EXERCISES

23.1. For each of the following power series, find the radius of convergence and determine the exact intervals of convergence.

(a) $\sum n^2 x^n$
(b) $\sum (x/n)^n$
(c) $\sum (2^n/n^2)x^n$
(d) $\sum (n^3/3^n)x^n$
(e) $\sum (2^n/n!)x^n$
(f) $\sum (1/(n+1)^2 2^n)x^n$
(g) $\sum (3^n/n \cdot 4^n)x^n$
(h) $\sum ((-1)^n/n^2 \cdot 4^n)x^n$

23.2. Repeat Exercise 23.1 for the following:

(a) $\sum \sqrt{n}\, x^n$
(b) $\sum (1/n^{\sqrt{n}})x^n$
(c) $\sum x^{n!}$
(d) $\sum (3^n/\sqrt{n})x^{2n+1}$

23.3. Find the exact interval of convergence for the series in Example 6.

23.4. For $n=0,1,2,3,\ldots$, let $a_n=[(4+2(-1)^n)/5]^n$.

(a) Find $\limsup(a_n)^{1/n}$, $\liminf(a_n)^{1/n}$, $\limsup|a_{n+1}/a_n|$ and $\liminf|a_{n+1}/a_n|$.
(b) Do the series $\sum a_n$ and $\sum(-1)^n a_n$ converge? Explain briefly.
(c) Now consider the power series $\sum a_n x^n$ with the coefficients a_n as above. Find the radius of convergence and determine the exact interval of convergence for the series.

23.5. Consider a power series $\sum a_n x^n$ with radius of convergence R.

(a) Prove that if all the coefficients a_n are integers and if infinitely many of them are nonzero, then $R \leqslant 1$.
(b) Prove that if $\limsup|a_n|>0$, then $R \leqslant 1$.

23.6. (a) Suppose that $\sum a_n x^n$ has finite radius of convergence R and that $a_n \geqslant 0$ for all n. Show that if the series converges at R, then it also converges at $-R$.
(b) Give an example of a power series whose interval of convergence is exactly $(-1, 1]$.

The next three exercises are designed to show that the notion of convergence of functions discussed prior to Example 8 has many defects.

23.7. For each $n \in \mathbb{N}$, let $f_n(x)=(\cos x)^n$. Each f_n is continuous. Nevertheless, show that

(a) $\lim f_n(x)=0$ unless x is a multiple of π,
(b) $\lim f_n(x)=1$ if x is an even multiple of π,
(c) $\lim f_n(x)$ does not exist if x is an odd multiple of π.

23.8. For each $n \in \mathbb{N}$, let $f_n(x)=(1/n)\sin nx$. Each f_n is differentiable. Show that

(a) $\lim f_n(x)=0$ for all $x \in \mathbb{R}$,
(b) but $\lim f_n'(x)$ need not exist [at $x=\pi$ for instance].

23.9. Let $f_n(x)=nx^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Show that

(a) $\lim f_n(x)=0$ for $x \in [0,1)$. *Hint*: Use Exercise 9.12.
(b) However, $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx=1$.

Exercises

24.1. Let $f_n(x) = [1 + 2\cos^2 nx]/\sqrt{n}$. Prove carefully that (f_n) converges uniformly to 0 on $\mathbb{R}$.

24.2. For $x \in [0, \infty)$, let $f_n(x) = x/n$.

(a) Find $f(x) = \lim f_n(x)$.
(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.
(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

24.3. Repeat Exercise 24.2 for $f_n(x) = 1/(1 + x^n)$.

24.4. Repeat Exercise 24.2 for $f_n(x) = x^n/(1 + x^n)$.

24.5. Repeat Exercise 24.2 for $f_n(x) = x^n/(n + x^n)$.

24.6. Let $f_n(x) = (x - 1/n)^2$ for $x \in [0, 1]$.

(a) Does (f_n) converge pointwise on the set $[0, 1]$? If so, give the limit function.
(b) Does the sequence (f_n) converge uniformly on $[0, 1]$? Prove your assertion.

24.7. Repeat Exercise 24.6 for $f_n(x) = x - x^n$.

24.8. Repeat Exercise 24.6 for $f_n(x) = \sum_{k=0}^{n} x^k$.