Expectations, Learning and Macroeconomic Policy

George W. Evans (University of Oregon and University of St. Andrews)

26 March 2010

J. C. Trichet: “Understanding expectations formation as a process underscores the strategic interdependence that exists between expectations formation and economics.” (Zolotas lecture, 2005)

Ben S. Bernanke: “In sum, many of the most interesting issues in contemporary monetary theory require an analytical framework that involves learning by private agents and possibly the central bank as well.” (NBER, July 2007).
Outline

Morning Session

Lecture 1: Review of techniques: E- stability & least squares learning

Lecture 2: Optimal monetary policy and learning
— Optimal policy with learning. Policy with perpetual learning

Afternoon session

Lecture 3: (i) Recurrent hyperinflations and learning
— (ii) Dynamic Predictor Selection and Endogenous Volatility

Lecture 4: Liquidity traps, learning and stagnation.
Lecture 1: Review of techniques: E-stability & least squares learning

Introduction

• Since Lucas (1972, 1976) and Sargent (1973) the standard assumption in the theory of economic policy is rational expectations (RE). This assumes, for both private agents and policymakers,

  – knowledge of the correct form of the model

  – knowledge of all parameters, and

  – knowledge that other agents are rational & know that others know . . . .
• RE assumes too much and is therefore implausible. We need an appropriate model of **bounded rationality** What form should this take?

• My general answer is given by the **Cognitive Consistency Principle**: economic agents should be about as smart as (good) economists. Economists forecast economic variables using econometric techniques, so a good starting point: *model agents as “econometricians.”*

• Neither private agents nor economists at central banks do know the true model. Instead economists formulate and estimate models. These models are re-estimated and possibly reformulated as new data becomes available. Economists engage in *processes of learning* about the economy.

• These processes of learning create **new tasks for macroeconomic policy**
Starting Points:

- The private sector is forward-looking (e.g. investment, savings decisions).

- Forecasts (including private forecasts) of future inflation and output have a key role in monetary policy:
  1. Empirical evidence e.g. by (Clarida, Gali and Gertler 1998).
  2. Bank of England and ECB discuss private forecasts in addition to internal macro projections.

- Private agents and/or policy-makers are learning.
Fundamental Problems:

- There may be multiple equilibria, depending on interest rate policy.

- Policy may lead to expectational instability if expectations are not always rational.

These problems necessitate careful design of interest rate rule: Bullard and Mitra (2002), Evans and Honkapohja (2003a, 2006).


- **Central message**: Policy should facilitate learning by private agents.
A Muth/Lucas-type Model

Before looking at issues of policy, we review the learning approach and the E-stability technique, beginning with a simple univariate reduced form:

\[ p_t = \mu + \alpha E_{t-1}^* p_t + \delta' w_{t-1} + \eta_t. \]  

(RF)

\( E_{t-1}^* p_t \) denotes expectations of \( p_t \) formed at \( t-1 \), \( w_{t-1} \) is a vector of exogenous shocks observed at \( t-1 \), \( \eta_t \) is an exogenous unobserved iid shock, and \( w_t \) follows an exogenous stationary VAR process.

**Muth example.** Demand and supply equations:

\[

d_t = m_I - m_p p_t + v_{1t}
\]

\[
s_t = r_I + r_p E_{t-1}^* p_t + r'_w w_{t-1} + v_{2t},
\]

Assuming market clearing, \( s_t = d_t \), yields (RF) where \( \mu = (m_I - r_I)/m_p \), \( \delta = -m_p^{-1} r_w \) and \( \alpha = -r_p/m_p \) and \( \eta_t = (v_{1t} - v_{2t})/m_p \).

