1. **C*-algebras.**

Recall that a *Banach algebra* $A$ is a Banach space that is also an associative algebra over $\mathbb{C}$, such that $\|ab\| \leq \|a\| \cdot \|b\|$ whenever $a, b \in A$, and $\|1_A\| = 1$ whenever $A$ is unital.

**Definition 1.1.** A *Banach *$*$-algebra* $A$, is a Banach algebra with a conjugate linear involution $*: A \to A$:

$$(a + \lambda \cdot b)^* = a^* + \overline{\lambda} \cdot b^*, \quad (a\cdot b)^* = a^* \cdot b^*, \quad (ab)^* = b^*a^*.$$  

If $a \in A$, then $a^*$ is usually called the adjoint of $a$. 

If the norm on $A$ also satisfies $\|a^*a\| = \|a\|^2$, then $A$ is called a *$C^*$-algebra.*

**Remark 1.2.** Notice that since $\|x\|^2 = \|x^*x\| \leq \|x\| \cdot \|x^*\|$ whenever $x \neq 0$. Since $x^{**} = x$, we get $\|x\| = \|x^*\|$ for all $x \in A$.

**Examples 1.3.** The following examples of $C^*$-algebras exhaust, in some sense, all $C^*$-algebras of certain kinds.

1. If $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a $C^*$-algebra, with the adjoint of $T$ being characterized by $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathcal{H}$.
2. More generally, any *$*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is naturally a $C^*$-algebra.
3. If $\mathcal{H}$ has finite dimension $n$, then $\mathcal{B}(\mathcal{H}) \cong M_n(\mathbb{C})$. Hence $M_n(\mathbb{C})$ is a *finite dimensional simple C*-algebra.*
4. More generally, $\oplus_{j=1}^{n} M_{n_j}(\mathbb{C})$ is a *finite dimensional C*-algebra.*
5. If $X$ is a locally compact and Hausdorff space, then $C_0(X)$ is a *commutative C*-algebra.* Notice that $C_0(X)$ is unital if and only if $X$ is compact, in which case $C_0(X) = C(X)$.

**Remark 1.4.** Notice that the last example is a particular case of the second one. Indeed, endow $X$ with any complex Borel measure $\mu$. Then $C_0(X) \leq B(L^2(X, \mu))$, where the action of $f \in C_0(X)$ on $g \in L^2(X, \mu)$ is given by multiplication: $f \cdot g = fg$. Notice that in this case,

$$\int_X |fg|^2d\mu \leq \|f\|_2^2 \int_X |g|^2d\mu = \|f\|_2^2 \cdot \|g\|_2^2 < \infty.$$  

The goal of this talk is to show that:

- Every $C^*$-algebra is (isomorphic to) a *$C^*$*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Hence, every $C^*$-algebra is as in (2).
- Every simple, finite-dimensional $C^*$-algebra is (isomorphic to) $M_n(\mathbb{C})$ for $n^2 = \text{dim}_\mathbb{C} A$. Hence, every simple, finite-dimensional $C^*$-algebra is as in (3).
- Every finite-dimensional $C^*$-algebra is (isomorphic to) a sum of matrices with coefficients in $\mathbb{C}$. Hence, every finite-dimensional $C^*$-algebra is as in (4).
- Every commutative $C^*$-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space $X$. Hence, every commutative $C^*$-algebra is as in (5).

Recall that given a ring $R$, its *Jacobson radical*, denoted by $J(A)$, is the intersection of all maximal ideals.

**Lemma 1.5.** If $A$ is a $C^*$-algebra, then $J(A) = 0$.

2. **Finite-dimensional C* -algebras.**

Assume that $A$ is a finite-dimensional $C^*$-algebra. Then $A$, regarded as an algebra over $\mathbb{C}$ is left Artinian, and since $J(A) = 0$, we conclude that $A$ is semisimple. Wedderburn-Artin theorem for algebras implies that there are positive integers $n_1, \ldots, n_m$ and an isomorphism of algebras over $\mathbb{C}$

$$A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_m}(\mathbb{C}).$$  

One can show that the above isomorphism is actually isometric and preserves the involution. We conclude that it is a $C^*$-algebra isomorphism. Hence, we’ve showed that

**Theorem 2.1.** Every finite-dimensional $C^*$-algebra is (isomorphic to) a sum of matrices.

**Corollary 2.2.** Up to isomorphism, the only simple finite-dimensional $C^*$-algebra is $M_n(\mathbb{C})$, where $n^2 = \text{dim}_\mathbb{C} A$.  

