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A Class of Measure-valued Branching Diffusions in a Random Medium *

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Abstract

For any positive integer n, an interacting branching particle system is considered in which m_t^n particles move in a random medium on $\bar{\mathbb{R}}$ at time t. In between branching times, the motion is governed by a singular, degenerate diffusion coefficient; each particle has mass $1/\theta^n$ and branches at rate $\gamma\theta^n$, where $\gamma \geq 0$ and $\theta \geq 2$ are fixed constants. As $n \to \infty$, the existence, uniqueness, Markovian property, and continuity of the limiting measure-valued processes are investigated.

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1 Introduction

The present paper is devoted to the construction and characterization of a new class of measure-valued diffusions which arise as limits in distribution of a sequence of interacting branching particle systems. (We will often use some of the constructions and ideas

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expounded in the notes by Dawson [6], which we recommend as an excellent general reference on this field.)

The model studied evolves as follows: for any positive integer n, we first consider a system of particles (initially, there are m_0^n particles) which move, die and produce offspring in a random medium on $\bar{\mathbb{R}} := \mathbb{R} \cup \{\partial\}$, the one point compactification of $\mathbb{R}$.

The diffusive part of such a system has the form

$$dx_i(t) = \int_{\mathbb{R}} g(y - x_i(t))W(dy, dt) + \epsilon\, dB_t^i, \tag{1.1}$$

where $W(\cdot,\cdot)$ is a cylindrical Brownian motion (refer to Example 7.1.2 in Dawson [6], to Walsh [18] or to Da Prato-Zabczyk [5]), $\{B_t^i\}$ are independent one-dimensional Brownian motions which are independent of W, ϵ is a real constant, and g is a measurable, symmetric function on $\bar{\mathbb{R}}$ such that

$$\lim_{x\to\partial} g(x) = g(\partial) = 0, \qquad \int_{\mathbb{R}} g^2(x)dx < \infty. \tag{1.2}$$

The quadratic variational process for the finite system given by (1.1) is

$$< x_i, x_j >_t = \int_0^t \rho(x_i(s) - x_j(s))ds + \epsilon^2\, t\, \delta_{ij}, \tag{1.3}$$

where we set $\delta_{ij} = 1$ or 0 according as $i = j$ or $i \neq j$ and

$$\rho(z) = \int_{\mathbb{R}} g(z - y)g(y)dy. \tag{1.4}$$

Here $x_i(t)$ is the location of the i^{th} particle. We assume that each particle has mass $1/\theta^n$ and branches at rate $\gamma\theta^n$, where $\gamma \geq 0$ and $\theta \geq 2$ are fixed constants. When a particle dies, it produces k particles with probability $p_k; k = 0, 1, 2, \cdots$. The offspring distribution is assumed to satisfy:

$$p_1 = 0, \qquad \sum_{k=0}^{\infty} kp_k = 1, \qquad \text{and} \quad m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty. \tag{1.5}$$

The second condition indicates that we are solely interested in the critical case. After branching, the resulting set of particles evolve in the same way as the parent and they start off from the parent particle's branching site. Let m_t^n denote the total number of particles at time t . Denote the empirical measure process by

$$\mu_t^n(\cdot) := \frac{1}{\theta^n} \sum_{i=1}^{m_t^n} \delta_{x_i^n(t)}(\cdot). \tag{1.6}$$

In order to obtain measure-valued processes by use of an appropriate scaling, we assume that there is a positive constant $\xi > 0$ such that $m_0^n/\theta^n \leq \xi$ for all $n \geq 0$ and that weak convergence of the initial laws $\mu_0^n \Rightarrow \tilde{\mu}$ holds, for some finite measure $\tilde{\mu}$ with compact support. In this model we suppose that there are no interactions in the branching mechanism itself.

Let $E := \mathbf{M}_F(\bar{\mathbb{R}})$ be the Polish space of all bounded Radon measures on $\bar{\mathbb{R}}$ with the weak topology defined by

$$\mu^n \Rightarrow \mu \quad \text{if and only if} \quad < f, \mu^n > \to < f, \mu > \text{ for } \forall\, f \in C(\bar{\mathbb{R}}) .$$

By Ito's formula and the independence of motions and branching, we can obtain the following formal generators (usually called pregenerators) for the limiting measure-valued processes:

$$\mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu), \tag{1.7}$$

$$\mathcal{B}F(\mu) := \tfrac{1}{2}\gamma(m_2 - 1)\int_{\bar{\mathbb{R}}} \frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx), \tag{1.8}$$

and

$$\begin{aligned} \mathcal{A}F(\mu) := {} & \tfrac{1}{2}\int_{\bar{\mathbb{R}}} \rho_\epsilon \left(\frac{d^2}{dx^2}\right)\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \\ & + \tfrac{1}{2}\int_{\bar{\mathbb{R}}}\int_{\bar{\mathbb{R}}} \rho(x-y)\left(\frac{d}{dx}\right)\left(\frac{d}{dy}\right)\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy) \end{aligned} \tag{1.9}$$

for $F(\mu) \in \mathcal{D}(\mathcal{L}) \subset C(E)$, where the variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta\mu(x)} := \lim_{h \downarrow 0} \frac{F(\mu + h\delta_x) - F(\mu)}{h}, \tag{1.10}$$

ρ_ϵ is a constant defined by

$$\rho_\epsilon = \rho(0) + \epsilon^2, \tag{1.11}$$

and $\mathcal{D}(\mathcal{L})$ is the domain of the pregenerator $\mathcal{L}$.

Clearly our model includes super-Brownian motion as a special case $(\rho(\cdot) \equiv 0$ and $\epsilon \neq 0)$. In the following we often evoque "the degenerate case" to indicate $\epsilon = 0$ and "the uniformly elliptic case" to indicate $\epsilon \neq 0$, by analogy with the classification of finite dimensional diffusions.

In the usual models (for example, (α, d, β)-superprocesses, see Chapter 4 of Dawson [6]) in which the motions of particles are independent and the motions are independent of branching, the particle systems have the multiplicative property, which is the fact that if two branching Markov processes start at m_1 and m_2 and evolve independently, then their sum is equal in law to the same process starting at m_1+m_2. It is well known that the log-Laplace functional (or evolution equation) technique can be applied to this model in order to construct the limiting measure-valued process. However, for our model and pregenerators, it is obvious that the motions of particles are not independent and this destroys the multiplicative property.

Let us prove this quickly in the case where $\rho(\cdot)$ is a smooth function, the system is branching-free ($\gamma = 0$) and there holds

$$\rho_0 = \rho(0) = \int_{\mathbb{R}} g^2(x)dx \quad \text{(i.e. } \epsilon = 0). \tag{1.12}$$

Suppose $(x_1(t), \cdots, x_N(t))$ is the finite dimensional diffusion process governed by generator

$$G^N f(x_1, \cdots, x_N) := \frac{1}{2} \sum_{i,j=1}^{N} \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \cdots, x_N). \tag{1.13}$$

Let $\eta_t = x_i(t) - x_j(t), i \neq j$; then $\{\eta_t\}$ is a diffusion process with state space $\mathbb{R}$, absorbing state 0 and generator

$$\mathcal{G} f(y) = (\rho(0) - \rho(y)) f''(y), \qquad f \in C_b^\infty(\mathbb{R}). \tag{1.14}$$

From Feller's criterion for accessibility (Ethier-Kurtz [11] p366), the probability that η reaches 0 is 0 or 1 according as

$$\int_0^1 \frac{y}{(\rho(0) - \rho(y))} dy$$

is ∞ or $< \infty$. We will see that $\rho(\cdot)$ is nonnegative definite (see the proof of Lemma 2.2), by the Bochner-Khinchin theorem there is a probability distribution function $F(\cdot)$ such that

$$0 \le 1 - \frac{\rho(y)}{\rho(0)} = \int_{\mathbb{R}} \{1 - \cos(xy)\} dF(x)$$

$$\le \int_{\mathbb{R}} \frac{1}{2}(xy)^2 dF(x) = \frac{1}{2\rho(0)} y^2 |\rho''(0)|$$

Since $\rho''(0)$ is finite, state 0 is inaccessible. So $(x_1(t), \cdots, x_N(t))$ has the property that for any $i \neq j$ and $t > 0$, if $x_i(0) \neq x_j(0)$, then $x_i(t)$ never collides with $x_j(t)$. So then, were the multiplicative property to hold, we would have, for any strictly positive, continuous function $f(\cdot)$,

$$\mathbb{E}_{\delta_{x_i(0)}+\delta_{x_j(0)}} e^{-<f,\delta_{x_i(t)}+\delta_{x_j(t)}>} = \mathbb{E}_{\delta_{x_i(0)}} e^{-<f,\delta_{x_i(t)}>} \mathbb{E}_{\delta_{x_j(0)}} e^{-<f,\delta_{x_j(t)}>}. \tag{1.15}$$

However, from the right hand side of (1.15), it clearly follows that the particles can be simultaneously located at same position. The contradiction implies that the multiplicative property is lost. In consequence, we cannot apply the evolution equation method. Therefore, questions such as existence, uniqueness, continuity, spatial structure properties of the limiting measure-valued processes and so on, arise and new techniques are needed to handle this interacting model. In this paper, we only consider the existence, uniqueness, Markovian property and continuity of the limiting measure-valued processes for the case in which the function $\rho(\cdot)$ is continuous and locally Lipschitz continuous away from 0 (i.e. Lipschitz outside each interval $(-c, c)$, for $c > 0$). Some of the further questions are discussed in [19], [8], and [9].

The paper is organized as follows. We introduce our model and describe the difficulties we encountered in Section 1. In Section 2, we will establish the fact that for any finite measure ν_0 with compact support, the $(\mathcal{A}, \delta_{\nu_0})$-martingale problem (MP) has a unique solution and a subspace $\mathcal{D}(\mathcal{A}) \subset C(E)$ can be chosen such that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the generator of a Feller semigroup. In Section 3, a sequence of finite particle systems will be constructed and for any given finite measure $\tilde{\mu}$ with compact support, we obtain a solution to the $(\mathcal{L}, \delta_{\tilde{\mu}})$-MP by a tightness argument, under the assumption that $\rho(\cdot)$ is smooth. The uniqueness of the $(\mathcal{L}, \delta_{\tilde{\mu}})$-MP is proved in that case by the duality method in Section 4. Sections 5 and 6 are devoted to the construction and characterization of the solution to the $\mathcal{L}$-MP under the assumption that $\rho(\cdot)$ is only continuous and locally Lipschitz continuous away from 0. Under this new assumption the existence can be established directly by the tightness argument. However, the proof of the uniqueness is not so easy. The reason is as follows: from Section 4 we see that the spatial diffusion semigroup is one of the two structural elements of the dual processes. Now that the function $g(\cdot)$ may be singular (by a singular function, we mean that there exists at least one point at which the function assumes infinite value), we cannot show that the closure of $(G^N, C_0^\infty(\mathbb{R}^N))$ (see (2.19) for definition) is the generator of a Feller semigroup. In consequence, we cannot directly construct the dual process such that the dual identity (cf. (4.50) of Section 4) holds as we do in the Section 4. In order to prove the uniqueness of the solution to the $(\mathcal{L}, \delta_{\mu_0})$-MP under the new assumption, in Section 6 we generalize the duality difference method, which was first introduced by Ethier-Kurtz in [12], to suit for this new situation. The main idea of the generalized duality difference method is to establish the dual identity by making use of the limiting argument and the existence of generalized dual processes. In Section 7, we will prove the continuity of the limiting measure-valued processes.

