us that \( C \) is the union of simple closed curves, and the connectedness of \( M \) tells us that \( C \) is either a simple closed curve or the union of two simple closed curves. We note that if \( C \) intersects the \( z \)-axis it must do so transversally, and at such a point \( C \) must be perpendicular to the \( z \)-axis.

At points where the tangent line to \( C \) is not parallel to the \( z \)-axis, we may write \( C \) as the curve \((x,\gamma(x))\) where \( \gamma \) satisfies the differential equation

\[
\frac{\gamma''}{1 + (\gamma')^2} = \left( \frac{1}{2} x - \frac{1}{x} \right) \gamma' - \frac{1}{2} \gamma.
\]

On the other hand, at a point where the tangent to \( C \) is parallel to the \( z \)-axis, we may write \( C \) in a neighborhood of this point as the curve \((\alpha(z),z)\) where \( \alpha \) is non-zero and satisfies the differential equation

\[
\frac{\alpha''}{1 + (\alpha')^2} = \left( \frac{1}{\alpha} - \frac{1}{2} \alpha \right) + \frac{1}{2} z \alpha'.
\]

We derive these differential equations in Appendix B.

The comparison arguments we use in this note resemble those presented in Chapter VII of [5], where it is shown that the spheres are the only compact embedded surfaces of constant mean curvature. Let \( d = (0,1) \), and let \( n \) denote the inward pointing unit normal to \( C \). Let \( A \) be the set of points in \( C \) where the normal to \( C \) is perpendicular to \( d \):

\[
A = \{ p \in C : d \cdot n(p) = 0 \}.
\]

Notice that \( A \) is the set of points in \( C \) whose tangent lines are parallel to the \( z \)-axis. In a neighborhood of such a point, we may write \( C \) as the curve \((\alpha(z),z)\), where \( \alpha \) is non-zero and satisfies (5).

**Lemma 1.** The set \( A \) is a finite union of points.
Proof. Let \( p_0 \in \mathcal{A} \). In a neighborhood of \( p_0 \), we may write \( \mathcal{C} \) as the curve \((\alpha(z), z)\), where \( \alpha \) is a non-zero and satisfies (5). Let \( p_0 = (x_0, z_0) \), so that \( \alpha'(z_0) = 0 \), and \( \alpha''(z_0) = (2 - x_0^2)/x_0 \). If \( x_0^2 \neq 2 \), then \( \alpha''(z_0) \neq 0 \) and there exists a neighborhood \( U_0 \) of \( p_0 \) so that \( p_0 \) is the only point of \( \mathcal{A} \) in \( U_0 \). Now, we know that in a small neighborhood of \( z_0 \), as long as \(|\alpha'|\) is less than some constant, then there exists a unique solution to the initial value problem. Since the constant function \( \alpha(z) = \sqrt{2} \) is a solution (which corresponds to the cylinder self-shrinker), we see that by the uniqueness of solutions, if \( x_0 = \sqrt{2} \), then \( \mathcal{C} \) must contain the line \( \{x = \sqrt{2}\} \). But this would contradict the compactness of \( M \), and therefore \( x_0 \neq \sqrt{2} \). Similarly, we have that \( x_0 \neq -\sqrt{2} \). Since \( x_0^2 \neq 2 \), it follows by compactness that \( \mathcal{A} \) is a finite union of points. \(\square\)

In order to understand the basic shape of \( \mathcal{C} \), we assume there is a point \((\tilde{x}, \tilde{z}) \in \mathcal{C} \) with a non-vertical tangent. We also assume \( \tilde{x} \geq 0 \), \( \tilde{z} > 0 \), and there is a curve \((x, \gamma(x))\) describing \( \mathcal{C} \) such that \( \gamma(\tilde{x}) = \tilde{z} \), \( \gamma'(\tilde{x}) \leq 0 \), and \( \gamma''(\tilde{x}) < 0 \). Recall that \( \gamma \) satisfies the differential equation:

\[
\frac{\gamma''}{1 + (\gamma')^2} = \frac{1}{2^{x} - \frac{1}{x}} \gamma' - \frac{1}{2} \gamma,
\]

and taking a derivative, we have a second order differential equation for \( \gamma' \):

\[
\frac{\gamma'''}{1 + (\gamma')^2} = \frac{2\gamma'/(\gamma')^2}{(1 + (\gamma')^2)} + \left( \frac{1}{2^{x} - \frac{1}{x}} \right) \gamma'' + \frac{1}{2^x} \gamma'.
\]

We see that as \( x \) increases (starting from \( x' \)) the curve \((x, \gamma(x))\) moves down (towards the \( x \)-axis) and right (away from the \( z \)-axis). Since \( \gamma''(x') < 0 \), the curve is concave down.

Claim 1. As long as it is defined, the curve \((x, \gamma(x))\) will be concave down for \( x > x' \).

Proof. Suppose there exists \( x' > \tilde{x} \) so that \( \gamma''(x') = 0 \) and \( \gamma''(x) < 0 \) for \( \tilde{x} < x < x' \). At such a point \( \gamma''(x') < 0 \), and \( \gamma''(x') < 0 \), which is impossible. \(\square\)

In particular, there exists \( x_0 > x' \) so that either \( \gamma(x_0) = 0 \) or \( \gamma(x_0) > 0 \) and \( \lim_{x \to x_0} \gamma(x) = -\infty \). We will see that as we leave the point \((x', z')\) and travel along \( \mathcal{C} \), the path we trace out along \( \mathcal{C} \) eventually reach the positive \( x \)-axis.

Claim 2. \( x_0 \geq \sqrt{2} \).