Note that \( \alpha < 0 \) if \( m_p, r_p > 0 \).
Lucas-type Monetary model. A simple Lucas-type model:

\[ q_t = \bar{q} + \pi(p_t - E^*_t p_{t-1}) + \zeta_t, \]

where \( \pi > 0 \), and aggregate demand function is given by

\[ m_t + v_t = p_t + q_t, \]
\[ v_t = \mu + \gamma' w_{t-1} + \xi_t, \]
\[ m_t = \bar{m} + u_t + \rho' w_{t-1}. \]

Here \( w_{t-1} \) are exogenous observables. The reduced form is again

\[ p_t = \mu + \alpha E^*_t p_{t-1} + \delta' w_{t-1} + \eta_t, \]

where

\[ \alpha = \pi(1 + \pi)^{-1} \text{ and } \delta = (1 + \pi)^{-1}(\rho + \gamma) \]

In this example \( 0 < \alpha < 1 \).
RATIONAL EXPECTATIONS

First consider the model under RE:

\[ p_t = \mu + \alpha E_{t-1}p_t + \delta' w_{t-1} + \eta_t. \]

The model has a unique RE solution since

\[
E_{t-1}p_t = \mu + \alpha E_{t-1}p_t + \delta' w_{t-1} \rightarrow \\
E_{t-1}p_t = (1 - \alpha)^{-1}\delta + (1 - \alpha)^{-1}\delta' w_{t-1}
\]

Hence the unique REE is

\[ p_t = \bar{a} + \bar{b}' w_{t-1} + \eta_t, \text{ where} \]

\[ \bar{a} = (1 - \alpha)^{-1}\delta \text{ and } \bar{b} = (1 - \alpha)^{-1}\delta. \]
LEAST-SQUARES LEARNING

Under learning, agents have the beliefs or perceived law of motion (PLM)

\[ p_t = a + bw_{t-1} + \eta_t, \]

but \( a, b \) are unknown. At the end of time \( t - 1 \) they estimate \( a, b \) by LS (Least Squares) using data through \( t - 1 \), i.e. Then they use the estimated coefficients to make forecasts \( E_{t-1}^* p_t \). Here the standard least squares (LS) formula are

\[
\left( \begin{array}{c}
  a_{t-1} \\
  b_{t-1}
\end{array} \right) = \left( \sum_{i=1}^{t-1} z_{i-1} z'_{i-1} \right)^{-1} \left( \sum_{i=1}^{t-1} z_{i-1} p_i \right), \text{ where}
\]

\[ z'_i = \left( \begin{array}{c}
  1 \\
  w'_i
\end{array} \right). \]
The timing is:

- End of $t - 1$: $w_{t-1}$ and $p_{t-1}$ observed. Agents update estimates of $a, b$ to $a_{t-1}, b_{t-1}$ using $\{p_s, w_{s-1}\}_{s=1}^{t-1}$. Agents make forecasts

$$E_{t-1}^* p_t = a_{t-1} + b'_{t-1} w_{t-1}.$$ 

- Period $t$: (i) The shock $\eta_t$ is realized, $p_t$ is determined as

$$p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1})' w_{t-1} + \eta_t$$

and $w_t$ is realized. (ii) agents update estimates to $a_t, b_t$ using $\{p_s, w_{s-1}\}_{s=1}^t$ and make forecasts

$$E_t^* p_{t+1} = a_t + b' w_t.$$ 

The system under learning is a fully specified dynamic system under learning.

Question: Will $(a_t, b_t) \to (\bar{a}, \bar{b})$ as $t \to \infty$?
LS updating and the system can be set up recursively. Letting

\[ \phi_t = \begin{pmatrix} a_t \\ b_t \end{pmatrix} \text{ and } z_t = \begin{pmatrix} 1 \\ w_t \end{pmatrix} \]

the system is:

\[
\begin{align*}
E_{t-1}^* p_t & = a_{t-1} + b'_{t-1} w_{t-1} \\
p_t & = \mu + \alpha E_{t-1}^* p_t + \delta' w_{t-1} + \eta_t, \\
\phi_t & = \phi_{t-1} + t^{-1} R_{t-1}^{-1} z_{t-1} (p_t - \phi'_{t-1} z_{t-1}) \\
R_t & = R_{t-1} + t^{-1} (z_{t-1} z'_{t-1} - R_{t-1}),
\end{align*}
\]