2 The Generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$

In this section, we prove that for any finite measure μ on $\bar{\mathbb{R}}$ with compact support, the $(\mathcal{A}, \delta_\mu)$-MP is uniquely solvable, and a subspace $\mathcal{D}(\mathcal{A}) \subset C(E)$ can be chosen such that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of a Feller semigroup $\{V_t : t \geq 0\}$. Now we begin by introducing a general setting for measure-valued processes and their dual processes. For any positive real number ℓ, let $E_\ell := \{\mu \in E :< 1, \mu >\leq \ell\}$, $\mathbb{N}$ be the set of natural numbers,

$$(2.16) \qquad \Pi_1^D(E_\ell) := \{F_f(\mu) : F_f(\mu) = \int_{\bar{\mathbb{R}}} \cdots \int_{\bar{\mathbb{R}}} f(x_1, \cdots, x_N)\mu^{\otimes N}(d\tilde{x}), \\ f(x_1, \cdots, x_N) \in C^\infty(\bar{\mathbb{R}}^N), \\ \mu^{\otimes N}(d\tilde{x}) = \Pi_{i=1}^N \mu(dx_i), \mu \in E_\ell, N \in \mathbb{N} \cup \{0\}\},$$

and $\Pi^D(E_\ell)$ be the smallest algebra of functions on E_ℓ which contains $\Pi_1^D(E_\ell)$. It follows from the Stone-Weiestrass theorem that $\Pi^D(E_\ell)$ is dense in $C(E_\ell)$ (see Dawson-Kurtz [7] p.93, Lemma 2.2). Let

$$(2.17) \qquad D(\bar{\mathbb{R}}^N) := C^\infty(\bar{\mathbb{R}}^N),$$

where $\bar{\mathbb{R}}^N$ is the one point compactification of $\mathbb{R}^N$ and $C^\infty(\bar{\mathbb{R}}^N)$ is the smallest subspace of $C(\bar{\mathbb{R}}^N)$ containing all constants, as well as those functions which are infinitely differentiable on $\mathbb{R}^N$ and vanish, together with all their derivatives, at ∂.

It is obvious that $D(\bar{\mathbb{R}}^N)$ is dense in $C(\bar{\mathbb{R}}^N)$.

Let $\tilde{C} := \cup_{N=0}^\infty C(\bar{\mathbb{R}}^N)$ (disjoint union), where $C(\bar{\mathbb{R}}^0) := \mathbb{R}^1$. We define a metric d on $\tilde{C}$ by $d(f,g) := \|f-g\|_n$ for n the larger of $N(f)$ and $N(g)$, where we set $N(f) := n$ if we have both $f \in C(\bar{\mathbb{R}}^n)$ and $f \notin C(\bar{\mathbb{R}}^{n-1})$, $\|f-g\|_n$ is the uniform norm on $C(\bar{\mathbb{R}}^n)$ and we use implicitly the canonical embedding $\bar{\mathbb{R}}^n \subset \bar{\mathbb{R}}^{n+1}$. Let $\tilde{D} := \cup_{N=0}^\infty D(\bar{\mathbb{R}}^N)$

(disjoint union). Then $\tilde{D}$ is a subspace of $\tilde{C}$. We will see $\tilde{C}$ is the state space of the dual processes. Define

$$(2.18)\qquad Q_0(\tilde{D}) := \{\ F_\mu(f) = \int_{\bar{\mathbb{R}}}\cdots\int_{\bar{\mathbb{R}}} f(x_1,\cdots,x_N)\mu^{\otimes N}(d\tilde{x}),\quad f\in D(\bar{\mathbb{R}}^N),\ \mu\in E,\ N\in\mathbb{N}\cup\{0\}\}.$$

Let $Q(\tilde{D})$ denote the linear space generated by $Q_0(\tilde{D})$.

The following facts are needed for proving the existence of the $(\mathcal{A},\delta_\mu)$-MP.

Lemma 2.1 *Let $X=(x_t,\mathcal{F}_t,P_x)$ be a Markov process on the state space $(H,\mathcal{H})$ with transition function $P(t,x,A)$. Let γ be a measurable transformation of $(H,\mathcal{H})$ into the state space $(\tilde{H},\tilde{\mathcal{H}})$ such that $\gamma H=\tilde{H}, \tilde{\mathcal{H}}\supseteq\gamma(\mathcal{H})$ and such that the following condition is satisfied: For all $A\in\tilde{\mathcal{H}}$ and $x,x'\in H$ such that $\gamma x=\gamma x'$,*

$$P(t,x,\gamma^{-1}A)=P(t,x',\gamma^{-1}A).$$

Set $\tilde{x}_t=\gamma x_t$ and denote by $\mathcal{N}$ the σ-algebra generated by the sets $\{\tilde{x}_t\in A\}$ $(t\geq 0, A\in\tilde{\mathcal{H}})$. Further, let $\tilde{\mathcal{F}}_t=\mathcal{F}_t\cap\mathcal{N}$ and

$$\tilde{P}_{\gamma x}(B)=P_x(B)\qquad \text{for }\ B\in\mathcal{N}.$$

The system $\tilde{X}=(\tilde{x}_t,\tilde{\mathcal{F}}_t,\tilde{P}_x)$ defines a Markov process on the state space $(\tilde{H},\tilde{\mathcal{H}})$ with transition function

$$\tilde{P}(t,\gamma x,A)=P(t,x,\gamma^{-1}A).$$

If X is a strong Markov process, then so is the process $\tilde{X}$.

Proof: see Dynkin([10] v1, p.325, Theorem 10.13) ■

Let $C_b^2(\mathbb{R}^N)$ be the space of all bounded, twice continuously differentiable functions on $\mathbb{R}^N$ with all first and second derivatives bounded; $C_c^\infty(\mathbb{R}^N)$, the space of all infinitely differentiable functions on $\mathbb{R}^N$ with compact support; and $C_0(\mathbb{R}^N)$ the space of all continuous functions on $\mathbb{R}^N$ vanishing at infinity.

Lemma 2.2 *For any $N\in\mathbb{N}$ and $f\in C_b^2(\mathbb{R}^N)$, let*

$$(2.19)\qquad G^N f(x_1,\cdots,x_N) := \frac{1}{2}\sum_{i=1}^N \epsilon^2\frac{\partial^2}{\partial x_i^2}f(x_1,\cdots,x_N) + \frac{1}{2}\sum_{i,j=1}^N \rho(x_i-x_j)\frac{\partial^2}{\partial x_i\partial x_j}f(x_1,\cdots,x_N),$$

where $\rho(\cdot)$ is defined by (1.4). Assume that $\rho(\cdot)$ is a smooth function.

(i) There exist a unique system of probabilities $\{P_x, x \in \bar{\mathbb{R}}^N\}$ on $(\bar{\mathbf{W}}^N, \mathcal{B}(\bar{\mathbf{W}}^N))$, which is strongly Markovian such that $P_x\{w; w(0) = x\} = 1$ for every $x \in \mathbb{R}^N$, and

$$f(w(t)) - f(w(0)) - \int_0^t (G^N f)(w(s))ds$$

is a $(P_x, \mathcal{B}_t(\bar{\mathbf{W}}^N))$-martingale for every $f \in C^2(\bar{\mathbb{R}}^N)$, where

$$\bar{\mathbf{W}}^N = \{w : w \in C([0,\infty), \bar{\mathbb{R}}^N) \text{ such that if } w(t) = \partial,$$

$$\text{then } w(u) = \partial, \text{ for all } u \geq t\},$$

$\mathcal{B}_t(\bar{\mathbf{W}}^N)$ is the σ-field generated by Borel cylinder sets before and including time t and $\mathcal{B}(\bar{\mathbf{W}}^N) := \sigma(\vee_t \mathcal{B}_t(\bar{\mathbf{W}}^N))$.

(ii) *Define*

$$\tilde{G}^N \tilde{f}(x) := \begin{cases} G^N f(x) & \text{if } x \neq \partial \\ 0 & \text{if } x = \partial, \end{cases}$$

where $\tilde{f} \in C^\infty(\bar{\mathbb{R}}^N)$ and $f := \tilde{f} - \tilde{f}(\partial) \in C_0^\infty(\mathbb{R}^N)$. Then the closure of $(\tilde{G}^N, C^\infty(\bar{\mathbb{R}}^N))$ is single-valued and generates a Feller semigroup on $C(\bar{\mathbb{R}}^N)$.

Remark Given any initial value $x \in \mathbb{R}^N$, by Ikeda-Watanabe([15] p.163, Theorem 2.4), the explosion time of the G^N-diffusion $e(\omega) = \infty$ *a.s.* This means that the G^N-diffusion always lives in $\mathbb{R}^N$.

Proof: (i) If $\epsilon \neq 0$, then this is the conclusion of Ikeda-Watanabe([15] p.155, Theorem 2.2 and p.177, Theorem 3.3). It remains to consider the case in which $\epsilon = 0$. Let $a^{ij}(x) := \rho(x_i - x_j)$ for $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ and $a(x) := (a^{ij}(x))$. Obviously $a(x)$ is a symmetric, nonnegative definite matrix. Indeed, for $\forall\, b = (b_1, \cdots, b_N) \in \mathbb{R}^N$,

$$\begin{aligned} ba(x)b^T &= \sum_{i,j=1}^N b_i b_j \int_{\mathbb{R}} g(x_i - x_j - z) g(z) dz \\ &= \sum_{i,j=1}^N b_i b_j \int_{\mathbb{R}} g(x_i - t) g(x_j - t) dt \\ &= \int_{\mathbb{R}} (\sum_{i=1}^N b_i g(x_i - t))^2 dt \geq 0. \end{aligned}$$

Given a matrix $\sigma(x)$ such that

$$(2.20) \qquad \begin{cases} x \to \sigma(x) & \text{is continuous} \\ a^{ij}(x) = \sum_{k=1}^{N} \sigma^{ik}(x)\sigma^{jk}(x), \end{cases}$$

let us consider the following stochastic differential equation

$$(2.21) \qquad dX^i(t) = \sum_{k=1}^{N} \sigma^{ik}(X(t))dB^k(t),$$

where $(B^1(t), \cdots, B^N(t))$ are independent Brownian motions.

It is well known that if (2.20) holds, then G^N-diffusion $\{P_x, x \in \bar{\mathbb{R}}^N\}$ exists uniquely if and only if (2.21) is uniquely solvable.

Since $\rho(\cdot)$ is a bounded smooth function, the result follows from the corollary of Ikeda-Watanabe([15] p.203).

(ii) Since $a^{ij}(x) \in C^2(\mathbb{R}^N)$ with $\partial_k\partial_l a^{ij}$ bounded for $i, j, k, l = 1, 2, \cdots N$, by Ethier-Kurtz ([11] p.373 Theorem 2.5), the closure of $(G^N, C_0^\infty(\mathbb{R}^N))$ is single-valued and generates a Feller semigroup $\{\hat{S}_t^N : t \geq 0\}$ on $C_0(\mathbb{R}^N)$. For $\bar{f} \in C(\bar{\mathbb{R}}^N)$, we define

$$S_t^N \bar{f} = \bar{f}(\partial) + \hat{S}_t^N(\bar{f} - \bar{f}(\partial)).$$

Then by Ethier-Kurtz ([11] p.166, Lemma 2.3), $\{S_t^N : t \geq 0\}$ is a Feller semigroup on $C(\bar{\mathbb{R}}^N)$ and for $\bar{h} \in C^\infty(\bar{\mathbb{R}}^N)$. Let $h := \bar{h} - \bar{h}(\partial) \in C_0^\infty(\mathbb{R}^N)$ and define $(\bar{G}^N, C^\infty(\bar{\mathbb{R}}^N))$ by

$$(2.22) \qquad \bar{G}^N\bar{h}(x) := \begin{cases} G^N h(x) & \text{if } x \neq \partial \\ 0 & \text{if } x = \partial. \end{cases}$$

Then the closure of $(\bar{G}^N, C^\infty(\bar{\mathbb{R}}^N))$ is the generator of $\{S_t^N : t \geq 0\}$. ■

In the following, we will not distinguish notationally between $\bar{G}^N$ and G^N.