Proof. First, if \( \lim_{x \to x_0} \gamma'(x) > -\infty \), then \( \gamma(x_0) = 0 \) and \( \frac{\gamma''(x_0)}{1 + (\gamma'(x_0))^2} = (x_0/2 - 1/x_0)\gamma'(x_0) \). Since \( \gamma''(x_0) \leq 0 \) and \( \gamma'(x_0) < 0 \), we have \( x_0 \geq \sqrt{2} \). Second, if \( \lim_{x \to x_0} \gamma'(x) = -\infty \) and \( x_0 < \sqrt{2} \), then \( \lim_{x \to x_0} \gamma''(x) = \infty \), but this impossible since \( \gamma''(x) < 0 \) for \( x < x_0 \). Thus, \( x \geq \sqrt{2} \). \(\square\)

Now, suppose we’re in the case where \( \gamma(x_0) > 0 \) and \( \lim_{x \to x_0} \gamma'(x) = -\infty \), so that the curve \((x, \gamma(x))\) did not reach the positive \( x \)-axis. In this case, \( \mathcal{C} \) has a vertical tangent at the point \( p_0 = (x_0, \gamma(x_0)) \), and we may describe \( \mathcal{C} \) as the curve \((\alpha(z), z)\), where \( \alpha > 0 \) satisfies the differential equation

\[
\frac{\alpha''}{1 + (\alpha')^2} = \frac{1}{2^{\alpha'}} + \frac{1}{\alpha} - \frac{1}{2} \alpha.
\]

At \( z_0 = \gamma(x_0) \), we have \( \alpha'(z_0) = 0 \) and \( \alpha''(z_0) = 1/x_0 - x_0/2 \). Since \( (x_0, z_0) \in \mathcal{A} \) and \( x \geq \sqrt{2} \), we have \( x_0 > \sqrt{2} \), and thus \( \alpha''(z_0) < 0 \).

This tells us that after travelling along \((x, \gamma(x))\) as \( x \) goes from \( \tilde{x} \) to \( x_0 \), we travel along \((\alpha(z), z)\), where \( \alpha'' < 0 \), so that the curve \( \mathcal{C} \) is still travelling down (towards the \( x \)-axis), and is now travelling left (back towards the \( z \)-axis). In particular, there exists \( x_1 < x_0 \) and a continuous function \( \beta \) defined on \([x_1, x_0]\) so that \( \mathcal{C} \) may be written as the curve \((x, \beta(x))\), \( \beta \) is smooth in \((x_1, x_0)\), \( \beta(x_0) = z_0 \), \( \lim_{x \to x_0} \beta'(x) = \infty \), and \( \beta \) satisfies the differential equation (4). We will see that either \( \beta(x_1) = 0 \) or \( \beta(x_1) > 0 \) and \( \lim_{x \to x_1} \beta'(x) = \infty \); however, this time the case \( \beta(x_1) = 0 \) can come from two different scenarios.

First, either using the equation for \( \beta \) or noticing that \((x, \beta(x))\) and \((\alpha(z), z)\) overlap, we see that there is an \( \varepsilon > 0 \) so that \( \beta''(x) > 0 \) for \( x \in (x_0 - \varepsilon, x_0) \). Now, if \( \beta'(x) = 0 \), then \( \beta''(x) = -\beta(x)/2 \) so
that $\beta'(x)$ does not equal 0 when $\beta''(x) > 0$ and $\beta(x) \geq 0$. In particular, there may exist $x_1 < x_0$ so that $\beta(x_1) = 0$ and $\beta''(x) > 0$ for $x_1 < x < x_0$. In this case, we know that $\beta'(x_1) \geq 0$. By the uniqueness of solutions to the initial value problem, we see that if $\beta''(x_1) = 0$, then $C$ contains the line $\{z = 0\}$ (which corresponds to the plane self-shrinker), and this contradicts the compactness of $C$. Thus $\beta'(x_1) > 0$. Also, from the equation for $\beta''$, we also see that $x_1 \geq \sqrt{2}$. This is the first scenario.

If the first scenario does not hold, then there exists $\tilde{x} < x_0$ so that $\beta''(\tilde{x}) = 0$ and $\beta''(x) > 0$ for $\tilde{x} < x < x_0$ and $\beta(\tilde{x}) > 0$. Then $\tilde{x} > \sqrt{2}$ (since $\beta'(\tilde{x}) > 0$). At $\tilde{x}$ we have $\beta''(\tilde{x}) = \beta(\tilde{x})/\tilde{x}^2 > 0$, and it follows that there is some $\varepsilon > 0$ so that $\beta''(x) < 0$ for $x \in (\tilde{x} - \varepsilon, \tilde{x})$.

Claim 3. $\beta''(x) < 0$ for $x < \tilde{x}$.

Proof. Suppose there exists a point $x'$ so that $\beta''(x') = 0$ and $\beta''(x) < 0$ for $x' < x < \tilde{x}$. Then $\beta'(x') > 0$ and hence $\beta''(x') > 0$, which is impossible.

Claim 4. Suppose $\beta$ is smooth on $(0, \tilde{x})$. If $\beta > 0$ and $\beta' > 0$ on $(0, \tilde{x})$, then $\lim_{x \to 0} \beta'(x) = \infty$. In particular, the curve $(x, \beta(x))$ intersects the $z$-axis tangentially at $(0, \beta(0))$.

Proof. For $x$ close to 0, we have $\beta'' \leq -\frac{1}{\tilde{x}} \beta'$. Integrating from $x$ to $\tilde{x}$, we see that

$$\log \beta'(\tilde{x}) - \log \beta'(x) \leq -\frac{1}{\tilde{x}} (\log \tilde{x} - \log x).$$

Thus, $\log \beta'(x) \geq -\frac{1}{\tilde{x}} \log x + \tilde{c}$, for some constant $\tilde{c}$. Taking $x \to 0$ proves the claim.

It follows from the previous claims (and arguing as we did for $\gamma$) that there exists $x_1 \in (0, \sqrt{2}]$ so that $\beta''(x) < 0$ for $x \in (x_1, \tilde{x})$ and either $\beta(x_1) = 0$ or $\beta(x_1) > 0$ and $\lim_{x \to x_1} \beta'(x) = \infty$. If the latter holds, then $x_1 < \sqrt{2}$.

Lemma 2. The mean curvature $H = \langle \vec{H}, \vec{N} \rangle$, where $\vec{N}$ is the unit inner normal to $M$, does not change sign along $\beta$.

Proof. The mean curvature $H(x)$ at the point $(x, 0, \beta(x)) \in M$ is zero if and only if the function $f(x) = x \beta'(x) - \beta(x)$ is zero (see Appendix B). We want to show that $f(x)$ is non-zero for $x < x_0$. Notice that in all the scenarios for $\beta$, we have $\lim_{x \to x_1} \beta'(x) > 0$, and hence $\lim_{x \to x_1} f(x) > 0$. Also, $\lim_{x \to x_0} f(x) = \infty$. If $f(x)$ is zero for some $x \in (x_1, x_0)$, then the infimum of $f$ over $(x_1, x_0)$ will be achieved by $f$ at some interior point $x_1 \in (x_1, x_0)$. In this case, we have $f'(x_*) = 0$ so that $\beta''(x_*) = 0$ and $f(x_*) = 2\beta'(x_*)/x_*> 0$. Since $f(x_*) \leq 0$, this is a contradiction. Therefore, $f$ is non-zero in $(x_1, x_0)$.