This recursive formulation is useful for (a) theoretical analysis, and (b) numerical simulations.
**Theorem:** Consider model (RF) with $E_{t-1}^*p_t = a_{t-1} + b'_{t-1}w_{t-1}$ and with $a_{t-1}, b_{t-1}$ updated over time using least-squares. If $\alpha < 1$ then $\left(\begin{array}{c} a_t \\ b_t \end{array}\right) \to \left(\begin{array}{c} \bar{a} \\ \bar{b} \end{array}\right)$ with probability 1. If $\alpha > 1$ convergence occurs with probability 0.

Thus the REE is stable under LS learning both for Muth model ($\alpha < 0$) and Lucas model ($0 < \alpha < 1$).

Example of an unstable REE: Muth model with $m_p < 0$ (Giffen good) and $|m_p| < r_p$. 
E-STABILITY

Proving the theorem is not easy. However, there is an easy way of deriving the stability condition $\alpha < 1$ that is quite general. Start with the PLM

$$p_t = a + b' w_{t-1} + \eta_t,$$

and consider what would happen if $(a, b)$ were fixed at some value possibly different from the RE values $(\bar{a}, \bar{b})$. The corresponding expectations are

$$E_{t-1}^* p_t = a + b' w_{t-1},$$

which would lead to the Actual Law of Motion (ALM)

$$p_t = \mu + \alpha(a + b' w_{t-1}) + \delta' w_{t-1} + \eta_t.$$

The implied ALM gives the mapping $T$: PLM $\rightarrow$ ALM:

$$T \left( \begin{array}{c} a \\ b \end{array} \right) = \left( \begin{array}{c} \mu + \alpha a \\ \delta + \alpha b \end{array} \right).$$
The REE $\bar{a}, \bar{b}$ is a fixed point of $T$. Expectational-stability ("E-stability") is defined by the differential equation

$$\frac{d}{d\tau} \begin{pmatrix} a \\ b \end{pmatrix} = T \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}.$$ 

Here $\tau$ denotes artificial or notional time. $\bar{a}, \bar{b}$ is said to be E-stable if it is stable under this differential equation.

In the current case the $T$-map is linear. Component by component we have

$$\frac{da}{d\tau} = \mu + (\alpha - 1)a,$$
$$\frac{db_i}{d\tau} = \delta + (\alpha - 1)b_i \text{ for } i = 1, \ldots, p.$$
\[
\frac{da}{d\tau} = \mu + (\alpha - 1)a \\
\frac{db_i}{d\tau} = \delta + (\alpha - 1)b_i \text{ for } i = 1, \ldots, p.
\]

It follows that the REE is E-stable if and only if \( \alpha < 1 \). This is the stability condition, given in the theorem, for stability under LS learning.

Intuition: under LS learning the parameters \( a_t, b_t \) are slowly adjusted, on average, in the direction of the corresponding ALM parameters.

Numerical simulation of learning in Muth model. \( \mu = 5, \delta = 1 \) and \( \alpha = -0.5 \). \( w_t \overset{iid}{\sim} N(0, 1) \) and \( \eta_t \overset{iid}{\sim} N(0, 1/4) \). Initial values \( a_0 = 1, b_0 = 2 \) and \( R_0 = \text{eye}(2) \). Convergence to the REE \( \bar{a} = 10/3 \) and \( \bar{b} = 2/3 \) is rapid.
A general definition of E-stability is a straightforward extension of the example in this chapter. Starting with an economic model, we consider its REE solutions. Assume that any particular solution can be described as a stochastic process with particular parameter values \( \bar{\phi} \). Here \( \phi \) might be, for example, the parameters of an ARMA process or of a VAR or the mean values at the different points in a \( k \)-cycle. Under adaptive learning the agents are assumed not to know \( \bar{\phi} \), but try to estimate it using data from the economy. This leads to statistical estimates \( \hat{\phi} \) at time \( t \), and the issue will be whether \( \phi_t \to \bar{\phi} \) as \( t \to \infty \). We will in each case set up the problem as an SRA in order to examine the stability of the solution \( \bar{\phi} \). We will continue to find that stability of \( \bar{\phi} \) under learning can be determined by the E-stability equation, i.e., by the stability of