Now we begin to consider the $(\mathcal{A}, \delta_\mu)$-MP and the operator $\mathcal{A}$. Let $(\bar{E}, r)$ be a metric space, $B(\bar{E})$ be the Banach space of real-valued, bounded, and Borel measurable functions on $\bar{E}$ with $\|f\| = \sup_{x \in \bar{E}} |f(x)|$ and $\bar{C}(\bar{E}) (\subset B(\bar{E}))$ be the space of bounded, continuous functions on the metric space $\bar{E}$.

Definition 2.1 *Consider a subset $A \subset B(\bar{E}) \times B(\bar{E})$ and $\mu \in \mathcal{P}(\bar{E})$, the space of all probability measures on $\bar{E}$. By a solution to the (A, μ)-MP, we mean a measurable stochastic process X with values in $\bar{E}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such* *that for each* $(f,g) \in A$,

$$f(X(t)) - \int_0^t g(X(s))ds$$

is a martingale with respect to the filtration

$$\hat{\mathcal{F}}_t^X := \mathcal{F}_t^X \vee \sigma\left(\int_0^s h(X(u))du, \quad s \le t, h \in B(\bar{E})\right)$$

and $\mathbb{P}X(0)^{-1} = \mu$.

Consider the canonical processes on the space $D([0,\infty),\bar{E})$ or $C([0,\infty),\bar{E})$, then we have following definition.

Definition 2.2 *A family of probability measures* $\{\mathbb{P}_x : x \in \bar{E}\}$ *on* $D([0,\infty),\bar{E})$ *(or* $C([0,\infty),\bar{E})$*) is said to be a solution to the martingale problem for A if there holds both the following:*

(1) *for each* $x \in \bar{E}$, $\mathbb{P}_x(X(0) = x) = 1$;

(2) *for each probability measure* $\mathbb{P}_x$, $\{X(t) : t \ge 0\}$ *is a solution to the* (A, δ_x)*-MP, where* $\{X(t) : t \ge 0\}$ *is the canonical process on* $D([0,\infty),\bar{E})$ *(or* $C([0,\infty),\bar{E})$*).*

We finally need the following definitions, in order to view and express the exchangeability of paths of our particles.

Definition 2.3 *Let* $\wp(m)$ *be the set of all permutations of* $\{1,\cdots,m\}$. *A family of probability measures* $\{P_y^m : y \in \mathbb{R}^m\}$ *on* $C([0,\infty),\mathbb{R}^m)$*(or on* $D([0,\infty),\mathbb{R}^m)$*) is called exchangeable if it satisfies* $P_{\Pi y}^m = P_y^m \circ \tilde{\Pi}^{-1}$ *for every* $y \in \mathbb{R}^m$ *and every* $\Pi \in \wp(m)$, *where* $\Pi y = \Pi(y_1, y_2, \cdots, y_m) := (y_{\Pi 1}, y_{\Pi 2}, \cdots, y_{\Pi m})$ *for each* $y \in \mathbb{R}^m$ *and the corresponding* $\tilde{\Pi}(X)(t) := \Pi(X(t))$ *for each* $X \in C([0,\infty),\mathbb{R}^m)$*(or* $D([0,\infty),\mathbb{R}^m)$*).*

Given a second order differential operator

$$A(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{d} b^i(x)\frac{\partial}{\partial x_i},$$

we define

$$\Pi A(x) := \frac{1}{2}\sum_{i,j=1}^{d} a^{\Pi i \Pi j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i}^{d} b^{\Pi i}(x)\frac{\partial}{\partial x_i}.$$

A is said to be symmetric if $\Pi A(x) = A(\Pi x)$.

Theorem 2.3 *Suppose* $\rho(\cdot)$ *defined by* (1.4) *is a bounded smooth function. Then we have:*

(I) *For any finite measure* μ *on* $\mathbb{R}$ *with compact support, the* $(\mathcal{A}, \delta_\mu)$*-MP is uniquely solvable with* $\Pi^D(E)$ *in the domain of* $\mathcal{A}$.

(II) *A subspace* $\mathcal{D}(\mathcal{A}) \subset C(E)$ *exists such that* $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ *is the generator of a Feller semigroup.*

Proof: (I) As for the existence, since μ is a finite measure, there exists a positive constant ξ such that $< 1, \mu > \leq \xi$.

(a) First, we consider only measures of the form $\mu = \frac{1}{\theta^n}\sum_{i=1}^m \delta_{x_i}$ and all the functions

$$F_f(\mu) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \cdots, x_N)\mu^{\otimes N}(d\tilde{x}) \in \Pi_1^D(E_\xi).$$

Let $f'_\alpha(x_1, \cdots, x_n)$ denote the derivative of $f(x_1, \cdots, x_n)$ with respect to the α^{th} variable. Now we change the form of $\mathcal{A}F_f(\mu)$ as follows.

$$\mathcal{A}F_f(\mu) \tag{2.23}$$

$$= \frac{1}{2}\rho_\epsilon \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i=1}^N f''_{ii}(y_1, \cdots, y_N)\mu^{\otimes N}(d\tilde{y})$$

$$+\frac{1}{2}\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1;i\neq j}^N \rho(y_i - y_j) f''_{ij}(y_1, \cdots, y_N)\mu^{\otimes N}(d\tilde{y})$$

$$= \frac{1}{2}\sum_{i=1}^m \rho_\epsilon \frac{\partial^2}{\partial x_i^2} g(x_1, \cdots, x_m)$$

$$+\frac{1}{2}\sum_{i,j=1;i\neq j}^m \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j} g(x_1, \cdots, x_m)$$

$$= G^m g(x_1, \cdots, x_m).$$

Here we write

$$g(x_1, \cdots, x_m) = (\frac{1}{\theta^n})^N \sum_{l_1,\cdots,l_N=1}^m f(x_{l_1}, \cdots, x_{l_N}), \tag{2.24}$$

and G^m is defined by (2.19).

For fixed integers n, m, let

$$E^{n,m} := \{\mu \in E | \mu = \frac{1}{\theta^n}\sum_{i=1}^m \delta_{x_i}, x_i \in \bar{\mathbb{R}}, i = 1, \cdots, m\}, \tag{2.25}$$

and

$$E^n := \cup_{m=0}^{\infty} E^{n,m}. \tag{2.26}$$

Both $E^{n,m}$ and E^n are endowed with subspace topologies of E.

We define a transformation from $\mathbb{R}^m$ to $E^{n,m}$ by

$$\Phi_n : (x_1, \cdots, x_m) \to \frac{1}{\theta^n} \sum_{i=1}^{m} \delta_{x_i}. \tag{2.27}$$

Since G^m is symmetric, the law of the corresponding diffusion process is exchangeable and the conditions of Lemma 2.1 are satisfied. By Lemma 2.1 and Lemma 2.2, we obtain a solution to the $(\mathcal{A}, \delta_\mu)$-MP.

(b) Now suppose μ is any finite measure with compact support, we define

$$B_n^k := B\left(\frac{k}{2^n}, \frac{1}{2^{n+1}}\right) := \{x \in \mathbb{R} : \frac{k}{2^n} - \frac{1}{2^{n+1}} \leq x < \frac{k}{2^n} + \frac{1}{2^{n+1}}\}, \tag{2.28}$$

$$N'(\mu(B_n^k)) := \begin{cases} i & \text{if existing } i \in \mathbb{N} \text{ such that} \\ & \frac{i}{\theta^n} \leq \mu(B_n^k)) < \frac{i+1}{\theta^n} \text{ and } |k| \leq n2^n \\ 0 & \text{otherwise,} \end{cases} \tag{2.29}$$

$$N_n := \sum_{|k| \in \mathbb{N}} N'(\mu(B_n^k)), \tag{2.30}$$

$$x_i^n = \begin{cases} \frac{-n2^n}{2^n} & \text{if } \quad 1 \leq i \leq N'(\mu(B_n^{-n2^n})) \\ \frac{-n2^n+1}{2^n} & \text{if } \quad N'(\mu(B_n^{-n2^n})) + 1 \leq i \leq N'(\mu(B_n^{-n2^n})) + N'(\mu(B_n^{-n2^n+1})) \\ \vdots & \qquad \vdots \\ \frac{n2^n}{2^n} & \text{if } \quad \sum_{k \leq (n2^n-1)} N'(\mu(B_n^k)) + 1 \leq i \leq \sum_{k \leq n2^n} N'(\mu(B_n^k)), \end{cases}$$

$$\mu_n := \frac{1}{\theta^n} \sum_{i=1}^{N_n} \delta_{x_i^n}(\cdot), \tag{2.31}$$

then it is obvious $\mu_n \Rightarrow \mu$. Without loss of generality, we can assume $< 1, \mu_n > \leq \xi$ for $n \in \mathbb{N}$. From the above argument, suppose Y_n is the solution to the $(\mathcal{A}, \delta_{\mu_n})$-MP. Since $\mathcal{A}$ does not change the total mass, for any $n \in \mathbb{N}$ and $t \geq 0$, $< 1, Y_n(t) > \leq \xi$. It follows from Ethier-Kurtz ([11] p.196, Lemma 5.1) that the $(\mathcal{A}, \delta_\mu)$-MP has a solution which is the weak convergence limit of Y_n.

(c) As for the uniqueness, let

(2.32) $$\mathcal{A}^{\#}F_{\mu'}(f) := F_{\mu'}(G^{N(f)}f) \quad \text{for } F_{\mu'}(f) \in Q_0(\tilde{D}),$$

where G^N is defined by (2.19) and $N(\cdot)$ is defined at the beginning of section 2. Then we have

(2.33) $$\mathcal{A}F_{\bar{f}}(\mu') = \frac{1}{2}\rho_\epsilon \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i=1}^{N} f''_{ii}(x_1, \cdots, x_N)\mu'^{\otimes N}(d\bar{x})$$

$$+\frac{1}{2}\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1; i\neq j}^{N} \rho(x_i - x_j) f''_{ij}(x_1, \cdots, x_N)\mu'^{\otimes N}(d\bar{x})$$

$$= \mathcal{A}^{\#}F_{\mu'}(f).$$

From Lemma 2.2, we know that for any $N \in \mathbb{N}$, the closure of $(\tilde{G}^N, D(\bar{\mathbb{R}}^N))$ is the generator of the Feller semigroup $\{S_t^N : t \geq 0\}$ and

$$S_t^N : D(\bar{\mathbb{R}}^N) \to D(\bar{\mathbb{R}}^N).$$

Therefore, $(\mathcal{A}^{\#}, Q(\tilde{D}))$ has the structure of the infinitesimal generator of a conservative $\tilde{D}$-valued Markov process restricted to functions in $Q(\tilde{D})$. If the initial state of this process is $f \in D(\bar{\mathbb{R}}^N)$, then it keeps evolving in a continuous and deterministic way on $D(\bar{\mathbb{R}}^N)$; namely: $f \to S_t^N f$. Then by Dawson-Kurtz([7] p.101, Theorem 4.4), it follows that the $(\mathcal{A}, \delta_\mu)$-MP is well posed.