We note that the mean curvature $H$ at a vertical tangent is never zero. Therefore, we have shown that the mean curvature of the closed, connected subset of $M$ obtained by rotating the curve $\{(x, \beta(x)) : x_1 \leq x \leq x_0\}$ about the $z$-axis does not vanish.

Further, in the case where $\beta(x_1) > 0$ and $\lim_{x \to x_1} \beta'(x) = \infty$, the curve $C$ has a vertical tangent at $(x_1, \beta(x_1))$. Following the argument we gave for $\beta$, we see that there is a curve, which to simplify notation we will also call $\gamma$, that travels down and to the right until it either reaches the positive $y$-axis or its slope approaches $-\infty$ at some point $(x_2, \gamma(x_2))$, where $\gamma(x_2) > 0$. As above, if we rotate the curve $\{(x, \gamma(x)) : x_1 \leq x \leq x_2\}$ about the $z$-axis we obtain a closed, connected subset of $M$ with non-zero mean curvature.

Lemma 3. Let $\beta : [x_1, x_0] \to [z_1, z_0]$ and $\gamma : [x_1, x_2] \to [z_2, z_1]$ be as above (so that either $\gamma(x_2) = 0$ or $\gamma(x_2) > 0$ and $\lim_{x \to x_2} \gamma'(x) = -\infty$). Then $x_2 \leq x_0$.

Proof. Suppose $x_2 > x_0$ and consider the positive function $\delta = \beta + \gamma$ on $[x_1, x_2]$. First, since $x_2 > x_0$, we have

$$\lim_{x \to x_0} \beta'(x) = \lim_{x \to x_0} \beta'(x) + \gamma'(x_0) = \infty,$$

and the minimum of $\delta$ in $[x_1, x_0]$ cannot occur at $x_0$.  

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Second, we study the behavior of $C$ at the point $(x_1, z_1)$. At this point, $C$ has a vertical tangent, $x_1 < \sqrt{2}$, and $C$ may be written as the curve $(\alpha(z), z)$, where

$$\frac{\alpha''}{1 + (\alpha')^2} = \left(\frac{1}{\alpha} - \frac{1}{2} \alpha'\right) + \frac{1}{2} \alpha'. $$

Since $\alpha(z_1) = x_1 < \sqrt{2}$, we have $\alpha''(z_1) > 0$. Also,

$$\frac{\alpha''}{1 + (\alpha')^2} = \frac{2\alpha'(\alpha'')^2}{(1 + (\alpha')^2)^2} - \frac{1}{\alpha^2} \alpha' + \frac{1}{2} \alpha^2,$$

so that $\alpha''(z_1) = z_1 \alpha''(z_1)/2 > 0$. Then

$$\alpha(z_1 + s) - \alpha(z_1 - s) = \frac{\alpha''(z_1)}{3} s^3 + O(s^4) \quad as \ s \to 0,$$

and $\alpha(z_1 + s) > \alpha(z_1 - s)$ for small $s > 0$. Therefore, for each fixed, small $s' > 0$, there exists $t' > s'$ so that $\alpha(z_1 + s') = \alpha(z_1 - t')$. Let $y' = \alpha(z_1 + s') = \alpha(z_1 - t')$. Then

$$\delta(x') = \beta'(x') + \gamma(x') = \beta'(\alpha(z_1 + s')) + \gamma(\alpha(z_1 - t')) = (z_1 + s') + (z_1 - t') < 2z_1 = \delta(x_1).$$

It follows that the minimum of $\delta$ on $[x_1, x_0]$ must be achieved at an interior point $x_* \in (x_1, x_0)$. Then $\delta'(x_*) = 0$ and $\delta''(x_*) \geq 0$ so that $\beta'(x_*) = -\gamma'(x_*)$ and

$$0 \leq \frac{\delta''(x_*)}{1 + \beta'(x_*)^2} = \left(\frac{1}{2} x_* - \frac{1}{x_*}\right) \delta'(x_*) - \frac{1}{2} \delta(x_*) = -\frac{1}{2} \delta(x_*).$$

This contradicts the positivity of $\delta$. \hfill $\square$

4 Embedded $S^2$ Self-Shrinkers with Rotational Symmetry

Let $F : S^2 \to \mathbb{R}^3$ be a compact, embedded self-shrinker. We will refer to the embedded manifold $F(S^2)$ as $M$. We assume that $M$ has rotational symmetry about the $z$-axis. Let $C$ be the slice of $M$ in the $(x, z)$-plane. The rotational symmetry of $M$ tells us that $C$ is the union of simple closed curves. Since $M$ is a genus 0 surface, any such simple closed curve must intersect the $z$-axis, and since $M$ is connected, $C$ will be a single, simple closed curve. At a point where $C$ intersects the $z$-axis, the tangent to $C$ must be perpendicular to the $z$-axis, and thus $C$ intersects the $z$-axis at precisely two points. Without loss of generality, we will assume that one of these points has a positive $z$-coordinate.

4.1 An arc from the positive $z$-axis to the positive $x$-axis

Let $(0, z_0)$ be a point on $C$, with $z_0 > 0$, and let $C$ be described by the curve $(x, \gamma_1(x))$ where $\gamma_1(0) = z_0$, $\gamma_1'(0) = 0$, and $\gamma_1$ is a solution of (4). Then $\gamma_1''(0) = -z_0/4$, and it follows from the discussion in Section 3 that $\gamma_1$ remains concave down as $x$ goes from 0 to $x_1 \geq \sqrt{2}$, and either $\gamma_1(x_1) = 0$ or $\gamma_1(x_1) > 0$ and $\lim_{x \to x_1} \gamma'_1(x) = -\infty$.

If the second case occurs, then $x_1 > \sqrt{2}$ and there is a curve $\beta_1$ defined on $[x_2, x_1]$ so that $\beta_1(x_1) = \gamma_1(x_1)$, $\lim_{x \to x_1} \beta_1'(x) = \infty$, and either $\beta_1(x_2) = 0$ or $\beta_1(x_2) > 0$ and $\lim_{x \to x_2} \beta_1'(x) = \infty$. Again, if the second case occurs, then $x_2 < \sqrt{2}$ and there is a curve $\gamma_2$ defined on $[x_2, x_3]$ so that $\gamma_2(x_2) = \beta_1(x_2)$, $\lim_{x \to x_2} \gamma_2'(x) = -\infty$, and either $\gamma_2(x_3) = 0$ or $\gamma_2(x_3) > 0$ and $\lim_{x \to x_3} \gamma_2'(x) = -\infty$. By Lemma 3, in this case, $x_3 \leq x_1$.

