Particle Physics Phenomenology

March 4, 2004
Relativity

- Proton Decay and Virtual Black Holes
- Special Relativity
- General Relativity
- The Equivalence Principle
- Gravitational Redshift and the Bending of Light
- Extra Dimensions and Black Holes
- Coordinates and Metric
- Curved Space-time
- Einstein’s Equations of Gravitation
- Particle Physics and Cosmology

- Reference: Bergstrom and Goobar, Cosmology and Particle Astrophysics
Proton Decay and Virtual Black Holes

- Consider proton decay in 4 dimensions where $M_{\text{pl}} \sim 10^{19}$ GeV
  - Virtual black holes will emerge in the vacuum
  - These virtual black holes live one Planck time $\sim 5 \times 10^{-44}$ sec
  - When two quarks fall into the black hole at the same time, $q+q \rightarrow \bar{q} + l$ processes can occur
  - Prob that 2 quarks are within one Planck length ($10^{-33}$ cm) within the proton is $(m_p/M_{\text{pl}})^3 \sim 10^{-57}$. This is a probability per proton crossing time $\sim 10^{-31}$ yr.
  - An additional factor of $m_p/M_{\text{pl}}$ accounts for the requirement that a virtual black hole be present when the two quarks are near each other.
  - So $\tau_p \sim m_p^{-1} \left( M_{\text{pl}}/m_p \right)^4 \sim 10^{45}$ yr, long compared to the expected proton lifetime $\tau_p \rightarrow$ this is okay

- WHAT ABOUT THE CASE FOR EXTRA DIMENSIONS?
Proton Decay and Virtual Black Holes

- Consider proton decay in 4 + n dimensions where $M_{\text{pl}} \sim 1 \text{ TeV}$
  - Two effects change the rate from the SM
    - The Planck mass is smaller, leading to more virtual BHs
    - The extra dimensions affect the rate if the fermions are allowed to propagate through d additional dimensions from the standard 3
  - Now $\tau_p \sim m_p^{-1} (M_*/m_p)^{4+d}$
    - if $d = 0$ and $M_* \sim 1 \text{ TeV}$, $\tau_p \sim 10^{-19}$ yr $\rightarrow$ fall too short
    - if $d = 7$ and $M_* \sim 1 \text{ TeV}$, $\tau_p \sim 100$ yr $\rightarrow$ still too short

- We know $\tau_p > 10^{33}$ yr
  - Therefore $M_* > 10^{64/(4+d)} \text{ GeV}$
  - For $d = 7$, $M_* > 700 \text{ TeV}$$^1$ hep-ph/0009154
Special Relativity

Postulates of Special Relativity

• The Laws of Physics take the same form in all inertial frames
• The velocity of light in vacuum is a universal constant, which has the same value in all inertial frames

• The line element is an invariant

\[ ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \]

- \( ds^2 > 0 \)  time-like separations
- \( ds^2 < 0 \)  space-like separations
- \( ds^2 = 0 \)  light-like separations

Physics 662, lecture 16
Special Relativity

• **Contravariant four-vectors**
  \[ x^\mu = (ct, \mathbf{r}) = (ct, x^i) \]

• **Covariant four-vectors**
  \[ x_\mu = \eta_{\mu\nu} x^\nu \]
  \[ \eta_{\mu\nu} = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  0 & 0 & 0 & -1
\end{pmatrix} \]

• **Lorentz Transformation**
  \[ x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \]
  \[ \Lambda(x^1; v/c) = \begin{pmatrix}
  \gamma & -\beta\gamma & 0 & 0 \\
  -\beta\gamma & \gamma & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix} \]

• **Lorentz scalar (invariant)**
  \[ S = A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu = A_\mu B^\mu \]
Special Relativity

• Electromagnetism is built from the four-potential
  \[ A^\mu = (\phi, A) \]
  and the current four-vector
  \[ j^\mu = (\rho, j) \]
  the fields are components of an antisymmetric second-rank tensor
  \[ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \]
  \[ F^{\mu\nu} = \begin{pmatrix}
  0 & E_x & E_y & E_z \\
  -E_x & 0 & B_z & -B_y \\
  -E_y & -B_z & 0 & B_x \\
  -E_z & B_y & -B_x & 0
\end{pmatrix} \]

• Maxwell’s equations are
  \[ \partial_\mu F^{\mu\nu} = j^\nu \]
  \[ \partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \]
General Relativity

• Non-inertial reference frames

• Gravitation
  - geometrical interpretation of gravity
**Equivalence Principle**

- Could an observer on board a window-less rocket far from sources of gravitation tell whether the rocket is accelerating at a rate $g$, or sitting on the Earth’s surface?