\[
\frac{d\phi}{d\tau} = T(\phi) - \phi, \tag{2.20}
\]

in a neighborhood of \( \bar{\phi} \), where \( T(\phi) \) is the mapping from the perceived law of motion \( \phi \) to the implied actual law of motion \( T(\phi) \). [Note that REEs corre-
The E-Stability Principle

– The E-stability technique works quite generally.

– To study convergence of LS learning to an REE, specify a PLM with parameters $\phi$. The PLM can be thought of as an econometric forecasting model. The REE is the PLM with $\phi = \bar{\phi}$.

– PLMs can take the form of ARMA or VARs or admit cycles or a dependence on sunspots.

– Compute the ALM for this PLM. This gives a map

$$\phi \rightarrow T(\phi),$$

with fixed point $\bar{\phi}$. 
E-stability is determined by local asymptotic stability of \( \bar{\phi} \) under

\[
\frac{d\phi}{d\tau} = T(\phi) - \phi.
\]

The E-stability condition is that all eigenvalues of \( DT(\bar{\phi}) \) have real parts less than 1.


E-stability can be used as a selection criterion in models with multiple REE.

These techniques can be applied to multivariate linearized models, and thus to RBC, OLG, New Keynesian and DSGE models.
The New Keynesian Model

• Log-linearized New Keynesian model (CGG 1999, Woodford 2003 etc.).

1. “IS” equation (IS curve)

\[ x_t = -\varphi (i_t - E_t^* \pi_{t+1}) + E_t^* x_{t+1} + g_t \]

2. the “New Phillips” equation (PC curve)

\[ \pi_t = \lambda x_t + \beta E_t^* \pi_{t+1} + u_t, \]

where \( x_t \) = output gap, \( \pi_t \) = inflation, \( i_t \) = nominal interest rate. \( E_t^* x_{t+1} \), \( E_t^* \pi_{t+1} \) are expectations. Parameters \( \varphi, \lambda > 0 \) and \( 0 < \beta < 1 \).
• Observable shocks follow

\[
\begin{pmatrix}
g_t \\
u_t
\end{pmatrix} = \begin{pmatrix}
F & 0 \\
0 & \rho
\end{pmatrix}
\begin{pmatrix}
g_{t-1} \\
u_{t-1}
\end{pmatrix} + \begin{pmatrix}
\tilde{g}_t \\
\tilde{u}_t
\end{pmatrix},
\]

where \(0 < |\mu|, |\rho| < 1\), and \(\tilde{g}_t \sim iid(0, \sigma_g^2), \tilde{u}_t \sim iid(0, \sigma_u^2)\).

• Interest rate setting by a standard \textbf{Taylor rule}, e.g.

\[
i_t = \chi_\pi \pi_t + \chi_x x_t \quad \text{where} \quad \chi_\pi, \chi_x > 0 \quad \text{or}
\]

\[
i_t = \chi_\pi \pi_{t-1} + \chi_x x_{t-1} \quad \text{or}
\]

\[
i_t = \chi_\pi E_t^\pi \pi_{t+1} + \chi_x E_t^x x_{t+1}
\]

• Under learning we treat the IS and PC Euler equations as behavioral equations. Explicit infinite-horizon formations also have been studied: Preston (IJCM, 2005 and JME, 2006), Evans, Honkapohja & Mitra (JME, 2009). For more on Euler equation learning see Evans and McGough (2009).
Determinacy and Stability under Learning