Thus, we have a sequence of curves $\gamma_i$ on $[x_{2i-2}, x_{2i-1}]$ and $\beta_i$ on $[x_{2i}, x_{2i-1}]$. Let $c_i = \{(x, \gamma_i(x)) : x_{2i-2} \leq x \leq x_{2i-1}\}$ and $b_i = \{(x, \beta_i(x)) : x_{2i} \leq x \leq x_{2i-1}\}$. We claim that $\gamma_i(x_{2i-1})$ or $\beta_i(x_{2i})$ must

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be zero for some \( i \). If not, then there is an infinite sequence of points \( p_i = (x_{2i-1}, \gamma_i(x_{2i-1})) \in A \), which is impossible. We conclude that there is an arc \( \Gamma^+ \) from \((0, z_0)\) to the positive \( y \)-axis, which is a finite union of the arcs \( c_i \) and \( b_i \). Furthermore, we know that the mean curvature is non-zero on the closed set obtained by rotating \( \Gamma^+ \) about the \( z \)-axis.

### 4.2 Behavior of \( C \) at points where the tangent line is perpendicular to the \( z \)-axis

We consider a point \((\bar{x}, \bar{z}) \in C \) with \( 0 < \bar{x} < \sqrt{2} \) and \( \bar{z} > 0 \). Suppose \( C \) is given by the curve \((x, \gamma(x))\), where \( \gamma(\bar{x}) = \bar{z} \) and \( \gamma'(\bar{x}) = 0 \), so that the tangent line to \( C \) at \((\bar{x}, \bar{z})\) is perpendicular to the \( z \)-axis. We know from Section 3 that \( \gamma \) decreases and remains concave down as \( x \) goes from \( \bar{x} \) to \( \sqrt{2} \) and also \( \gamma(\sqrt{2}) \geq 0 \). By Claim 3, we know that \( \gamma \) remains concave down when \( x < \bar{x} \).

**Claim 5.** There is some \( x' > 0 \) so that \( \lim_{x \to x'} \gamma'(x) = \infty \).

**Proof.** The proof is similar to the proof of Claim 4. Suppose \( \gamma \) is smooth in \((0, x')\). Since \( M \) is compact, we know that there is some \( m > 0 \) so that \( \gamma(x) \geq -m \) for \( x \in (0, x') \). Also, since \( \gamma \) is concave down, we know that there is an \( \varepsilon > 0 \) and a \( C > 0 \) so that, \( \gamma'(x) \geq C \) when \( x < \varepsilon \). Then for \( x \) close to 0, we have \( \beta'' \leq -\frac{1}{15} \beta' \), and as in the proof of Claim 4, we deduce that \( \gamma \) intersects the \( z \)-axis tangentially, which is a contradiction. Therefore, \( \gamma' \) must blow up somewhere between \( 0 \) and \( x' \).

As an application of the previous discussion we show that the point \((0, 0)\) is not in \( C \).

**Lemma 4.** \((0, 0) \notin C \).

**Proof.** Suppose \((0, 0) \in C \). Then there exists a curve \( \gamma(x) \) so that \( C \) is given by \((x, \gamma(x))\) and \( \gamma'(0) = 0 \). Suppose there exists \( \varepsilon > 0 \) so that \( \gamma'(\varepsilon) = 0 \). The previous discussion shows that \( \gamma'(\varepsilon) \) cannot be positive (for otherwise there would be a point between 0 and \( \varepsilon \) so that \( \gamma \) has a vertical tangent). Similarly, \( \gamma'(\varepsilon) \) cannot be negative. Thus \( \gamma'(\varepsilon) = 0 \), and by uniqueness of the initial value problem, \( \gamma(x) = 0 \) for \( x \geq 0 \). Since \( M \) is compact, this cannot happen, and we deduce that \( \gamma'(x) \) is non-zero for \( x > 0 \). To see that this cannot happen, we notice that if \( \gamma'(x) > 0 \) for small \( x > 0 \), then \( \gamma(x) > 0 \) for small \( x > 0 \), and the equation for \( \gamma'' \) shows that \( \gamma''(x) < 0 \) for small \( x > 0 \). But this cannot happen when \( \gamma'(0) = 0 \). Similarly, we cannot have \( \gamma'(x) < 0 \) for small \( x > 0 \). \( \square \)

### 5 A Comparison Principle

In this section we derive a comparison principle for solutions to the differential equation

\[
g'' \frac{1}{1 + (\gamma')^2} = \left( \frac{1}{2} x - \frac{1}{x} \right) \gamma' - \frac{1}{2} \gamma,
\]

with \( \gamma'(0) = 0 \) and \( \gamma(0) > 0 \). An application of this comparison principle shows that \( C \) cannot intersect the positive \( z \)-axis at two different points.

**Proposition 1.** Let \( \gamma \) and \( \beta \) be two solutions of (4) with \( \gamma'(0) = \beta'(0) = 0 \) and \( \gamma'(0) > \beta'(0) > 0 \). Suppose there exist \( x_1, x_2 > 0 \) so that \( \lim_{x \to x_1} \gamma'(x) = -\infty \) and \( \lim_{x \to x_2} \beta'(x) = -\infty \). Then \( \gamma \) and \( \beta \) intersect or \( x_2 < x_1 \).

**Proof.** Suppose \( \gamma \) and \( \beta \) do not intersect and \( x_2 < x_1 \). Then we consider \( \delta = \gamma - \beta \) on \([0, x_2]\). If \( \gamma \) and \( \beta \) do not intersect, then \( \delta > 0 \) on \([0, x_2]\), and if \( x_2 < x_1 \) then \( \lim_{x \to x_2} \delta'(x) = -\infty \). Also, \( \delta'(0) = 0 \).