(II) First, since the $(\mathcal{A}, \delta_\mu)$-MP has a unique solution, by Ethier-Kurtz([11] p.178, Prop. 3.5), $\mathcal{A}$ is a dissipative operator. Secondly, given any real number $\lambda > 0$ and any monomial function $< g(x_1, \cdots, x_N), \mu^{\otimes N} >$, we consider the range condition of $(\lambda - \mathcal{A})$. Since G^N is the generator of a Feller semigroup, for above $g \in C(\mathbb{R}^N)$ and $\lambda > 0$, by the Hille-Yosida theorem, there exists $f \in \mathcal{D}(G^N)$ such that $(\lambda - G^N)f = g$. Therefore,

(2.34) $$(\lambda - \mathcal{A}) \int \cdots \int f(x_1, \cdots, x_N)\mu^{\otimes N}(d\bar{x})$$

$$= \int \cdots \int (\lambda - G^N) f(x_1, \cdots, x_N)\mu^{\otimes N}(d\bar{x})$$

$$= \int \cdots \int g(x_1, \cdots, x_N)\mu^{\otimes N}(d\bar{x}).$$

Since $\Pi^D(E_l)$ is dense in $C(E_l)$, $(\lambda - \mathcal{A})$ satisfies the range condition. Note that l is an arbitrary positive number, thus the closure of $\mathcal{A}$ generates a strongly continuous contraction semigroup V_t on $C(E)$ by the Hille-Yosida theorem. It is obvious that V_t is conservative. Since for any nonnegative $F(\mu) \in C(E)$, we have

$$\|F\| - V_tF(\mu) = V_t(\|F\| - F(\mu)) \leq \|\|F\| - F\| \leq \|F\|$$

for all $\mu \in E$ and $t \geq 0$, the positivity of V_t follows. Hence the proof of (II) is complete.
■

3 Finite Particle Systems and Tightness

In this section, we will first construct a sequence of finite particle systems and prove that they satisfy tightness conditions, then we will show that their limiting measure-valued processes are solutions to the $(\mathcal{L}, \delta_\mu)$-MP. Before doing so, we want to find conditions to guarantee the existence of the moments for the finite particle systems and for their limiting measure-valued processes.

Given $n \in \mathbb{N}$ and $0 \leq \gamma < \infty$, $2 \leq \theta < \infty$. let $\{Q_n(t); t \geq 0\}$ be a one dimensional continuous time Markov branching process with state space $\Theta_n := \{\frac{k}{\theta^n}; k \in \mathbb{N} \cup \{0\}\}$. Its time homogeneous transition functions

$$P^n_{ij}(t) = P^n_{ij}(s, s+t) = P\{Q_n(s+t) = \frac{j}{\theta^n} | Q_n(t) = \frac{i}{\theta^n}\}$$

are determined by the following infinitesimal probabilities:

$$p_1 = 0, \quad \sum_{k=0}^{\infty} p_k = 1, \quad \sum_{k=0}^{\infty} kp_k = 1,$$
$$m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty.$$

These transition functions are solution to the Kolmogorov forward equation

$$\frac{d}{dt}P^n_{ij}(t) = -j\gamma\theta^n P^n_{ij}(t) + \gamma\theta^n \sum_{k=1}^{j+1} kP^n_{ik}(t)p_{j-k+1} \tag{3.35}$$

and to the Kolmogorov backward equation

$$\frac{d}{dt}P^n_{ij}(t) = -i\gamma\theta^n P^n_{ij}(t) + \gamma\theta^n \sum_{k=i-1}^{\infty} ip_{k-i+1}P^n_{kj}(t) \tag{3.36}$$

with boundary condition

$$P^n_{ij}(0+) = \delta_{ij}.$$

The following results are well-known.

Lemma 3.1 *Let $k \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ and $t > 0$, $\mathbb{E}Q_n(t)^k < \infty$ if and only if $\sum_{j=0}^{\infty} j^k p_j < \infty$.*

Proof: refer to Athreya-Ney([1] p.111, Corollary 1). ■

Under the assumptions given in Section 1, we have $Q_n(0) := \frac{m_0^n}{\theta^n} \leq \xi$ for $\forall n \in \mathbb{N}$. According to the theory of continuous state branching processes (see [1], [13], [16]), the limiting process $\{Q(t); t \geq 0\}$ of $\{Q_n(t); t \geq 0\}$ as $n \to \infty$ exists and is a diffusion process.

Lemma 3.2 *If $\sum_{k=0}^{\infty} kp_k = 1, m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty$, then $\mathbb{E}(Q(t)^k) < \infty$ for any $k \in \mathbb{N}$ and $t \geq 0$.*

Proof: By direct calculation, we know that the density function of the transition function $P_t(x, A)$ of $Q(t)$ is the solution of the backward Kolmogorov equation

$$\frac{\partial \varphi}{\partial t} = \frac{1}{2}(m_2 - 1)\gamma x \frac{\partial^2 \varphi}{\partial x^2}.$$

Because of the branching property, the Laplace transform of $P_t(x, A)$ can be written in the form

$$\int_0^{\infty} exp(-\lambda y) P_t(x, dy) = exp(-x\psi_t(\lambda)), \quad \text{for } \lambda \geq 0$$

and

$$\psi_t(\lambda) = \lambda[1 + \frac{1}{2}(m_2 - 1)\gamma t \lambda]^{-1}.$$

Our conclusion follows directly from

$$\mathbb{E}[Q(t)^k | Q(0) = x] = (-1)^k \frac{\partial^k}{\partial \lambda^k} \{exp(-x\psi_t(\lambda))\}|_{\lambda=0}$$

and the fact that $\psi_t(\lambda)$ is infinitely differentiable with respect to λ . ■

Now we begin to construct a sequence of finite particle systems.

Define the bounded operator $\mathcal{B}_n : B(E^n) \to B(E^n)$ by

$$\mathcal{B}_n F(\nu_n) := \gamma \theta^n (\theta^{2n} \wedge m) \int_{E^n} (F(\nu') - F(\nu_n)) \Gamma(\nu_n, d\nu'), \tag{3.37}$$

where E^n is defined by (2.26), $B(E^n)$ is the space of all bounded measurable functions on E^n, $x \wedge y := min(x, y)$, and

$$\Gamma(\nu_n, d\nu') := \frac{1}{m}\sum_{i=1}^{m}\sum_{k=0}^{\infty} p_k \delta_{(\nu_n + \frac{k-1}{\theta^n}\delta_{x_i})}(d\nu')$$

$$\text{for } \nu_n = \frac{1}{\theta^n}\sum_{i=1}^{m}\delta_{x_i} \in E^n.$$

Lemma 3.3 *For μ_n defined as in* (2.31), *the $(\mathcal{A}+\mathcal{B}_n, \delta_{\mu_n})$-MP has a unique solution with state space E^n.*

Proof: If $\mathcal{B}_n F_f(\mu_n)$ is defined as in (3.37), then $(\mathcal{B}_n, B(E^n))$ is the generator of a pure jump process. It follows from the bounded perturbation theorem ([11] p.37. Theorem 7.1), Trotter's product formula ([11] p.33. Corollary 6.7), and Ethier-Kurtz ([11] p.166, Lemma 2.3, p.169. Theorem 2.7 and p.182, Theorem 4.1) that the $(\mathcal{A}+\mathcal{B}_n, \delta_{\mu_n})$-MP has a unique solution with state space E^n. ■

Let us write $X_n(t) = \frac{1}{\theta^n}\sum_{i=1}^{m_t^n}\delta_{x_i(t)}$ for the canonical process associated with the solution to this martingale problem. This means that m_t^n is the total number of particles at time t with initial $m_0^n = N_n$.

Lemma 3.4 *For μ_n defined as in* (2.31), *we have the following results:*

(1) *For any positive η, we have*

$$\tilde{\mathbb{P}}(\sup_t < 1, X_n(t) > > \eta) \leq \frac{< 1, \mu_n >}{\eta} \tag{3.38}$$

and $\{X_n(t) : t \geq 0\}$ is tight.

(2) *Suppose that the law $\tilde{\mathbb{P}}^{X_n}$ of X_n converges weakly to a probability measure $\tilde{\mathbb{P}}$ on $D([0,\infty), E)$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $D([0,\infty), E)$-valued random variables $Z_n, n = 1, 2, \cdots$, and Z, all defined on it, such that $\tilde{\mathbb{P}}^{X_n} = \mathbb{P}^{Z_n}$ holds, $\tilde{\mathbb{P}} = \mathbb{P}^Z$ holds and $\{Z_n : n > 0\}$ converges to Z on $D([0,\infty), E)$ almost surely with respect to probability $\mathbb{P}$.*

(3) *There comes*

$$\lim_{n\to\infty}\sup_{0\leq t<\infty}\frac{< 1, Z_n(t) >}{\theta^n} = 0 \qquad \mathbb{P}-a.s.$$

Proof: (1). First let us prove inequality (3.38). For $\lambda > 0$, let $F_\lambda(\mu) = e^{-<\lambda,\mu>}$, $\vartheta(z) = \sum_{k=0}^{\infty} p_k z^k$. Define $|X_n(t)| :=< 1, X_n(t) >$. Observe that, since there comes $\mathcal{A}F_\lambda = 0$ (motion-free), the process

$$M_\lambda(t) := e^{-\lambda|X_n(t)|} - \int_0^t \mathcal{B}_n F_\lambda(X_n(s))ds \tag{3.39}$$

$$= e^{-\lambda|X_n(t)|} + \int_0^t \gamma\theta^{2n}(\theta^n \wedge |X_n(s)|)e^{-\lambda|X_n(s)|}[1 - e^{\lambda/\theta^n}\vartheta(e^{-\lambda/\theta^n})]ds$$

is a martingale. So

$$\tilde{\mathbb{E}}e^{-\lambda|X_n(t)|} = e^{-\lambda|\mu_n|} - \int_0^t \tilde{\mathbb{E}}\gamma\theta^{2n}(\theta^n \wedge |X_n(s)|)e^{-\lambda|X_n(s)|}[1 - e^{\lambda/\theta^n}\vartheta(e^{-\lambda/\theta^n})]ds.$$

Since $e^x \geq 1 + x$ and $1 - \vartheta(e^{\lambda/\theta^n}) \leq \vartheta'(1)(\lambda/\theta^n)$, we have

$$\tilde{\mathbb{E}}(e^{-\lambda|X_n(t)|}\lambda|X_n(t)|) \leq \tilde{\mathbb{E}}(1 - e^{-\lambda|X_n(t)|})$$

$$= 1 - e^{-\lambda|\mu_n|} + \int_0^t \tilde{\mathbb{E}}\gamma\theta^{2n}(\theta^n \wedge |X_n(s)|)e^{-\lambda|X_n(s)|}[1 - e^{\lambda/\theta^n}\vartheta(e^{-\lambda/\theta^n})]ds$$

$$\leq 1 - e^{-\lambda|\mu_n|} + \int_0^t \tilde{\mathbb{E}}\gamma\theta^{2n}(\theta^n \wedge |X_n(s)|)e^{-\lambda|X_n(s)|}[1 - \vartheta(e^{-\lambda/\theta^n})]\frac{\lambda}{\theta^n}ds$$

$$\leq 1 - e^{-\lambda|\mu_n|} + \int_0^t \gamma\theta^n[1 - \vartheta(e^{-\lambda/\theta^n})]\tilde{\mathbb{E}}\lambda|X_n(s)|e^{-\lambda|X_n(s)|}ds.$$

By Gronwall's inequality,

$$\tilde{\mathbb{E}}e^{-\lambda|X_n(t)|}\lambda|X_n(t)| \leq (1 - e^{-\lambda|\mu_n|})\exp\{t\gamma\theta^n[1 - \vartheta(e^{-\lambda/\theta^n})]\}.$$

Let $\lambda \to 0$, we have

$$\tilde{\mathbb{E}}|X_n(t)| \leq |\mu_n| \tag{3.40}$$

and $|X_n(t)| = \lim_{\lambda\to 0}\frac{1-M_\lambda(t)}{\lambda}$ is a martingale. Hence (3.38) follows from Doob's supermartingale inequality. Now we use Ethier-Kurtz ([11] p.142, Theorem 9.1 and p.145, Theorem 9.4) to prove the tightness. First let us check the compact containment condition (9.1) of Ethier-Kurtz ([11] p.142, Theorem 9.1). Since $\bar{\mathbb{R}}$ is compact, by Prokhorov([17] Theorem 1.12) , a set $\ell \subset M_F(\bar{\mathbb{R}})$ is compact if and only if $\sup_{\mu\in\ell}\mu(\bar{\mathbb{R}}) < \infty$. Recall the construction of μ_n given by (2.31) for a finite measure μ with compact support, then $\mu_n \Rightarrow \mu$ and there exists a constant $\xi > 0$ such that $< 1, \mu_n > \leq \xi$ for any $n \in \mathbb{N}$. By (3.38), the compact containment condition (9.1) of Ethier-Kurtz ([11] p.142, Theorem 9.1) holds. Now let us check the conditions of Ethier-Kurtz ([11] p.145, Theorem 9.4). By (3.40), the condition (9.16) of Ethier-Kurtz ([11] p.145, Theorem 9.4) is satisfied. For a more general function $f(< \phi_1, \mu >, \cdots, < \phi_k, \mu >)$, where $f \in C^\infty(\mathbb{R}^k)$ and $\phi_i \in C_0^\infty(\mathbb{R})$, $1 \leq i \leq k$, by Taylor's expansion we have