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Throughout this and the following sections, $A$ will denote a $C^*$-algebra. Moreover, ideals of $C^*$-algebras are always assumed to be two-sided and closed.

The goal of this section is to prove that if $A$ is commutative, then $A \cong C_0(X)$ for some space $X$. If we start with an isomorphism $A \cong C_0(X)$, we can recover $X$ from $A$ by considering the set of maximal ideals of $A$, because each maximal ideal $M$ has the form $M = \{f \in C_0(X) : f(x_0) = 0\}$ for some $x_0 \in X$. This is the motivating idea for this section.

**Definition 3.1.** A character on $A$ is a nonzero $*$-preserving linear homomorphism from $A$ into $\mathbb{C}$. If $A$ is unital, we require this homomorphism to be unital.

The set of all characters on $A$ is called the maximal ideal space of $A$, and it is denoted by Max($A$).

**Remark 3.2.** In the above definition, preservation of the involution is redundant. Indeed, any $*$-homomorphism $A \to \mathbb{C}$ is automatically $*$-preserving.

**Theorem 3.3.** Let $A$ be unital.

(a) If $\omega \in$ Max($A$), then $\|\omega\| = 1$.

(b) The map $\omega \mapsto \ker \omega$ defines a bijection from Max($A$) onto the set of all maximal ideals of $A$.

**Remark 3.4.** If $A$ is unital, then Max($A$) is nonempty. However, Max($A$) may be empty if $A$ is nonunital, although this is still true for commutative $C^*$-algebras.

Endow Max($A$) with the weak* topology (pointwise convergence). Since Max($A$) is a closed subset (that is, the pointwise limit of a net of characters is again a character – only algebra is involved here) of the unit ball of $A^*$, the following theorem is a consequence of Alaoglu’s theorem.

**Theorem 3.5.** If $A$ is unital and commutative, then Max($A$) is a nonempty compact Hausdorff space.

**Definition 3.6.** Let $A$ be unital and commutative. If $a \in A$, define $\varepsilon_a : \text{Max}(A) \to \mathbb{C}$ by

$$\varepsilon_a(\omega) = \omega(a).$$

**Proposition 3.7.** The identity (1) defines a homomorphism $\varepsilon : A \to C(\text{Max}(A))$, given by $a \mapsto \varepsilon_a$.

**Proof.** Let’s check that $\varepsilon_a$ is continuous. Notice that if $F \subseteq \mathbb{C}$ is closed, then

$$\varepsilon_a^{-1}(F) = \{\omega \in \text{Max}(A) : \omega(a) \in F\}$$

is weak* closed (recall that $\omega_i \to \omega$ weak* if $\omega_i(b) \to (\omega(b))$). Hence $\varepsilon_a \in C(\text{Max}(A))$.

$\varepsilon$ is a unital $*$-homomorphism:

$$\varepsilon_{ab}(\omega) = \omega(ab) = \omega(a)\omega(b) = \varepsilon_a(\omega)\varepsilon_b(\omega) \quad \varepsilon_{a^*}(\omega) = \omega(a^*) = \overline{\omega(a)} = \overline{\varepsilon_a(\omega)} \quad \varepsilon_1(\omega) = \omega(1) = 1.$$

The unital homomorphism $\varepsilon : A \to C(\text{Max}(A))$ is called the Gelfand transform.

**Theorem 3.8.** Gelfand Theorem

Suppose that $A$ is commutative and unital. Then $\varepsilon : A \to C(\text{Max}(A))$ is a $*$-isomorphism. Hence $A \cong C(\text{Max}(A))$.

**Proof.** We need to show that it is isometric and surjective. To show that it is isometric, it is enough to show that it is injective (this is a general result in $C^*$-algebras). Hence, we will only show that it is bijective.

Injectivity: assume that $\varepsilon_a = 0$, that is, $\omega(a) = 0$ for all characters $\omega$. Then $a$ is in the intersection of all maximal ideals, that is, $a \in J(A)$. Since $J(A) = 0$ for any $C^*$-algebra $A$, we conclude that $\varepsilon$ is injective.

Surjectivity: notice that $\varepsilon(A)$ is a subalgebra of $C(\text{Max}(A))$ that separates points of Max($A$) (because two different characters on $A$ must differ at some point of $A$), contains the constants (because $\varepsilon$ is unital) and is closed (because $\varepsilon$ is an isometry). Hence the Stone-Weierstrass theorem implies that $\varepsilon(A) = C(\text{Max}(A))$. □

In the above proof, we needed to know that $\varepsilon$ is an isometry. This follows using Spectral Theory for Banach algebras in both $A$ and $C(\text{Max}(A))$. It also follows from the fact that every homomorphism between $C^*$-algebras is norm-decreasing, and it is injective if and only if it is an isometry.