Using the equations for \( \gamma'' \) and \( \beta'' \), we have

\[
\delta''(x) = \left( \frac{1}{2} x - \frac{1}{x} \right) (1 + \gamma'' + \gamma' \beta' + \beta'' \delta' - \frac{1}{2} \beta' \delta - \frac{1}{2} \beta (\gamma' + \beta')) - \frac{1}{2} \beta (\gamma' + \beta') \delta' - \frac{1}{2} (1 + \gamma' \delta).
\]
Since \( \delta > 0 \) and \( \delta'(0) = 0 \), Hopf’s lemma tells us that the minimum of \( \delta \) cannot occur at 0. Also, since \( \lim_{x \to x_2} \delta'(x) = \infty \), the minimum cannot occur at \( x_2 \). Therefore, the minimum of \( \delta \) on \([0, x_2]\) must occur at some point \( x' \in (0, x_2) \), but this contradicts the maximum principle.

As an application of the previous proposition we use the embeddedness of \( M \) to show that \( C \) cannot intersect the positive \( z \)-axis at two different points.

**Lemma 5.** \( C \) cannot intersect the positive \( z \)-axis at two different points.

*Proof.* Suppose \( C \) intersects the positive \( z \)-axis at two different points. Then there are two curves \( \gamma \) and \( \beta \) satisfying the conditions of the previous proposition. Namely, \( \gamma \) and \( \beta \) are solutions of (4) with \( \gamma'(0) = \beta'(0) = 0 \) and \( \gamma(0) > \beta(0) > 0 \). Furthermore, the curves \((x, \gamma(x))\) and \((x, \beta(x))\) are concave down components of the same simple closed curve and cannot intersect. Thus, applying the previous proposition, we have \( x_2 \geq x_1 \).

Let \( \Gamma^+ \) be the arc from the positive \( z \)-axis to the positive \( x \)-axis which begins at the point \((0, \gamma(0))\). We treat two cases. First, suppose \( \beta(x_2) \geq 0 \). Let \((x_0, 0) \in \Gamma^+ \) be the point where \( \Gamma^+ \) intersects the positive \( x \)-axis. We know \( x_0 \leq x_1 \leq x_2 \) (the \( x \)-coordinates of the points in \( \Gamma^+ \) are less than or equal to \( x_1 \)). Let \( S \) be the simple closed curve formed by \( \Gamma^+ \), the line segment \([0, \gamma(0)]\), and the line segment \([0, x_0] \times \{0\} \). If \( x_0 < x_2 \), then the \( \beta \) curve must intersect \( S \) at some point \((x', \beta(x'))\) where \( 0 < x' < x_2 \). But then \( \beta'(x') > 0 \) and \( (x', \beta(x')) \in \Gamma^+ \). If \( x_0 = x_2 \), then either \( \beta(x_2) = 0 \) and the \( \beta \) curve intersects \( S \) at \((x_2, 0) \) or \( \beta(x_2) > 0 \) and the \( \beta \) curve intersects \( S \) at some point \((x', \beta(x'))\) where \( \beta'(x') > 0 \). In all these cases, the \( \beta \) curve starts below \( \Gamma^+ \), is decreasing, and eventually intersects \( \Gamma^+ \), which contradicts the assumption that \( M \) is an embedded surface.

Second, we treat the case where \( \beta(x_2) < 0 \). As above, let \((x_0, 0) \) be the point where \( \Gamma^+ \) intersects the positive \( x \)-axis. Since \( \beta(x_2) < 0 \), the \( \beta \) curve intersects the positive \( x \)-axis at some point with \( x \)-coordinate greater than or equal to \( \sqrt{2} \). We know that \( \Gamma^+ \) and the \( \beta \) curve are disjoint, and thus \( x_0 > \sqrt{2} \) and \( \beta(x_0) < 0 \). Let \( \tilde{\gamma} \) be the curve which describes \( C \) as it leaves \((x_0, 0)\) and enters the lower half-plane. If \( \tilde{\gamma} \) is increasing at \( x_0 \), it must be concave up, and it will be defined when \( x = \sqrt{2} \). Moreover, \( \tilde{\gamma}(\sqrt{2}) < 0 \). In this case, \( \beta(\sqrt{2}) > \tilde{\gamma}(\sqrt{2}) \) and \( \beta(x_0) < \tilde{\gamma}(x_0) \), so there exists a point \( \sqrt{2} < x' < x_0 \) where \( \beta(x') = \tilde{\gamma}(x') \), which contradicts the assumption that \( M \) is an embedded surface.

Finally, if \( \tilde{\gamma} \) from the previous case is decreasing, then it will be defined on some interval \([x_3, x_4]\) where \( \lim_{x \to x_3} \tilde{\gamma}(x) = -\infty \) and \( \lim_{x \to x_4} \tilde{\gamma}(x) = -\infty \). Since \( \beta \) is also decreasing (and \( M \) is an embedded surface), we have \( \tilde{\gamma} > \beta \) on their common domain. The argument used in proving the comparison principle shows that \( x_4 \leq x_2 \), which reduces this case to the previous one (namely, there exists a solution which is increasing and above \( \beta \) as \( x \) approaches \( x_4 \)).

### 6 Proof of Mean-Convexity

In this section, we show that an embedded, compact \( S^2 \) self-shrinker with rotational symmetry is mean convex.

**Theorem 2.** An embedded, compact \( S^2 \) self-shrinker in \( \mathbb{R}^3 \) with rotational symmetry is mean-convex.