(3.41) $\mathcal{B}_n f(< \phi_1, X_n(t) >, \cdots, < \phi_k, X_n(t) >) =$

$$\frac{1}{2}\gamma(m_2-1)\int_{\mathbb{R}}\frac{\delta^2 f(<\phi_1,X_n(t)>,\cdots,<\phi_k,X_n(t)>)}{\delta X_n(t)(x)^2}X_n(t)(dx)+O(\frac{1}{\theta^{2n}})$$

$$=\mathcal{B}f(<\phi_1,X_n(t)>,\cdots,<\phi_k,X_n(t)>)+O(\frac{1}{\theta^{2n}}).$$

For a monomial function $F_f(\cdot)$, we have

$$(3.42)\qquad (\mathcal{A}+\mathcal{B})F_f(\mu):=F_{G^{N(f)}f}(\mu)+\frac{1}{2}\gamma(m_2-1)\sum_{j,k=1;j\neq k}^{N(f)}F_{\Phi_{j,k}f}(\mu),$$

where $G^{N(f)}$ is defined by (2.19) and its closure is the generator of the Feller semigroup $\{S_t^{N(f)}:t\geq 0\}$:

$$(3.43)\quad \Phi_{j,i}f(x_1,\cdots,x_N):=\begin{cases} f(x_1,\cdots,x_j,\cdots,x_{i-1},x_j,x_{i+1},\cdots,x_N) & \text{if } j<i \\ f(x_1,\cdots,x_{i-1},x_j,x_{i+1},\cdots,x_j,\cdots x_N) & \text{if } i<j. \end{cases}$$

Lemma 3.2 implies the condition (9.18) of Ethier-Kurtz ([11] p.145, Theorem 9.4) and the tightness is proved.

(2). These results follow from Ikeda-Watanabe([15] p.9, Theorem 2.7).

(3). (3.38) implies $\sup_{0\leq t}<1,Z(t)><\infty$ almost surely with respect to $\mathbb{P}$. So

$$\lim_{n\to\infty}\sup_{t\geq 0}\frac{<1,Z_n(t)>}{\theta^n}=0 \qquad \mathbb{P}-a.s. \qquad \blacksquare$$

Theorem 3.5 *If $\rho(\cdot)$ is a bounded, smooth function, then for any finite measure μ with compact support, the $(\mathcal{A}+\mathcal{B},\delta_\mu)$-MP has at least one solution.*

Proof: First from Lemma 3.4, for $\forall t\in K(Z):=\{t\geq 0: P\{Z(t)=Z(t-)\}=1\}$ and any

$$f(<\varphi,\mu>),\quad \text{for } f\in C^\infty(\bar{\mathbb{R}}),\ \varphi\in C_0^\infty(\mathbb{R})$$

by Ethier-Kurtz([11] p.118, Proposition 5.2), we have

$$(3.44)\qquad \lim_{n\to\infty}f(<\varphi,Z_n(t)>)=f(<\varphi,Z(t)>)\qquad \mathbb{P}-a.s.$$

From (3.41), we have

$$(3.45)\qquad \lim_{n\to\infty}\mathcal{B}_n f(<\varphi,Z_n(t)>)$$

$$=\frac{1}{2}\gamma(m_2-1)\int_{\mathbb{R}}\frac{\delta^2 f(<\varphi,Z(t)>)}{\delta Z(t)(x)^2}Z(t)(dx)\qquad \mathbb{P}-a.s.$$

$$= \mathcal{B}f(<\varphi, Z(t)>).$$

From Ethier-Kurtz([11] p.131, Lemma 7.7), we know that the complement in $[0,\infty)$ of $K(Z)$ is at most countable . Let Λ be Lebesgue measure on $[0,\infty)$, then from (3.45) we have

$$\lim_{n\to\infty}(\mathcal{A}+\mathcal{B}_n)f(<\varphi, Z_n(u)>) = (\mathcal{A}+\mathcal{B})f(<\varphi, Z(u)>) \quad \mathbb{P}\otimes\Lambda - a.s. \tag{3.46}$$

For any $0 \le t_i \le s < t,\ t_i, s, t \in K(Z)$ and any $h_i \in C(E), i = 1, \cdots, k$, by Lemma 3.2,

$$\{[f(<\varphi, Z(t)>) - f(<\varphi, Z(s)>) - \int_s^t (\mathcal{A}+\mathcal{B})f(<\varphi, Z(u)>)du]\prod_{i=1}^k h_i(Z(t_i))\}$$

is integrable. So by the dominated convergence theorem, we have

$$\begin{aligned}&\mathbb{E}\{[f(<\varphi, Z(t)>) - f(<\varphi, Z(s)>)\\ &- \int_s^t (\mathcal{A}+\mathcal{B})f(<\varphi, Z(u)>)du]\prod_{i=1}^k h_i(Z(t_i))\}\\ &= \lim_{n\to\infty}\mathbb{E}\{[f(<\varphi, Z_n(t)>) - f(<\varphi, Z_n(s)>)\\ &- \int_s^t (\mathcal{A}+\mathcal{B}_n)f(<\varphi, Z_n(u)>)du]\prod_{i=1}^k h_i(Z_n(t_i))\} = 0.\end{aligned}$$

The right continuity of Z implies the above equality holds for all $0 \le t_i \le s < t$, and hence Z is a solution to the $(\mathcal{A}+\mathcal{B}, \delta_\mu)$-MP, since the set of functions $\{f(<\varphi, \cdot>)\}$ defined above has a linear span which is dense in $C(E)$. ∎

4 Uniqueness for the Smooth Case

In this section, we want to prove that the $(\mathcal{A}+\mathcal{B}, \delta_\mu)$-MP is well posed by the duality method.

Theorem 4.1 *If $\rho(\cdot)$ defined by (1.4) is a bounded and smooth function, then the $(\mathcal{A}+\mathcal{B}, \delta_\mu)$-MP is well posed and the solution to the $(\mathcal{A}+\mathcal{B}, \delta_\mu)$-MP is a strong Markov process.*

To prove the above theorem, we need the following lemma which is discussed in Dawson-Kurtz([7]). We omit its proof here.

Lemma 4.2 *Given $A \subset B(E) \times B(E)$, suppose the (A, δ_μ)-MP has at least one solution. Assume that any solution to the (A, δ_μ)-MP satisfies both the conditions*

(i) *for $t_0 > 0$ and $0 \le t \le t_0, f \in \tilde{D}$ ($\tilde{D}$ is defined in Section 2) and*

$$F_f(\nu) := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \cdots, x_{N(f)}) \nu^{\otimes N(f)}(d\bar{x}),$$

there is a function $\aleph_{f,t}(\cdot)$ which only depends on f and t and is a measurable function on $E := M_F(\mathbb{R})$, bounded on $E_\xi := \{\mu \in E| < 1, \mu > \le \xi\}$ for $0 < \xi < \infty$, and

$$\mathbb{E}^{\mathbb{P}_\mu}(F_f(\mu_t)) = \aleph_{f,t}(\mu)$$

holds for any solution $\mathbb{P}_\mu$ to the (A, δ_μ)-MP;

(ii) *for $t_0 > 0$ and $0 \le t \le t_0$, the moment problem for $< 1, \mu(t) >$ with distribution determined by $\{\mathbb{P}_\mu : \mu \in E\}$ has a unique solution.*

Then the (A, δ_μ)-MP is well posed. The solution is measurable and satisfies the strong Markov property.

Proof of the Theorem 4.1 : Let $\mathbb{P}_\mu$ be a solution to $(\mathcal{A}+\mathcal{B}, \delta_\mu)$-MP on $D([0,\infty), E)$ and $\{\mu_t\}$ be the canonical process. Let $\tau_\xi := inf\{t : \mu_t \notin E_\xi\}$. For $F_f(\nu) := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \cdots, x_{N(f)}) \nu^{\otimes N(f)}(d\bar{x})$, let

$$\text{(4.47)} \quad \mathcal{L}^* F_\mu(f) := F_\mu(G^{N(f)} f) + \frac{1}{2}\gamma(m_2 - 1) \sum_{j,k=1; j \ne k}^{N(f)} (F_\mu(\Phi_{j,k} f) - F_\mu(f)),$$

where $G^{N(f)}$ is defined by (2.19), $\Phi_{j,k}$ is defined by (3.43), and

$$\text{(4.48)} \qquad \mathcal{L}^{\#} F_\mu(f) := \mathcal{L}^* F_\mu(f) + \frac{1}{2}\gamma(m_2 - 1) N(f)(N(f) - 1) F_\mu(f).$$

Then $\mathcal{L}^*$ has the structure of the infinitesimal generator of a $\tilde{D}$-valued Markov process restricted to functions in $Q(\tilde{D})$. The dynamic of the $\tilde{D}$-valued Markov process $\{Y(t) : t \ge 0\}$, defined on probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, with generator $\mathcal{L}^*$ involves two basic mechanisms:

(m1) (Jump mechanism) The function valued dual process $Y(t)$ jumps $f \to \Phi_{j,k} f$ with a rate of $\frac{1}{2}\gamma(m_2 - 1)$, where the pair (j, k), satisfying $j \ne k$, $j, k = 1, \cdots, N(f)$, is randomly chosen with equal probability.

(m2) (Spatial diffusion semigroup) In between jumps , $Y(t)$ evolves in a continuous and deterministic way; namely $f \to S_t^{N(f)} f$.