- **Weak equivalence principle:** gravitational and inertial masses are equal

\[ a = \frac{m_g}{m_i} g \]
Equivalence Principle

- **Strong equivalence principle**: the results of all local experiments in a freely falling frame are independent of the state of motion, and the same in all such frames. The results in freely falling frames are consistent with the special theory of relativity.

- The **strong equivalence principle** has two consequences in general relativity:
  - time moves slower in the presence of a gravitational field
  - a straight light-ray is bent by a gravitating body near its path
Gravitational Redshift

Rocket in free-fall in Earth’s gravitational field

- Clock within the freely falling rocket will move more slowly than a clock at rest
Pound Rebka Experiment

22.6 m high tower

clock difference should be $2.46 \times 10^{-15}$
(1 second in $10^8$ years)

Pound and Rebka found $2.57 \pm 0.26 \times 10^{-15}$
by employing the Mossbauer effect
Bending of Light

- 1919 Eddington mounts two expeditions during Solar eclipse
  - Principe (West coast of Africa)
  - Northern Brazil
- Einstein’s prediction confirmed
- BIG NEWS
  - (New York Times)
Gravitational Lenses

- Double quasar discovered in 1979
Gravitational Arcs

- Photos from Hubble Space Telescope
Coordinates and Metric

Coordinates label points in a manifold (eg. plane, or spherical surface)

The distance between points requires us to know the metric, $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} d\xi^\mu d\xi^\nu$$

Every isotropic, homogeneous three-space can be parametrized with coordinates giving the metric equation

$$ds^2 = \alpha^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \quad d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2$$

$k = 1, 0, -1$ (closed, flat, open)
Curved Space-time

- The three-dimensional description can be generalized to four-dimensions by allowing the scale factor $a$ to be a function of time, $a(t)$

$$ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$

- $k=0$, clearly corresponds to flat, Minkowski space

- The radius of a sphere can be determined ($d\theta=0$, $d\phi=0$)

$$r_{\text{phys}} = a(t) \int_0^r \frac{dr}{\sqrt{1 - kr^2}}$$

- therefore, for $k=+1$, Universe will be closed
- and for $k=0$, -1, Universe is open and infinite
Summary of Curved Space-time

- Local free fall frame at point P \([\xi^\mu = (\xi^0, \xi^1, \xi^2, \xi^3)]\)
  - find a reference frame with \(g_{\mu\nu}(P) = \eta_{\mu\nu}\)
  - meaning \(\frac{d^2\xi^\mu}{dT^2} = 0\)
    \(dT^2 = \eta_{\mu\nu}d\xi^\mu d\xi^\nu\)

  \[\eta_{\mu\nu} = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 \\
    0 & 0 & -1 & 0 \\
    0 & 0 & 0 & -1
  \end{pmatrix}\]

- In the general frame:
  \[\frac{d^2x^\sigma}{dT^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{dT} \frac{dx^\nu}{dT} = 0\]

  \[\Gamma^\sigma_{\mu\nu} = \frac{g^{\rho\sigma}}{2} \left( \frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\rho} \right)\]

  \(g_{\rho\mu}g^{\mu\nu} = \delta^\nu_\rho\)

  (affine connections or Christoffel symbols)
Summary of Curved Space-time

- Weak-field limit

\[ g_{00} = 1 + 2\phi \quad g_{ii} = \eta_{ii} = -1 \]

\( \phi \) being the ordinary Newtonian gravitational potential
Summary of Curved Space-time

- A measure of the curvature of space-time is the Riemann curvature tensor $R^\mu_{\alpha \beta \gamma}$
  - tells how much the direction of a vector is changed when it is parallel-transported around a curved surfaces

\[
R^\alpha_{\beta \gamma \delta} \equiv -\Gamma^\alpha_{\beta \gamma, \delta} + \Gamma^\alpha_{\beta \delta, \gamma} + \Gamma^\alpha_{\epsilon \gamma} \Gamma^\epsilon_{\beta \delta} - \Gamma^\alpha_{\epsilon \delta} \Gamma^\epsilon_{\beta \gamma}
\]