*Proof.* Let \( F : S^2 \to \mathbb{R}^3 \) be a compact, embedded self-shrinker. We will refer to the embedded manifold \( F(S^2) \) as \( M \). We assume that \( M \) has rotational symmetry about the \( z \)-axis. Let \( C \) be the slice of \( M \) in the \((x, z)\)-plane. The rotational symmetry of \( M \) tells us that \( C \) is the union of simple closed curves. Since \( M \) is a genus 0 surface, any such simple closed curve must intersect the \( z \)-axis, and since \( M \) is connected, \( C \) will be a single, simple closed curve. At a point where \( C \) intersects the \( z \)-axis, the tangent to \( C \) must be perpendicular to the \( z \)-axis, and thus \( C \) intersects the \( z \)-axis at precisely two points. Applying Lemma 4 and Lemma 5, we see that \( C \) intersects the \( z \)-axis at a point \((0, s_+)\) with \( s_+ > 0 \) and at a point \((0, s_-)\) with \( s_- < 0 \). Let \( \Gamma^+ \) be the arc from \((0, s_+)\) to the positive \( x \)-axis (see Section 4.1), and let \((x_+, 0)\) be the point where \( \Gamma^+ \) intersects the positive \( x \)-axis. Similarly, we can define an arc \( \Gamma^- \) from \((0, s_-)\) to the positive \( x \)-axis. Let \((x_-, 0)\) be the point where \( \Gamma^- \) intersects the positive \( x \)-axis.
We claim that $x_\pm = x_\mp$. Suppose not. Then without loss of generality we assume $x_- < x_+$. Let $S$ be the simple closed curve formed by $\Gamma^+$, the line segment $\{0\} \times [0,s_+]$, and the line segment $[0,x_+] \times \{0\}$. As we travel along $\Gamma^-$, we enter $S$ at the point $(x_- , 0)$. Since $\mathcal{C}$ is a simple closed curve, which only intersects the $z$-axis at $(0,s_+)$ and $(0,s_-)$, we see that there is a curve that starts at $(x_- , 0)$, enters the interior of $S$, and first leaves $S$ at some point $(x_0, 0)$ with $x_0 < x_+$. In other words, there exists a curve $\rho : [0,1] \to \mathbb{R}^2$ such that $\rho(0) = (x_-, 0)$ and $\rho(1) = (x_0, 0)$ and $\rho(t)$ is in the interior of $S$ for $t \in (0,1)$. In addition, we note that that $\rho$ does not intersect $\Gamma^+$.

We write $\rho = (\rho_1, \rho_2)$. Let $t_0$ be the point where $\rho_2$ achieves its maximum on $[0,1]$. Since the maximum of $\rho_2$ is positive, this occurs at some $t_0 \in (0,1)$. Let $\beta$ be the curve which describes $C$ in a neighborhood of $\rho(t_0)$, and let $x' = \rho_1(t_0)$. Then $\beta$ is a solution of (4) and $\beta'(x') = 0$. From the discussion in Section 4.2, we know that there exist $x_1 < x'$ and $x_2 > x'$ so that $\lim_{x \to x_1} \beta'(x) = \infty$ and $\lim_{x \to x_2} \beta'(x) = -\infty$. Let $\gamma$ be the curve that describes $\Gamma$ at the point $(0,s_+)$, so that $\gamma$ is a solution of (4), $\gamma(0) = s_+$, and $\gamma'(0) = 0$. Then there is a point $x_\alpha > \sqrt{2}$ so that $\lim_{x \to x_\alpha} \gamma'(x) = -\infty$. The argument used to prove Lemma 5 (in particular, the third paragraph of the proof) leads to a contradiction. Therefore, we must have $x_- = x_+.$

Let $M^+(M^-)$ be the connected, closed set obtained by rotating the arc $\Gamma^+(\Gamma^-)$ about the $z$-axis. From Section 4.1, we know that the mean curvature does no vanish on $M^+$ and $M^-$. Since $x_+ = x_\pm$, we see that $M = M^+ \cup M^-$ and $M^+ \cap M^-$ is non-empty. Therefore, the mean curvature $H = \langle \overrightarrow{H}, \overrightarrow{N} \rangle$, where $\overrightarrow{N}$ is the unit inner normal to $M$, does not vanish on $M$. Since $M$ is compact, $H$ is positive at some point, and we conclude that $M$ is mean-convex.

**Corollary 1.** By Huisken’s classification of mean-convex compact self-shrinkers, we conclude that a compact, embedded $S^2$ self-shrinker in $\mathbb{R}^3$ with rotational symmetry must be the standard sphere of radius 2 centered at the origin.

## 7 Appendix A: The Distance Between Compact Solutions of Mean Curvature Flow

Let $M_1$ and $M_2$ be compact $n$-dimensional manifolds, and let $F : M_1 \times I \to \mathbb{R}^{n+1}$ and $G : M_2 \times I \to \mathbb{R}^{n+1}$ be solutions of mean curvature flow. We assume that $F(M_1,t)$ and $G(M_2,t)$ are disjoint for $t \in I$, and we want to show that the difference $d(t)$ between $F(M_1,t)$ and $G(M_2,t)$ is increasing in $t$.

We consider the function $f(x,y,t) = |F(x,t) - G(y,t)|^2$. First, we fix $t \in I$, and let $x_0 \in M_1$, $y_0 \in M_2$ be such that $f(x_0,y_0,t) \leq f(x,y,t)$ for all $x \in M_1$, $y \in M_2$. In a neighborhood of $F(x_0,t)$, we choose a (smooth) unit normal $\overrightarrow{N}_F$. Similarly, we choose $\overrightarrow{N}_G$.

**Claim 6.** There exist constants $a, b$ so that $F(x_0,t) - G(y_0,t) = a\overrightarrow{N}_F(x_0,t) = b\overrightarrow{N}_G(y_0,t)$.

**Proof.** Let $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ be coordinates on $M_1$ and $M_2$, respectively. Then

$$0 = \frac{\partial f}{\partial x^i}(x_0,y_0,t) = 2(F(x_0,t) - G(y_0,t), \frac{\partial F}{\partial x^i}(x_0,t))$$

and

$$0 = \frac{\partial f}{\partial y^i}(x_0,y_0,t) = -2(F(x_0,t) - G(y_0,t), \frac{\partial G}{\partial y^i}(x_0,t)).$$

Thus, $F(x_0,t) - G(y_0,t)$ is normal to $F(M_1,t)$ at $F(x_0,t)$ and to $G(M_2,t)$ at $G(x_0,t)$. □

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Now, we can choose coordinates \((x^1, \ldots, x^{n+1})\) on \(\mathbb{R}^{n+1}\) so that \(F(x_0, t) = 0\) and the tangent space to \(F(M_1, t)\) at 0 is the span of \(\frac{\partial}{\partial x^i}|_0, \ldots, \frac{\partial}{\partial x^{n+1}}|_0\). Furthermore, in a neighborhood of \(F(x_0, t)\), we may write \(F(\cdot, t)\) as \((x, \phi(x))\). By the previous claim, we see that \(G(y_0, t) = (0, \ldots, 0, d) = d\), and the tangent space to \(G(M_2, t)\) at \(d\) is the span of \(\frac{\partial}{\partial x^i}|_d, \ldots, \frac{\partial}{\partial x^{n+1}}|_d\). As above, in a neighborhood of \((y_0, t)\), we may write \(G(\cdot, t)\) as \((y, \psi(y))\). Without loss of generality, we may assume \(d > 0\).