The above theorem reduces the study of commutative $C^*$-algebras to the study of topological spaces. Therefore the study of $C^*$-algebras is usually thought of as the study of noncommutative topology.

**Definition 4.1.** A representation of the $C^*$-algebra $A$ is a pair $(\pi, \mathcal{H})$, where $\pi : A \to \mathcal{B}(\mathcal{H})$ is a homomorphism of $C^*$-algebras. We usually say that $\pi$ is a representation of $A$ (by bounded operators) on $\mathcal{H}$.

In this last section we will show that for every $C^*$-algebra there is a Hilbert space $\mathcal{H}$ and a faithful representation of $A$ on $\mathcal{H}$.

Notice that to prove this fact, it will be enough to prove that for every $a \in A$, there exists a homomorphism $\pi_a : A \to \mathcal{B}(\mathcal{H})$ such that $\pi_a(a) \neq 0$. Indeed, if we had proven this, we could simply take $(\pi, \mathcal{H}) = (\oplus_{a \in A} \pi_a, \oplus_{a \in A} \mathcal{H}_a)$.

**Definition 4.2.** An element $a \in A$ is said to be positive if $\text{sp}(a) \subseteq \mathbb{R}^+$. The set of all positive elements of $A$ is denoted by $A^+$. Moreover, a linear map $\tau : A \to \mathbb{C}$ is said to be positive if $\tau(A^+) \subseteq \mathbb{R}^+$.

**Examples 4.3.** Positive elements in some $C^*$-algebras.

1. If $A = \mathcal{B}(\mathcal{H})$, then an operator $T$ is positive if and only if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$.
2. If $A = M_n(\mathbb{C})$ and a matrix $a$ is positive if and only if $a$ is self-adjoint and all its eigenvalues are nonnegative.
3. If $A = C_0(X)$, then a function $f$ on $X$ is positive if and only if $f \geq 0$.

We will assume the following fact, which is essentially equivalent to the fact that $J(A) = 0$.

**Lemma 4.4.** For every $a \in A$, there exists a positive map $\tau_a : A \to \mathbb{C}$ such that $\tau_a(a^*a) \neq 0$.

Let $\tau : A \to \mathbb{C}$ be a (continuous) positive linear map, and let $N_\tau = \{a \in A^* : \tau(a^*a) = 0\}$. Then $N_\tau$ is a closed left ideal of $A$ (closedness follows from the continuity of $\tau$), and the map

$$\frac{A}{N_\tau} \times \frac{A}{N_\tau} \to \mathbb{C} : (a + N_\tau, b + N_\tau) \mapsto \tau(b^*a)$$

is an inner product in $\frac{A}{N_\tau}$. We denote by $\mathcal{H}_\tau$ its completion.

Fix a (continuous) positive linear map $\tau : A \to \mathbb{C}$. If $a \in A$, define the map $\pi(a) \in \mathcal{B}(\frac{A}{N_\tau})$ by $\pi(a)(b + N_\tau) = ab + N_\tau$. Then $\pi(a)$ is bounded on the normed vector space $\frac{A}{N_\tau}$, and hence it has a unique extension to $\mathcal{H}_\tau$. This defines a representation of $A$ on $\mathcal{H}_\tau$, given by $\pi_a : A \to \mathcal{B}(\mathcal{H}_\tau)$.

**Definition 4.5.** Given a $C^*$-algebra $A$, the universal representation is $(\pi_U, \mathcal{H}_U) = (\oplus_{\tau} \pi_{\tau}, \oplus_{\tau} \mathcal{H}_\tau)$.

**Theorem 4.6.** Gelfand-Naimark Representation Theorem

Every $C^*$-algebra has a faithful representation. In other words, every $C^*$-algebra is (isomorphic to) a $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

**Proof.** We will show that the universal representation is faithful. Let $a \neq 0$ in $A$. Then

$$\|\pi_U(a)\| \geq \|\pi_{\tau_a}(a)\| \geq \|\pi_{\tau_a}(a)(1)\| = \|a + N_{\tau_a}\| > 0$$

where the last step follows from the fact that $a \notin N_{\tau_a}$ because $\tau_a(a^*a) \neq 0$. \qed