- This rank 4 tensor appears to have 256 independent components, but symmetries reduce the number of independent components to 20

\[
R_{\alpha \beta \gamma \delta} = R_{\gamma \delta \alpha \beta} \\
R_{\alpha \beta \gamma \delta} = -R_{\beta \alpha \gamma \delta} = -R_{\alpha \beta \delta \gamma} \\
R_{\alpha \beta \gamma \delta} + R_{\alpha \delta \beta \gamma} + R_{\alpha \gamma \delta \beta} = 0
\]
Energy-momentum tensor

- Matter curves space-time
  - This is the basic concept of general relativity
  - the matter density is expressed by the energy-momentum tensor

\[ T^{\mu 0} = \sum_{i=1}^{N} p_i^\mu(t) \delta^{(3)}(\mathbf{r} - \mathbf{r}_i(t)) \]

\[ T^{\mu k} = \sum_{i=1}^{N} p_i^\mu(t) \frac{dx_i^k(t)}{dt} \delta^{(3)}(\mathbf{r} - \mathbf{r}_i(t)) \]

\[ T^{\mu \nu} = \sum_{i=1}^{N} \frac{p_i^\mu p_i^\nu}{E_i} \delta^{(3)}(\mathbf{r} - \mathbf{r}_i(t)) \]
Einstein’s Equations of Gravitation

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \]

where the Ricci tensor is \( R_{\mu\nu} = g^{\alpha\gamma} R_{\alpha\mu\gamma\nu} \)

and the Ricci scalar is \( R = g^{\mu\nu} R_{\mu\nu} \)

It is easy to show

\[ R_{\mu\nu} = 8\pi G_N (T_{\mu\nu} - \frac{1}{2} T^\rho_\rho g_{\mu\nu}) \]
Einstein’s Equations of Gravitation

“No doubt about it, Ellington—we’ve mathematically expressed the purpose of the universe. God, how I love the thrill of scientific discovery!”
Schwarzschild Solution

Schwarzschild solved Einstein’s equations for the space-time metric outside a massive body of mass $M$ in 1916:

$$ds^2 = (1 - r_s/r)dt^2 - \frac{1}{1 - r_s/r}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$r_s = 2GM$$

where $r_s$, the Schwarzschild radius, is the “radius” of a black hole

(note: $r_s = 2 (M/M_p) l_p$ )
The Schwarzschild solution has several famous consequences

- **Perihelion advance**
  - eg. orbit of Mercury

- **Deflection of light**
  - $\Delta \theta = 4GM/c^2$
  - (1.74 arcsec for limb of Sun)

- **Time required for photon to escape from $r = r_s$ is infinite**
  - black hole
Extra Dimensions and Black Holes

- Supposed in a high energy collision, the particles come within their common Schwarzschild radius

\[ b < 2 \, G_N \, M \, c^2 = 4 \, G_N \, M_p \, c^2 \]
  \[ \cdot \, G_N = 6.67 \times 10^{-11} \, \text{m}^3 /\text{kg} \, \text{s}^2 \]
  \[ \cdot \, M_p = 1.67 \times 10^{-27} \, \text{kg} \]
- \[ b < 5 \times 10^{-54} \, \text{m} = 5 \times 10^{-29} \, \text{fm} \]

- This is verrrrry small

- (can also arrive at this by \( r_s = 2 \, (M/M_p) \, l_p \))
  \[ = 2 \, (2 \, \text{GeV}/1.2 \times 10^{19} \, \text{GeV}) \, 1.6 \times 10^{-35} \, \text{m} \]
Extra Dimensions and Black Holes

- However, when the extra dimensions (n) are considered, and the fundamental Planck scale is reduced ($M^*$), the equation for the Schwarzschild radius becomes:

$$r_H = \frac{1}{\sqrt{\pi} M_*} \left( \frac{M_{BH}}{M_*} \right) \left( \frac{8 \Gamma \left( \frac{n+3}{2} \right)}{n+2} \right)^{\frac{1}{n+1}}$$

- So if $M_* = 1$ TeV:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_H$ (10^{-4} fm)</td>
<td>4.06</td>
<td>2.63</td>
<td>2.22</td>
<td>2.07</td>
<td>2.00</td>
<td>1.99</td>
<td>1.99</td>
</tr>
</tbody>
</table>