Let \(g_F\) and \(g_G\) denote the induced metrics on \(F(M_1, t)\) and \(G(M_2, t)\), respectively. Then \(g_F(x_0) = g_G(y_0) = \text{Id}\). Now, the mean curvature \(\overline{H}_F\) of \(F(M_1, t)\) at the point \(F(x, t)\) is given by \(\overline{N}_F(x, t) = \Delta_F F(x, t)\), where \(\Delta_F\) denotes the Laplace-Beltrami operator on \(F(M_1, t)\) with respect to the induced metric \(g_F\). Similarly \(\overline{H}_G(y, t) = \Delta_G G(y, t)\). Using Claim 6, and suppressing the \(t\) variable to simplify the notation, we have

\[
\langle F(x_0) - G(y_0), \overline{H}_F(x_0) - \overline{H}_G(y_0) \rangle = \langle F(x_0) - G(y_0), \Delta_F F(x_0) - \Delta_G G(y_0) \rangle = \langle F(x_0) - G(y_0), g_F^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j}(x_0) - g_G^{ij} \frac{\partial^2 G}{\partial y^i \partial y^j}(y_0) \rangle = -d \left( g_F^{ij} \frac{\partial^2 \phi}{\partial x^i \partial x^j}(0) - g_G^{ij} \frac{\partial^2 \psi}{\partial y^i \partial y^j}(0) \right).
\]

**Claim 7.**

\[
g_F^{ij} \frac{\partial^2 \phi}{\partial x^i \partial x^j}(0) \leq g_G^{ij} \frac{\partial^2 \psi}{\partial y^i \partial y^j}(0)
\]

**Proof.** We know that \(f(x, y, t) \geq f(x_0, y_0, t) = d^2\). In particular, \(|\psi(x) - \phi(x)|^2 \geq d^2\), and since \(\psi(0) - \psi(0) = d > 0\), we have \(\psi(x) - \psi(x) \geq d\). Therefore \(\psi - \phi\) has a local minimum at 0 (here we are using \(x\) as the variable for \(\psi\)). In particular,

\[
\frac{\partial^2 \psi}{\partial x^2}(0) - \frac{\partial^2 \phi}{\partial x^2}(0) \geq 0.
\]

Since \(g_F(0) = g_G(0) = \text{Id}\), we conclude that

\[
g_F^{ij} \frac{\partial^2 \psi}{\partial y^i \partial y^j}(0) - g_G^{ij} \frac{\partial^2 \phi}{\partial y^i \partial y^j}(0) = \sum_{i=1}^n \frac{\partial^2 \psi}{\partial x^2}(0) - \frac{\partial^2 \phi}{\partial x^2}(0) \geq 0.
\]

It follows from the above discussion and the previous claim that

\[
(F(x_0, t) - G(y_0, t), \overline{H}_F(x_0, t) - \overline{H}_G(y_0, t)) \geq 0.
\]

Since \(F\) and \(G\) are solutions of mean curvature flow, we have \(\frac{\partial F}{\partial t}(x, t) = \overline{H}_F(x, t)\) and \(\frac{\partial G}{\partial t}(y, t) = \overline{H}_G(y, t)\). Therefore,

\[
\frac{\partial f}{\partial t}(x_0, y_0, t) = 2\langle F(x_0, t) - G(y_0, t), \overline{H}_F(x_0, t) - \overline{H}_G(y_0, t) \rangle \geq 0.
\]

Now, we introduce a function \(f = f(t)\) defined by

\[
f(t) = \min_{x \in M_1, y \in M_2} f(x, y, t).
\]

If we restrict to a closed interval \([t_1, t_2]\) in \(I\), then \(f(t)\) is a Lipschitz function, and consequently, \(f\) is differentiable almost everywhere. Let \(t_0\) be a point where \(f(t)\) is differentiable and take \(x_0\) and \(y_0\) so that \(f(t_0) = f(x_0, y_0, t_0)\). Since \(f(t)\) is differentiable at \(t_0\), we have \(f'(t_0) = \frac{\partial f}{\partial t}(x_0, y_0, t_0)\), and from the previous discussion, we know that \(\frac{\partial f}{\partial t}(x_0, y_0, t_0) \geq 0\). Therefore, \(f'(t_0) \geq 0\). Integrating \(f'\) from \(t_1\) to \(t_2\), we conclude that \(f(t_2) \geq f(t_1)\). In particular, the distance between \(F(M_1, t)\) and \(G(M_2, t)\) is non-decreasing.
Proposition 2. Let $M_1$ and $M_2$ be compact $n$-dimensional manifolds, and let $F : M_1 \times I \to \mathbb{R}^{n+1}$ and $G : M_2 \times I \to \mathbb{R}^{n+1}$ be solutions of mean curvature flow. Let $d(t)$ denote the distance between $F(M_1, t)$ and $G(M_2, t)$. If $t_2 \geq t_1$, then $d(t_2) \geq d(t_1)$.

As an application of the above proposition, we show that a compact self-shrinker must be connected. Suppose $F : M \to \mathbb{R}^{n+1}$ is a self-shrinker with two (disjoint) components: $M_1$ and $M_2$. Now, the one parameter family of surfaces $\sqrt{-t}F(M)$ is evolving by mean curvature flow as $t$ goes from, say, $-1$ to $-1/2$. In particular, the distance $d(t)$ from $\sqrt{-t}M_1$ to $\sqrt{-t}M_2$ is given by $d(t) = \sqrt{-t} \operatorname{dist}(M_1, M_2)$. However, as $t$ goes from $-1$ to $-1/2$, $d(t)$ is decreasing, which contradicts the previous proposition. Therefore, a compact self-shrinker must be connected.

8 Appendix B: Derivation of the Differential Equations

Let $C$ be a curve in the $(x, z)$-plane, and let $M$ be that rotation of $C$ about the $z$-axis. We assume that if $C$ intersects the $z$-axis, then it does so perpendicularly.