DETERMINACY

Combining IS, PC and the $i_t$ rule leads to a bivariate reduced form in $x_t$ and $\pi_t$. Letting $y_t' = (x_t, \pi_t)'$ and $v_t' = (g_t, u_t)'$ the model can be written

$$
\begin{pmatrix}
x_t \\
\pi_t
\end{pmatrix} = M \begin{pmatrix}
E_t^* x_{t+1} \\
E_t^* \pi_{t+1}
\end{pmatrix} + N \begin{pmatrix}
x_{t-1} \\
\pi_{t-1}
\end{pmatrix} + P \begin{pmatrix}
g_t \\
u_t
\end{pmatrix},
$$

$$
y_t = ME_t y_{t+1} + Ny_{t-1} + P v_t.
$$

If the model is “determinate” there exists a unique stationary REE of the form

$$
y_t = \bar{b} y_{t-1} + \bar{c} v_t.
$$

Determinacy condition: compare # of stable eigenvalues of matrix of stacked first-order system to # of predetermined variables. If “indeterminate” there are multiple solutions, which include stationary sunspot solutions.
LEARNING

Under learning, agents have beliefs or a perceived law of motion (PLM)

\[ y_t = a + by_{t-1} + cv_t, \]

where we now allow for an intercept, and estimate \((a_t, b_t, c_t)\) in period \(t\) based on past data.

- Forecasts are computed from the estimated PLM.
- New data is generated according to the model with the given forecasts.
- Estimates are updated to \((a_{t+1}, b_{t+1}, c_{t+1})\) using least squares.

Question: when is it the case that

\[ (a_t, b_t, c_t) \rightarrow (0, \bar{b}, \bar{c})? \]

We determine this using E-stability
E-STABILITY

Reduced form

\[ y_t = ME_t^* y_{t+1} + Ny_{t-1} + Pv_t. \]

Under the PLM (Perceived Law of Motion)

\[ y_t = a + by_{t-1} + cv_t. \]

\[ E_t^* y_{t+1} = (I + b)a + b^2 y_{t-1} + (bc + cF)v_t. \]

This \( \rightarrow \) ALM (Actual Law of Motion)

\[ y_t = M(I + b)a + (Mb^2 + N)y_{t-1} + (Mbc + NcF + P)v_t. \]
This gives a mapping from PLM to ALM:

\[ T(a, b, c) = (M(I + b)a, Mb^2 + N, Mbc + NcF + P). \]

The optimal REE is a fixed point of \( T(a, b, c) \). If

\[ \frac{d}{d\tau}(a, b, c) = T(a, b, c) - (a, b, c) \]

is locally asymptotically stable at the REE it is said to be E-stable. See EH, Chapter 10, for details. The E-stability conditions can be stated in terms of the derivative matrices

\[
\begin{align*}
DT_a &= M(I + \bar{b}) \\
DT_b &= \bar{b}' \otimes M + I \otimes M\bar{b} \\
DT_c &= F' \otimes M + I \otimes M\bar{b},
\end{align*}
\]

where \( \otimes \) denotes the Kronecker product and \( \bar{b} \) denotes the REE value of \( b \).

E-stability governs stability under LS learning.
Results for Taylor-rules in NK model (Bullard & Mitra, JME 2002)

\[ i_t = \chi_\pi \pi_t + \chi_x x_t \] yields **determinacy and stability** under LS learning if

\[ \lambda(\chi_\pi - 1) + (1 - \beta)\chi_x > 0. \]

Note that \( \chi_\pi > 1 \) is sufficient.
With $i_t = \chi_{\pi} \pi_{t-1} + \chi_{x} x_{t-1}$, determinacy & E-stability for $\chi_{\pi} > 1$ and $\chi_{x} > 0$ small. Also an explosive region ($\chi_{\pi} > 1$ and $\chi_{x}$ large) and a determinate E-unstable region ($\chi_{\pi} < 1$ and $\chi_{x}$ moderate).
For $i_t = \chi_\pi E_t^* \pi_{t+1} + \chi_x E_t^* x_{t+1}$, determinacy & E-stability for $\chi_\pi > 1$ and $\chi_x > 0$ small. Indeterminate & E-stable for $\chi_\pi > 1$ and $\chi_x$ large. Recent work has shown stable sunspot solutions in that region.