A direct calculation yields $\mathcal{L}F_f(\mu) = \mathcal{L}^{\#}F_\mu(f)$. Let

$$\aleph_{f,t}(\mu) := \mathbb{E}_1[< Y(t), \mu^{N(Y(t))} > exp(\frac{1}{2}\gamma(m_2-1)\int_0^t N(Y(u))(N(Y(u))-1)du)],$$

where function $N(\cdot)$ is defined in Section 2. Then by the Feynman-Kac formula (see ([11] p.189, Proposition 4.7 and p.184, Theorem 4.2), we have

$$\mathbb{E}_\mu[F_f(\mu_{t\wedge\tau_\xi})1_{(<1,\mu_t>\le\xi)}] = \mathbb{E}_\mu[F_f(\mu_t)1_{(<1,\mu_t>\le\xi)}] \tag{4.49}$$

$$= \aleph_{f,t}(\mu)\mathbb{E}_\mu[1_{(<1,\mu_t>\le\xi)}].$$

Since $\mathbb{E}_\mu(< 1, \mu_t >) < \infty$, $\lim_{\xi\to\infty}\mathbb{P}_\mu(\tau_\xi \le t) = 0$. Note that G^N is conservative and $\mathcal{L}F_{1_{(N)}}(\mu) \ge 0$ for $F_{1_{(N)}}(\mu) \in \Pi_1^D$ and $1_{(N)}$ is the function of N variables which assumes the constant value 1. By Lemma 3.2, $(< 1, \mu_t >)^N$ is a submartingale for each $N \ge 1$. Therefore,

$$\lim_{\xi\to\infty}\mathbb{E}_\mu(< 1, \mu_{t\wedge\tau_\xi} >^N 1_{\{\tau_\xi\le t\}})$$

$$\le \lim_{\xi\to\infty}\mathbb{E}_\mu(< 1, \mu_t >^N 1_{\{\tau_\xi\le t\}}) = 0.$$

Then letting $\xi \to \infty$ in (4.49), we obtain

$$\mathbb{E}_\mu[F_f(\mu_t)] = \aleph_{f,t}(\mu). \tag{4.50}$$

Since $\aleph_{f,t}(\cdot)$ is a measurable function which is bounded on the sets E_ξ, it remains to verify that the moment problem for $< 1, \mu_t >$ has a unique solution. Let $Q(t)$ be the continuous state branching process constructed in Section 3 and

$$\psi_t(\lambda) = \lambda[1 + \frac{1}{2}(m_2-1)\gamma t\lambda]^{-1}.$$

From the proof of Lemma 3.2 and $Q(t) =< 1, \mu_t >$, it is easy to conclude that the moment power series is

$$\sum_k \alpha_k \frac{\lambda^k}{k!} = exp(x\psi_t(\lambda)).$$

It follows from Billingsley ([4] p.342, Theorem 30.1) that the moment problem for $< 1, \mu_t >$ is well-posed. This completes the proof of Theorem 4.1. ■

5 Existence for the Singular Case

In the previous sections, we have constructed and characterized the limiting measure-valued processes of a sequence of interacting branching particle systems with smooth function $g(\cdot)$. In the following sections, we will consider the same model and questions as before; however, $g(\cdot)$ may be a singular function. Here we assume that $\rho(\cdot)$ is continuous and locally Lipschitz continuous away from 0 which means that $\rho(\cdot)$ is Lipschitz outside each interval $(-c, c), c > 0$. From Section 2, we know $\rho(\cdot)$ is nonnegative definite. In the following sections, our operators $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{B}$ are formally defined same as (1.7), (1.9), and (1.8), respectively except that the assumption for the coefficient $\rho(\cdot)$ is replaced by assuming that $\rho(\cdot)$ is continuous and locally Lipschitz continuous away from 0. If $F_f(\mu) \in \Pi_1^P(E)$, then

$$\mathcal{A}F_f(\mu) = F_{G^N f}(\mu), \tag{5.51}$$

where G^N is defined by (2.19).

First let us show how to approximate $\rho(\cdot)$ uniformly by smooth functions.

Definition 5.1 *Let*

$$\alpha(x) := \begin{cases} \frac{1}{c} exp(\frac{1}{|x|^2 - 1}) & |x| < 1 \\ 0 & |x| \geq 1, \end{cases} \tag{5.52}$$

where c is a constant such that $\int_{\mathbb{R}} \alpha(x)dx = 1$ *and*

$$\alpha_\varepsilon(x) := \varepsilon^{-1}\alpha(\frac{x}{\varepsilon}). \tag{5.53}$$

For any locally integrable function u on $\mathbb{R}$, *define*

$$J_\varepsilon u(x) := \int_{\mathbb{R}} u(x-y)\alpha_\varepsilon(y)dy.$$

$J_\varepsilon u$ *is called mollifier of u.*

Lemma 5.1 *Suppose $g(\cdot)$ satisfies the conditions in (1.2) and $\rho(\cdot)$, defined by (1.4), is continuous and locally Lipschitz continuous away from 0. Define $g_k(x) := J_{1/k}g(x)$ on $\mathbb{R}$ and $\rho_k(z) := \int_{\mathbb{R}} g_k(z-y)g_k(y)dy$. Then the functions $\{\rho_k(x)\}$ are bounded and smooth, while the functions $\{g_k(x)\}$ are bounded and symmetric. Also there holds*

$$\rho_k \to \rho \qquad \textit{uniformly as} \qquad k \to \infty.$$

Proof: The smoothness of $\rho_k \in C^\infty$ is a classical result — see Barros-Neto ([3] p.6). The boundedness of both ρ_k and g_k comes straight out of the Cauchy-Schwarz inequality. Since there holds

$$g_k(-x) = \int_{|y|<1} g(-x-y)\alpha_{1/k}(y)dy = g_k(x), \tag{5.54}$$

g_k is also symmetric. The uniform convergence follows from the following inequalities:

$$|\rho_k(x) - \rho(x)| \le |\int_{\mathbb{R}} g_k^2(x-y)dy|^{1/2}|\int_{\mathbb{R}} |g_k(y) - g(y)|^2 dy|^{1/2} \tag{5.55}$$

$$+[\int_{\mathbb{R}} g^2(y)dy]^{1/2}[\int_{\mathbb{R}} |g_k(x-y) - g(x-y)|^2 dy]^{1/2}.$$

■

If in (1.7) we substitute ρ_n for ρ, the resulting operator is denoted by $\mathcal{L}_n F(\mu)$.

Theorem 5.2 *If $\rho(\cdot)$ defined by (1.4) is continuous and locally Lipschitz continuous away from 0, then for any finite measure μ_0 with compact support, the $(\mathcal{L}, \delta_{\mu_0})$-MP has at least one solution.*

Proof: By Theorem 3.5 and Theorem 4.1 , the $(\mathcal{L}_n, \delta_{\mu_0})$-MP has a unique solution. Since the present branching mechanism is the same as in the previous sections, a similar argument to the proofs of Lemma 3.4 and Theorem 3.5 can reach the conclusion. ■

6 Uniqueness for the Singular Case

In this section, we want to prove that the $(\mathcal{L}, \delta_{\mu_0})$-MP has at most one solution. Recall $\tilde{C} := \cup_{N=0}^{\infty} C(\bar{\mathbb{R}}^N)$ (disjoint union), $\tilde{D} := \cup_{N=0}^{\infty} D(\bar{\mathbb{R}}^N))$ (disjoint union).

We define $H : \tilde{D} \times E \to \mathbb{R}$ by $H(f,\mu) = < f, \mu^m >$ for $f \in D(\bar{\mathbb{R}}^m)$, where we have simplied the notation $\mu^{\otimes m}$ by μ^m. Then

$$\mathcal{L}H(f,\mu) = < G^m f, \mu^m > + \tfrac{1}{2}\gamma(m_2 - 1)\textstyle\sum_{j,k=1,j\ne k}^m < \Phi_{jk} f, \mu^{m-1} > \tag{6.56}$$

$$= < G^m f, \mu^m > + \tfrac{1}{2}\gamma(m_2 - 1)\textstyle\sum_{j,k=1,j\ne k}^m \{< \Phi_{jk} f, \mu^{m-1} >$$

$$- < f, \mu^m >\} + \tfrac{1}{2}\gamma(m_2 - 1)m(m-1) < f, \mu^m >,$$

where G^m is defined by (2.19) and $\Phi_{j,k}$ is defined by (3.43).

In the following, we want to use Lemma 4.2 to prove the uniqueness. Since the branching mechanism did not change, if we can prove that condition (i) of Lemma 4.2 holds, then condition (ii) of the Lemma 4.2 will hold automatically. To prove condition (i) of Lemma 4.2, the key point is to construct a generalized dual process for the measure-valued processes, solutions to the $(\mathcal{L}, \delta_{\mu_0})$-MP, such that the duality identity holds. In order to construct the generalized dual processes, we need to present C-diffusions (coalescing diffusions) which were first introduced by T.E.Harris [14] for constructing Brownian flow. Let $\Omega = \Omega_d = C([0,\infty), \mathbb{R}^d)$ be the space of continuous mapping $\omega : [0,\infty) \to \mathbb{R}^d$ with the topology of uniform convergence on compact sets. Putting $\xi_t = \omega(t) = \omega_t$, let $\mathcal{F}_t^d = \sigma(\xi_s, 0 \le s \le t)$, $\mathcal{F}_\infty^d = \vee_t \mathcal{F}_t^d$ and

$$\bar{G}^d f(x) = \frac{1}{2}\sum_{i,j=1}^{d} b_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \beta_i(x)\frac{\partial f(x)}{\partial x_i}, \tag{6.57}$$

$$x = (x_1, \cdots, x_d), \text{ for } f \in C_b^\infty(\mathbb{R}^d),$$

where $B = (b_{ij})$ is nonnegative definite, while both B and $\beta = (\beta_1, \cdots, \beta_d)$ are bounded and continuous.

Definition 6.1 *A probability measure $\mathbb{P}_x$ on $\mathcal{F}_\infty^d$, governing a process $\{\xi_t : t \ge 0\}$ in $\mathbb{R}^d$, is a solution to the martingale problem for $\bar{G}^d$ from $x \in \mathbb{R}^d$ if*
$\mathbb{P}_x(\xi_0 = x) = 1$ and

$$f(\xi_t) - \int_0^t \bar{G}^d f(\xi_s)ds$$

is a $(\mathbb{P}_x, \mathcal{F}_t^d)$-martingale for every $f \in C_b^\infty(\mathbb{R}^d)$.

Definition 6.2 *Define a shift operator $S_t : \Omega_d \to \Omega_d$ by*

$$(S_t\omega)(s) = \omega(t+s), \quad \textit{for } s, t \ge 0.$$

By a C-diffusion, we mean a probability measure $\mathbb{P}$ on $\mathcal{F}_\infty^d$ which solves the martingale problem for $\bar{G}^d$ and for which there is a $C \in \mathcal{F}_\infty^d$ such that $S_t C \subset C$ and $\mathbb{P}(C) = 1$. Henceforth C is specialized to be the set of ω in Ω_d such that if $\omega_i(t) = \omega_j(t)$ for some $i \ne j$ and $t \ge 0$, then $\omega_i(t+s) = \omega_j(t+s)$ for all $s \ge 0$.

Let

$$D_k := \{(x_1, \cdots, x_k) : (x_1, \cdots, x_k) \in \mathbb{R}^k, \quad x_i \ne x_j \text{ for } i \ne j\} \text{ and } H_k := \mathbb{R}^k \backslash D_k .$$

Consider the following conditions on a real-valued function $b(\cdot)$:

(c.1) $b(0) = a > 0$, a is a constant,

(c.2) $b(x)$ is continuous on $\mathbb{R}$ and locally Lipschitz continuous away from 0,

(c.3) For every $k \geq 2$ and $(x_1, \cdots, x_k) \in D_k$, matrix $(b(x_i - x_j))$ is strictly positive definite. Define

$$G^d f(x) = \frac{1}{2} \sum_{i,j=1}^{d} b_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \quad \text{for } f \in C_b^\infty(\mathbb{R}^d), \tag{6.58}$$

where $x = (x_1, \cdots, x_d)$ and $b_{ij}(x) := b(x_i - x_j)$.

Theorem 6.1 *If $b(\cdot)$ satisfies conditions (c.1) to (c.3), then for each $x \in \mathbb{R}^d$, there is a unique solution, C-diffusion $\mathbb{P}_x$, to the MP for G^d from x. If $A \in \mathcal{F}_\infty^d$, $\mathbb{P}_x(A)$ is measurable in x. The family $\{\mathbb{P}_x : x \in \mathbb{R}^d\}$ is strong Markov and Feller.*

Proof: see T.E.Harris [14] p.191, Lemma(3.2). ■

If $\rho(\cdot)$ defined by (1.4) is continuous and locally Lipschitz continuous away from 0, then $\rho(\cdot)$ satisfies all the conditions of Theorem 6.1. By Theorem 6.1, suppose $\{\mathbb{P}_z\}$ is the C-diffusion governed by G^n and suppose $\xi(t) = w(t)$ is the canonical process in the space $C([0, \infty), \mathbb{R}^n)$. Define

$$T_n(t) f(z) = \mathbb{E}^{\mathbb{P}_z} f(\xi(t)) \qquad \text{for} \quad f \in C(\bar{\mathbb{R}}^n), \tag{6.59}$$

we have the following lemma.