8.1 Differential equation at a non-vertical tangent

At a point $(x', z')$ where $C$ is given by the curve $(x, \gamma(x))$, we want to relate the differential equation $\gamma$ satisfies with the mean curvature of $M$. Since $M$ has rotational symmetry about the $z$-axis, it suffices to consider the point $(x', 0, z')$ in $M$. Near the point $(x', 0, z')$, the surface $M$ may be written as

$$F(x, y) = (x, y, \gamma(\sqrt{x^2 + y^2})).$$

Then

$$\frac{\partial F}{\partial x} = \begin{pmatrix} 1, 0, \frac{x}{r} \gamma'(r) \end{pmatrix}$$

and

$$\frac{\partial F}{\partial y} = \begin{pmatrix} 0, 1, \frac{y}{r} \gamma'(r) \end{pmatrix},$$

where $r = \sqrt{x^2 + y^2}$. Also, a unit normal to $M$ at $F(x, y)$ is given by

$$\vec{N} = \frac{1}{\sqrt{1 + (\gamma')^2}} \begin{pmatrix} \frac{x}{r} \gamma', \frac{y}{r} \gamma', -1 \end{pmatrix}.$$

Consider the mean curvature $\vec{H} = \Delta_g F$. We set $x^1 = x$, $x^2 = y$ to compute $\Delta_g F$:

$$g = \text{Id} + \left(\frac{(\gamma')^2}{r^2}\right) (x^i x^j),$$

$$\det(g) = 1 + (\gamma')^2,$$

$$g^{-1} = \text{Id} - \frac{1}{\det(g)} \left(\frac{(\gamma')^2}{r^2}\right) (x^i x^j),$$

$$\frac{\partial^2 F}{\partial x^i \partial x^j} = \begin{pmatrix} 0, 0, \left[\frac{\delta_{ij}}{r} - \frac{x^i x^j}{r^3}\right] \gamma' + \frac{x^i x^j}{r^2} \gamma'' \end{pmatrix}. $$

Then

$$\langle \Delta_g F, \vec{N} \rangle = \langle g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j}, \vec{N} \rangle$$

$$= -\frac{1}{\sqrt{1 + (\gamma')^2}} \left(\frac{1}{r} \gamma' + \frac{1}{1 + (\gamma')^2}\right),$$

and

$$\Delta_g F = -\frac{1}{\sqrt{1 + (\gamma')^2}} \left(\frac{\gamma''}{1 + (\gamma')^2} + \frac{1}{r} \gamma'\right) \vec{N}. $$

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Also,

\[-\frac{1}{2} F^\perp = -\frac{1}{2} \frac{1}{\sqrt{1 + (\gamma')^2}} (r\gamma' - \gamma) \mathbf{N}. \]

Therefore, we see that \( M \) satisfies \( \Delta_g F = -\frac{1}{2} F^\perp \) if and only if \( \gamma(x) \) satisfies

\[ \frac{\gamma''}{1 + (\gamma')^2} + \frac{1}{r} \frac{\gamma'}{r^2} = \frac{1}{2} \frac{\gamma'}{r} - \frac{1}{2} \gamma. \]

When we restrict to the \((x, z)\)-plane this gives us the differential equation for \( \gamma(x) \).

### 8.2 Differential equation at a vertical tangent

At a point \((x', z')\) where \( C \) is given by the curve \((\alpha(z), z)\), we want to relate the differential equation \( \alpha \) satisfies with the mean curvature of \( M \). Since \( M \) has rotational symmetry about the \( z \)-axis, it suffices to consider the point \((x', 0, z')\) in \( M \). Also, since \( C \) only intersects the \( z \)-axis perpendicularly, we know that \( x' \neq 0 \). We assume that \( x' > 0 \). Then, near the point \((x', 0, z')\), the surface \( M \) may be written as

\[ F(y, z) = (\sqrt{\alpha(y)^2 - y^2}, y, z). \]

We have

\[ \frac{\partial F}{\partial y} = \left( \frac{-y}{\sqrt{\alpha^2 - y^2}}, 1, 0 \right) \]

and

\[ \frac{\partial F}{\partial z} = \left( \frac{\alpha'}{\sqrt{\alpha^2 - y^2}}, 0, 1 \right). \]

Also, a unit normal to \( M \) at \( F(y, z) \) is given by

\[ \mathbf{N} = \frac{1}{\sqrt{1 + (\alpha')^2}} \left( -\frac{\sqrt{\alpha^2 - y^2}}{\alpha}, -\frac{y}{\alpha}, \alpha' \right). \]

Let \( V(y, z) = (y, -\alpha(z)\alpha'(z)) \) so that

\[ g = \text{Id} + \frac{1}{\alpha^2 - y^2} V^T \cdot V, \]

\[ \det(g) = \frac{\alpha^2 (1 + (\alpha')^2)}{\alpha^2 - y^2}, \]

and

\[ g^{-1} = \text{Id} - \frac{1}{\alpha^2 (1 + (\alpha')^2)} V^T \cdot V. \]

The second derivatives of \( F \) are:

\[ \frac{\partial^2 F}{\partial y^2} = \left( \frac{-\alpha^2}{(\alpha^2 - y^2)^{3/2}}, 0, 0 \right), \]

\[ \frac{\partial^2 F}{\partial y \partial z} = \left( \frac{y \alpha'}{(\alpha^2 - y^2)^{3/2}}, 0, 0 \right), \]

and

\[ \frac{\partial^2 F}{\partial z^2} = \left( \frac{(\alpha^2 - y^2)\alpha'' - y^2 (\alpha')^2}{(\alpha^2 - y^2)^{3/2}}, 0, 0 \right). \]

To compute \( \Delta_g F \), we set \( x^1 = y, x^2 = z \). Then

\[ g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j} = \frac{1}{\sqrt{\alpha^2 - y^2}} \left( \frac{\alpha \alpha''}{1 + (\alpha')^2} - 1, 0, 0 \right) \]
so that
\[ \langle \Delta_g F, \mathbf{N} \rangle = -\frac{1}{\sqrt{1 + (\alpha')^2}} \left( \frac{\alpha''}{1 + (\alpha')^2} - \frac{1}{\alpha}, 0, 0 \right), \]
and
\[ \Delta_g F = -\frac{1}{\sqrt{1 + (\alpha')^2}} \left( \frac{\alpha''}{1 + (\alpha')^2} - \frac{1}{\alpha}, 0, 0 \right) \mathbf{N}. \]
Also,
\[ -\frac{1}{2} F^\perp = -\frac{1}{2} \frac{1}{\sqrt{1 + (\alpha')^2}} (\frac{z \alpha'}{2} - \alpha) \mathbf{N}. \]
Therefore, we see that \( M \) satisfies \( \Delta_g F = -\frac{1}{2} F^\perp \) if and only if \( \alpha(z) \) satisfies
\[ \frac{\alpha''}{1 + (\alpha')^2} - \frac{1}{\alpha} = \frac{1}{2} z \alpha' - \frac{1}{2} \alpha. \]

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**References**


