1. Consider the standard least squares formula

\[ c = (\sum_{i=1}^{T} x_i x'_i) - 1 (\sum_{i=1}^{T} x_i y_i). \]

This formula arises in fitting the regression equation \( y_i = c' x_i + e_i \) using data \( i = 1, ..., T \) on the \( k \times 1 \) independent vector \( x_i \) and the dependent variable \( y_i \), so that \( c \) minimizes \( \sum_{i=1}^{T} e_i^2 \). Writing

\[ R_t = t^{-1} \sum_{i=1}^{t} x_i x'_i \]
\[ c_t = t^{-1} R_t^{-1} \sum_{i=1}^{t} x_i y_i \]

for the moment matrix and the coefficient vector show by mathematical induction on the number of data points that \( c \) can instead be computed using the recursive least squares (RLS) formulae

\[ c_t = c_{t-1} + t^{-1} R_t^{-1} x_t (y_t - c'_t x_t) \]
\[ R_t = R_{t-1} + t^{-1} (x_t x'_t - R_{t-1}). \]

2. Consider the model

\[ p_t = f(p_{t+1}). \]

Suppose that \( p_{t+1} = a_t \), where \( a_t = a_{t-1} + \gamma_t (p_{t-1} - a_{t-1}) \), so that

\[ a_t = a_{t-1} + \gamma_t (f(a_{t-1}) - a_{t-1}). \]

(a) For the decreasing gain case \( \gamma_t \to 0 \), apply the stochastic approximation technique to obtain the condition for stability under learning of a steady state \( \bar{a} = f(\bar{a}) \). That is, treating the above equation as an SRA, show how to obtain the associated ODE and find the corresponding stability condition.

(b) For the constant gain case \( \gamma_t = \gamma \) for \( 0 < \gamma \leq 1 \), derive the local stability condition for a steady state \( \bar{a} = f(\bar{a}) \) using the standard results on the local stability of difference equations.

3. Consider a variation of the cobweb model in which \( p_t \) depends on the observable exogenous variable \( w_t \) rather than \( w_{t-1} \):

\[ p_t = \mu + \alpha p^e_t + \delta w_t + \eta_t, \quad \text{where } \alpha \neq 1, \]
\[ w_t = k + \lambda w_{t-1} + \varepsilon_t, \quad \text{where } |\lambda| < 1, \]

where \( \varepsilon_t \) and \( \eta_t \) are independent white noise processes. (Here \( w_t \) is univariate and \( \eta_t \) is unobserved).
(a) Obtain the unique REE and show that it can be written in the form
\[ p_t = \bar{a} + \bar{b}w_{t-1} + u_t, \]
for suitable \( \bar{a}, \bar{b} \) and \( u_t \) white noise.

(b) Suppose agents forecast according to
\[ p_t^c = a_{t-1} + b_{t-1}w_{t-1}, \]
where \( a_{t-1}, b_{t-1} \) are estimated in the usual way by RLS ("least squares learning").

(i) For the PLM \( p_t = a + bw_{t-1} + u_t \), obtain the T-mapping from the PLM to the ALM and find the E-stability condition for the REE.

(ii) Outline the steps of the stochastic approximation argument used to obtain the stability condition for the REE under least squares learning and show that the condition is identical to the E-stability condition.

4. Consider the Cagan model
\[ y_{t+1} = \mu + \beta E_t y_{t+1} + \delta w_t \]
\[ w_t = \lambda w_{t-1} + v_t. \]
The parameter \( \lambda \) is assumed to be known.

(a) Show that generically there exists a unique REE of the form
\[ y_t = \bar{a} + \bar{b}w_t. \]

(b) Suppose agents have a PLM of the form \( y_t = a + bw_t \) and forecast accordingly. Derive the \( T \)-mapping and the E-stability conditions.

5. Continuing on Problem 4, suppose agents update the parameters of their PLM
\[ y_t = a_{t-1} + b_{t-1}w_t \]
using the RLS algorithm. Write down the details of the algorithm. Using numerical parameter values \( \mu = 5, \beta = 0.4, \lambda = 0.7 \) and \( \delta = 1 \), write a Matlab routine to simulate the RLS learning and show the results for \( a_t, b_t \) and \( y_t \). Now change \( \beta \) to \( \beta = 0.8 \) and run another simulation. Which value of \( \beta \) appear to give quicker convergence to the REE?