Lemma 6.2 *If $\rho(\cdot)$ defined by (1.4) is continuous and locally Lipschitz continuous away from 0, then $\{T_n(t)\}$ defined by (6.59) is a strongly continuous non-negative contraction semigroup on $C(\bar{\mathbb{R}}^n)$. Let A^n be its infinitesimal generator, then $A^n|_{C^2(\bar{\mathbb{R}}^n)} = G^n$.*

Proof: The proof is elementary. ■

Remark $T_n(t) : C(\bar{\mathbb{R}}^n) \to C(\bar{\mathbb{R}}^n)$ because $\mathbb{P}_z$ is a Feller process.

Lemma 6.3 *Suppose $\rho(\cdot)$ defined by (1.4) is continuous and locally Lipschitz continuous away from 0. If $\{\mu_t\}$ is a solution to the $(\mathcal{L}, \delta_{\mu_0})$-MP on a probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, and if we define filtration*

$$\mathcal{F}_t^\mu := \sigma(\mu(s), s \leq t) \vee \sigma\left(\int_0^s h(\mu(u))du, s \leq t, h \in B(E)\right),$$

where $B(E)$ is the space of all bounded measurable functions on E; then

$$< f, \mu_t^n > - \int_0^t [< A^n f, \mu_u^n > + \sigma \sum_{i,j=1, i \neq j}^{n} < \Phi_{ij} f, \mu_u^{n-1} >] du$$

$$\textit{for any} \quad f \in C(\bar{\mathbb{R}}^n) \cap \mathcal{D}(A^n)$$

is a $\mathcal{F}_t^\mu$*-martingale, where* $\sigma := \frac{1}{2}\gamma(m_2 - 1)$.

Proof: Since $A^n|_{C^2(\bar{\mathbb{R}}^n)} = G^n$ and $\{\mu_t\}$ is a solution to the $(\mathcal{L}, \delta_{\mu_0})$-MP, we have

$$\mathbb{E}^{\mathbb{P}_2}\{< f, \mu_t^n > - \int_0^t [< A^n f, \mu_u^n > + \sigma \sum_{i,j, i \neq j}^{n} < \Phi_{ij} f, \mu_u^{n-1} >] du$$

(6.60)

$$- < f, \mu_s^n > + \int_0^s [< A^n f, \mu_u^n > + \sigma \sum_{i,j, i \neq j}^{n} < \Phi_{ij} f, \mu_u^{n-1} >] du | \mathcal{F}_s^\mu\} = 0$$

$$\text{for} \quad t \geq s \geq 0 \quad \text{and for any} \quad f \in D(\bar{\mathbb{R}}^n).$$

For any $g \in C(\bar{\mathbb{R}}^n) \cap \mathcal{D}(A^n)$, there exist a sequence of smooth functions $g_n \in D(\bar{\mathbb{R}}^n)$ such that g_n uniformly converges to g on $\bar{\mathbb{R}}^n$. Since A^n is the generator of the semigroup $\{T_n(t)\}$, by the Hille-Yosida Theorem, A^n is a closed linear operator (refer to Ethier-Kurtz [11] p.8 for definition). Therefore, $A^n g_k \to A^n g$ in $C(\bar{\mathbb{R}}^n)$ as $k \to \infty$. In (6.60), replace f by g_k and let $k \to \infty$, we reach the conclusion. ■

Now we turn to the construction of the dual process. Let $\text{M} = \{M(t) : \ t \geq 0\}$ be a pure jump Markov process on a probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ with state space $\mathbb{N}$ and transition intensities $q_{m,m-1} = \sigma m(m-1), q_{i,j} = 0$ for all other pairs (i,j). Let $\{\tau_k\}$ be the sequence of jump times of M (take $\tau_0 = 0$) and $\{\Gamma_k\}$ be a sequence of random operators which are conditionally independent given M and satisfy

$$\text{(6.61)} \qquad \mathbb{P}_1(\Gamma_k = \Phi_{i,j} | M) = \frac{1}{m(m-1)} 1_{\{M(\tau_k -) = m, M(\tau_k) = m-1\}}$$

for $1 \leq i, j \leq m$ and $j \neq i$, where $\Phi_{i,j}$ is defined as in (3.43), then the dual process is given by

$$\text{(6.62)} \qquad Y(t) = T_{M(\tau_k)}(t - \tau_k)\Gamma_k T_{M(\tau_{k-1})}(\tau_k - \tau_{k-1})\Gamma_{k-1} \cdots \Gamma_1 T_{M(0)}(\tau_1) Y(0)$$

$$\text{for} \quad \tau_k \leq t < \tau_{k+1} \qquad k = 0, 1, 2, \cdots,$$

where $Y(0) \in D(\bar{\mathbb{R}}^{M(0)})$.

For any $f \in D(\bar{\mathbb{R}}^m)$, take $Y(0) = f, M(0) = m$, let $\{\mu_t\}$ be a solution to the $(\mathcal{L}, \delta_{\mu_0})$-MP defined on $(\Omega_2, \mathcal{F}_2, \mathcal{F}_t^\mu, \mathbb{P}_2)$ and let $\mathbb{E} = \mathbb{E}^{\mathbb{P}_1 \times \mathbb{P}_2}$ be the expectation on

$(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$. If the following duality identity

$$\mathbb{E}[< f, \mu_t^m >] = \mathbb{E}[< Y(t), \mu_0^{M(t)} > exp\{\sigma \int_0^t M(u)(M(u)-1)du\}] \tag{6.63}$$

holds for any solution $\{\mu_t\}$ to the $(\mathcal{L}, \delta_{\mu_0})$-MP, then the $(\mathcal{L}, \delta_{\mu_0})$-MP is well-posed by Lemma 4.2.

Theorem 6.4 *If $\rho(\cdot)$ defined by (1.4) is continuous and locally Lipschitz continuous away from 0, then for any finite measure μ_0 with compact support, the $(\mathcal{L}, \delta_{\mu_0})$-MP has a unique solution.*

Proof: We only need to establish the identity (6.63). Writing the difference of the two sides of (6.63) as a telescoping sum, we only need to show that each term of the telescoping sum is $o(h)$ and this is equivalent to showing

$$\begin{aligned} &\mathbb{E}\{[< Y(s+h), \mu_{t-s-h}^{M(s+h)} > exp\{\sigma \int_0^{s+h} M(u)(M(u)-1)du\}] \\ &-[< Y(s), \mu_{t-s}^{M(s)} > exp\{\sigma \int_0^s M(u)(M(u)-1)du\}]\} = o(h). \end{aligned} \tag{6.64}$$

Let

$$\mathcal{F}_t^Y := \sigma(Y(s), s \le t) \vee \sigma(\int_0^s h(Y(u))du, s \le t, h \in B(\tilde{D})),$$

where $B(\tilde{D})$ is the space of all bounded measurable functions on $\tilde{D}$. Using the definition of Y and some basic results of Q-process M, we can decompose the first term in (6.64) in terms of the values of $M(s+h) - M(s)$ to obtain

$$\mathbb{E}\left\{\mathbb{E}[< Y(s+h), \mu_{t-s-h}^{M(s+h)} > exp\{\sigma \int_0^{s+h} M(u)(M(u)-1)du\} | \mathcal{F}_s^Y]\right\}$$

$$= \mathbb{E}\left[< T_{M(s)}(h)Y(s), \mu_{t-s-h}^{M(s)} > exp\{\sigma \int_0^s M(u)(M(u)-1)du\}\right]$$

$$+\mathbb{E}\left\{\textstyle\int_0^h \sum_{i \ne j} < T_{M(s)-1}(h-r)\Phi_{ij}T_{M(s)}(r)Y(s), \mu_{t-s-h}^{M(s)-1} > \cdot\right.$$

$$exp\{\sigma \int_0^s M(u)(M(u)-1)du + \sigma M(s)(M(s)-1)r$$

$$+\sigma(M(s)-1)(M(s)-2)(h-r) - \sigma M(s)(M(s)-1)r\}\sigma dr\} + o(h)$$

$$= \mathbb{E}[< T_{M(s)}(h)Y(s), \mu_{t-s-h}^{M(s)} > exp\{\int_0^s M(u)(M(u)-1)du]$$

$$(6.65)\qquad +\mathbb{E}[\int_0^h \sigma \sum_{i\neq j} exp\{\sigma(M(s)-1)(M(s)-2)(h-r)+$$

$$\sigma\int_0^s M(u)(M(u)-1)du\} < T_{M(s)-1}(h-r)\Phi_{ij}T_{M(s)}(r)Y(s), \mu_{t-s-h}^{M(s)-1} > dr] + o(h).$$

Using Lemma 6.3, we have

Lemma 6.5 *For above $\{T_m(u)\}$, and any positive t, s provided $t \geq s$,*

$$(6.66)\qquad < T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)} >$$

$$-\int_0^v \sigma \sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij}T_{M(s)}(t-s-l)Y(s), \mu_l^{M(s)-1} > dl$$

is a $\{\mathcal{F}_v^\mu\}$-martingale for every $\omega_1 \in \Omega_1$ and $0 \leq v \leq t-s$ on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, where $\{\mathcal{F}_v^\mu\}$ is defined in Lemma 6.3.

Proof of Lemma 6.5: Our conclusion is equivalent to showing that:

for $t-s \geq u \geq v \geq 0$,

$$\mathbb{E}^{\mathbb{P}_2}\Big[< T_{M(s)}(t-s-u)Y(s), \mu_u^{M(s)} >$$

$$-\int_0^u \sigma \sum_{i,j=1,i\neq j}^{M(s)} [< \Phi_{ij}T_{M(s)}(t-s-l)Y(s), \mu_l^{M(s)-1} >]dl$$

$$- < T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)} >$$

$$+\int_0^v \sigma \sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij}T_{M(s)}(t-s-l)Y(s), \mu_l^{M(s)-1} > dl|\mathcal{F}_v^\mu\Big] = 0.$$

Above equality holds if for any $h > 0$ satisfying $t-s \geq v+h$, we have

$$\mathbb{E}^{\mathbb{P}_2}\Big[< T_{M(s)}(t-s-v-h)Y(s), \mu_{v+h}^{M(s)} >$$

$$-\int_0^{v+h} \sigma \sum_{i,j,i\neq j}^{M(s)} [< \Phi_{ij}T_{M(s)}(t-s-r)Y(s), \mu_r^{M(s)-1} > dr$$

$$- < T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)} >$$

$$+\int_0^v \sigma \sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij}T_{M(s)}(t-s-r)Y(s), \mu_r^{M(s)-1} > dr|\tilde{\mathcal{F}}_v^\mu\Big] = o(h)$$

or

$$\lim_{h\to 0}\frac{1}{h}\Big\{\mathbb{E}^{\mathbb{P}_2}\Big[(< T_{M(s)}(t-s-v-h)Y(s),\mu^{M(s)}_{v+h} >$$

(6.67)

$$- < T_{M(s)}(t-s-v)Y(s),\mu^{M(s)}_{v+h} >)$$

$$+ < T_{M(s)}(t-s-v)Y(s),\mu^{M(s)}_{v+h} > -$$

$$< T_{M(s)}(t-s-v)Y(s),\mu^{M(s)}_{v} > -$$

$$\int_v^{v+h}\sigma\sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij}T_{M(s)}(t-s-r)Y(s),\ \mu_r^{M(s)-1} > dr|\mathcal{F}_v^\mu\Big]\Big\} = 0.$$

By the dominated convergence theorem and Lemma 6.3, (6.67) is equivalent to

$$\mathbb{E}^{P_2}\Big\{- < A^{M(s)}T_{M(s)}(t-s-v)Y(s),\mu^{M(s)}_v >$$

$$+ < A^{M(s)}T_{M(s)}(t-s-v)Y(s),\mu^{M(s)}_v >$$

(6.68)

$$+\sigma\textstyle\sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij}T_{M(s)}(t-s-v)Y(s),\mu^{M(s)-1}_v >$$

$$-\sigma\textstyle\sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij}T_{M(s)}(t-s-v)Y(s),\mu^{M(s)-1}_v >\Big\} = 0.$$

(6.68) is obvious, So Lemma 6.5 is proved.

Now we return to prove the Theorem 6.4. From Lemma 6.5, the second term on left hand side of (6.64) can be decomposed as

(6.69)

$$\mathbb{E}\Big[< Y(s),\mu^{M(s)}_{t-s} > exp\left\{\sigma\textstyle\int_0^s M(u)(M(u)-1)du\right\}\Big]$$

$$= \mathbb{E}\Big\{\Big[< T_{M(s)}(h)Y(s),\mu^{M(s)}_{t-s-h} >$$

$$+\int_0^h\sigma\sum_{i,j,i\neq j}^{M(s)} < \Phi_{ij}T_{M(s)}(r)Y(s),\ \mu^{M(s)-1}_{t-s-r} > dr\Big]$$

$$\cdot exp\left[\sigma\int_0^s M(u)(M(u)-1)du\right]\Big\}.$$

Indeed, by Lemma 6.5, we have

$$\mathbb{E}\Big\{\Big[< T_{M(s)}(t-s-v-h)Y(s),\mu^{M(s)}_{v+h} >$$

$$- < T_{M(s)}(t-s-v)Y(s), \mu_v^{M(s)} > \tag{6.70}$$

$$-\int_v^{v+h} \sigma \sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(t-s-r)Y(s),\ \mu_r^{M(s)-1} > dr\Big] |\mathcal{F}_v^{\mu}\Big\} = 0.$$

Let $h = t - s - v$, then above equality is equivalent to

$$\mathbb{E}\Big\{\Big[< T_{M(s)}(0)Y(s), \mu_{t-s}^{M(s)} >$$

$$- < T_{M(s)}(h)Y(s), \mu_{t-s-h}^{M(s)} > \tag{6.71}$$

$$-\int_{t-s-h}^{t-s} \sigma \sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(t-s-r)Y(s), \mu_r^{M(s)-1} > dr\Big] |\mathcal{F}_{t-s-h}^{\mu}\Big\} = 0$$

or

$$\mathbb{E}\Big\{\Big[< Y(s), \mu_{t-s}^{M(s)} > - < T_{M(s)}(h)Y(s), \mu_{t-s-h}^{M(s)} >$$

$$-\int_0^h \sigma \sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(u)Y(s),\ \mu_{t-s-u}^{M(s)-1} > du\Big] |\mathcal{F}_{t-s-h}^{\mu}\Big\} = 0, \tag{6.72}$$

which implies (6.69).

Comparing (6.64),(6.65) and (6.69), we see that (6.64) is equivalent to

$$\frac{1}{h}\mathbb{E}\Big\{\int_0^h \Big[\sigma \sum_{i,j=1,i\neq j}^{M(s)} < T_{M(s)-1}(h-r)\Phi_{ij} T_{M(s)}(r)Y(s), \mu_{t-s-h}^{M(s)-1} > \cdot \tag{6.73}$$

$$exp\{\sigma \int_0^s M(u)(M(u)-1)du + \sigma(M(s)-1)(M(s)-2)(h-r)\}\Big] dr$$

$$-\int_0^h \sigma \sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(r)Y(s), \mu_{t-s-r}^{M(s)-1} >$$

$$\cdot exp\Big[\sigma \int_0^s M(u)(M(u)-1)du\Big] dr\Big\} \to 0 \quad \text{as} \quad h \to 0$$

or

$$\frac{1}{h}\mathbb{E}\Big\{\int_0^h \Big[\sigma \sum_{i,j=1,i\neq j}^{M(s)} < T_{M(s)-1}(h-r)\Phi_{ij} T_{M(s)}(r)Y(s), \mu_{t-s-h}^{M(s)-1} > \tag{6.74}$$

$$-\sigma \sum_{i,j=1,i\neq j}^{M(s)} < \Phi_{ij} T_{M(s)}(r)Y(s), \mu_{t-s-r}^{M(s)-1} >\Big]$$

$$\cdot exp\Big[\sigma \int^s M(u)(M(u)-1)du\Big] dr\Big\} \to 0 \quad \text{as} \quad h \to 0.$$

From Lemma 6.5, similar to the proof of (6.69), we can get

$$\mathbb{E}\left\{\left[< T_{M(s)-1}(v)\Phi_{ij}T_{M(s)}(r)Y(s), \mu_{t-s-v-r}^{M(s)-1} > \right.\right.$$

$$(6.75) \qquad \left.\left. - < \Phi_{ij}T_{M(s)}(r)Y(s), \mu_{t-s-r}^{M(s)-1} > \right] |\mathcal{F}_{t-s-v-r}^{\mu}\right\} =$$

$$\mathbb{E}\left\{-\left[\int_0^v \sigma \sum_{k,l,k\neq l}^{M(s)-1} < \Phi_{kl}T_{M(s)-1}(u)\Phi_{ij}T_{M(s)}(r)Y(s), \mu_{t-s-r-u}^{M(s)-2} > du\right] |\mathcal{F}_{t-s-v-r}^{\mu}\right\}.$$

From (6.75), we see that (6.74) holds. So does (6.64) and we reach the conclusion of Theorem 6.4. ■

7 Continuity

In this Section, we want to prove that if $\rho(\cdot)$ defined by (1.4) is continuous and locally Lipschitz continuous away from 0, then the sample paths of $\{\mu_t; t \geq 0\}$, the solution to the $(\mathcal{L}, \delta_{\mu_0})$-MP, are continuous. This will prove that $\{\mu_t; t \geq 0\}$ is a diffusion process. Let Δ and ∇ denote Laplacian and gradient operators on $\mathbb{R}$, respectively. Let μ_0 be a finite measure on $\mathbb{R}$ with compact support. $\mathbb{P}_{\mu_0}$ denotes the solution to the $(\mathcal{L}, \delta_{\mu_0})$-MP with canonical process μ_t on the space $D([0,\infty), E)$, where $E = M_F(\bar{\mathbb{R}})$. First, let us cite the following Lemma. Let $D(\mathcal{Q})$ denote the class of functions on E of the following form

$$F(\mu) = \psi(< f_1, \mu >, \cdots, < f_n, \mu >),$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$, $f_i \in C^2(\mathbb{R})$, and $n \geq 1$. Suppose $\mathcal{Q}$ is an operator on space $C(E)$ and for any $\mu \in E$,

$$M_t^F := F(X_t) - F(X_0) - \int_0^t \mathcal{Q}F(X_s)ds \text{ for all } F \in D(\mathcal{Q}) \text{ and } t \geq 0$$

is a locally bounded $\mathbb{P}_\mu$-martingale and is càdlàg a.s. with respect to $\mathbb{P}_\mu$, where X_t is the canonical process on the space $D([0,\infty), E)$. Define

$$\mathcal{R}_\mathcal{Q}(F, G) := \mathcal{Q}(FG) - F\mathcal{Q}(G) - G\mathcal{Q}(F) \text{ for any } F, G \in D(\mathcal{Q}).$$

Lemma 7.1 *Suppose that $\mathcal{R}_\mathcal{Q}$ has the derivation property:*

$$(7.76) \qquad \mathcal{R}_\mathcal{Q}(FG, H) := F\mathcal{R}_\mathcal{Q}(G, H) + G\mathcal{R}_\mathcal{Q}(F, H) \text{ for any } F, G, H \in D(\mathcal{Q}).$$

Then the processes M^F and $F(X.)$ have continuous paths a.s. with respect to $\mathbb{P}_\mu$ for all $F \in D(\mathcal{Q})$.

Proof: see Bakry-Emery ([2] Prop.2) ■

Theorem 7.2 *Suppose $\rho(\cdot)$ defined by (1.4) is continuous and locally Lipschitz continuous away from 0, $\mathcal{L}$ is the generator defined by (1.7). Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_{\mu_0}, \mu_t)$ be the unique solution to the $(\mathcal{L}, \delta_{\mu_0})$-MP. Then $\{\mu_t\}$ is a measure-valued diffusion process.*

Proof: Since

$$\mathcal{R}_{\mathcal{L}}(F,G)(\mu) = \gamma(m_2-1)\int_{\bar{\mathbb{R}}} \frac{\delta F(\mu)}{\delta\mu(x)}\frac{\delta G(\mu)}{\delta\mu(x)}\mu(dx)$$

$$+\int_{\bar{\mathbb{R}}}\int_{\bar{\mathbb{R}}} \rho(x-y)(\frac{d}{dx})(\frac{d}{dy})\frac{\delta F(\mu)}{\delta\mu(x)}\frac{\delta G(\mu)}{\delta\mu(y)}\mu(dx)\mu(dy),$$

$\mathcal{L}$ has the derivation property. Therefore, the continuity of $\{\mu_t; t \geq 0\}$ follows from the Lemma 7.1 and the fact that $C^\infty(\bar{\mathbb{R}})$ is convergence determining. ■

Since μ_t is a measure-valued diffusion process, we have the following corollary.

Corollary 7.3 *The following two conditions are equivalent:*

(1) *For any $\phi \in C_c^2(\mathbb{R})$*

$$M_t^\phi := <\phi, \mu_t> - <\phi, \mu_0> - \frac{1}{2}\int_0^t \rho_\epsilon <\Delta\phi, \mu_s> ds \tag{7.77}$$

is a $\mathbb{P}_{\mu_0}$-martingale with increasing process

$$\gamma(m_2-1)\int_0^t <\phi^2, \mu_s> ds + \int_0^t\int_{\mathbb{R}} <g(y-x)\nabla\phi, \mu_s>^2 dyds. \tag{7.78}$$

(2) *For any $\phi \in C_c^2(\mathbb{R})$ and $G \in C^2(\mathbb{R})$*

$$G(<\phi,\mu_t>) - G(<\phi,\mu_0>) - \int_0^t \mathcal{L}G(<\phi,\mu_s>)ds \tag{7.79}$$

is a $\mathbb{P}_{\mu_0}$-martingale.

Proof: (1)⇒(2). This follows from Ito's formula.

(2)⇒(1). Let $G(x) = x$, then

$$M_t^\phi = <\phi, \mu_t> - <\phi, \mu_0> - \frac{1}{2}\int_0^t \rho_\epsilon <\Delta\phi, \mu_s> ds$$

is a $\mathbb{P}_{\mu_0}$-martingale and

$$<\phi, \mu_t> = M_t^\phi + <\phi, \mu_0> + \frac{1}{2}\int_0^t \rho_\epsilon <\Delta\phi, \mu_s> ds$$

is a semimartingale. For any $G \in C^2(\mathbb{R})$, by Ito's formula again, then

$$G(<\phi,\mu_t>) - G(<\phi,\mu_0>) = \int_0^t G'(<\phi,\mu_s>)dM_s^\phi \tag{7.80}$$

$$+\frac{\rho_\epsilon}{2}\int_0^t G'(<\phi,\mu_s>)<\Delta\phi,\mu_s>ds$$

$$+\frac{1}{2}\int_0^t G''(<\phi,\mu_s>)d<M^\phi>_s,$$

i.e.,

$$\int_0^t G'(<\phi,\mu_s>)dM_s^\phi = G(<\phi,\mu_t>) - G(<\phi,\mu_0>) \tag{7.81}$$

$$-\frac{\rho_\epsilon}{2}\int_0^t G'(<\phi,\mu_s>)<\Delta\phi,\mu_s>ds$$

$$-\frac{1}{2}\int_0^t G''(<\phi,\mu_s>)d<M^\phi>_s$$

is a $\mathbb{P}_{\mu_0}$-martingale.

By uniqueness, comparing (7.79) with (7.81), we obtain

$$<M^\phi>_t = \gamma(m_2-1)\int_0^t <\phi^2,\mu_s>ds + \int_0^t\int_{\mathbb{R}} <g(y-x)\nabla\phi,\mu_s>^2 dyds.$$

So the proof is complete. ■

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