*hep-ph/0402168*
Particle Physics and Cosmology

- The ‘Standard Model’ of the early universe is supported by the following understood pieces of observational evidence:
  - Hubble’s Law
  - the cosmic microwave background radiation
  - the cosmic abundances of the light elements
  - anisotropies in the background radiation
Cosmological Models

• The Einstein equations when applied to an homogeneous and isotropic Universe do not permit static solutions
  - The Universe is either expanding or contracting

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \]

• This was a problem for Einstein, since it was generally assumed that the Universe was static

• He could not find an argument against adding another term, which would lead to a static Universe, so he added it in

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \]

• \( \Lambda \) is known as the “cosmological constant”
Cosmological Models

- The energy-momentum tensor can generally be expressed as

\[ T^{\mu\nu} = (p + \rho) u^\mu u^\nu - pg^{\mu\nu} \]

where \( u^\mu = (1,0,0,0) \) in the rest frame of the fluid

- One can think of the addition of the cosmological constant as an addition to the energy momentum tensor of the form

\[
T^\Lambda_{\mu\nu} = \begin{pmatrix}
\rho_\Lambda & 0 & 0 & 0 \\
0 & -\rho_\Lambda & 0 & 0 \\
0 & 0 & -\rho_\Lambda & 0 \\
0 & 0 & 0 & -\rho_\Lambda
\end{pmatrix}
\]

\[ \rho_\Lambda = \frac{3\Lambda}{8\pi G} \]

\( \rho_\Lambda \) acts as a negative pressure if \( \Lambda \) is positive
**Hubble’s Law and the Expanding Universe**

\[ \lambda' = \lambda \sqrt{\frac{1 + \beta}{1 - \beta}} = \lambda (1 + z) \]

**Hubble’s Law**

\[ v = Hr \]

H is the Hubble constant

1 Mpc = 3.1 \times 10^{19} \text{ km} = 3.26 \times 10^6 \text{ light-years}
Hubble’s Law and the Expanding Universe

- The expansion of the Universe means that the distance to a distant galaxy has a cosmological time dependence

\[ r(t) = R(t)r_0 \]

where the subscript 0 refers to the present time \( t=t_0 \)
also \( R(t_0) = R_0 = 1 \)

\[ v(t) = \dot{R}(t)r_0 \qquad H = \frac{\ddot{R}}{R} \]

- \( H \) is time dependent. Current measurements find

\[ H_0 = 70 \pm 10 \text{ km s}^{-1} \text{ Mpc}^{-1} \qquad 1 \text{ Mpc} = 3.1 \times 10^{19} \text{ km} \]

- which can be expressed

\[ H_0 = 100h_0 \text{ km s}^{-1} \text{ Mpc}^{-1} \qquad h_0 = 0.7 \pm 0.1 \]
Friedmann Equation

- Once Hubble had discovered that the Universe was not static, but rather expanding, Einstein considering the introduction of $\Lambda$ (the cosmological constant) his greatest blunder.

- Today it is of renewed interest as it appears to be a significant force in the Universe.

- Einstein’s equations can be solved for a homogeneous and isotropic Universe to yield the temporal development of the Universe, known as the Friedmann equation

$$H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G_N \rho}{3} - \frac{K c^2}{R^2} + \frac{\Lambda}{3}$$
A classical example is useful in illustrating the meaning of $K$ in the Friedmann equation.

Consider a point mass $m$ at distance $R$ from the Earth being attracted by the mass of within a sphere of radius $R$, with mass density $\rho$. The total attractive mass is $M = 4\pi R^3 \rho / 3$.

The Newtonian equation of motion $(ma=F)$ is

$$m\ddot{R} = -\frac{MmG_N}{R^2}$$

Integration yields

$$\frac{1}{2} m \dot{R}^2 - \frac{mMG_N}{R} = \text{constant}$$
Friedmann Equation

\[ \frac{1}{2} m \dot{R}^2 - \frac{m M G_N}{R} = \text{constant} \quad \text{where} \quad M = 4\pi R^3 \rho / 3 \]

- The first term of this classical solution is the kinetic energy and the second term is the potential energy, so the constant is the total energy.

- Compare this to the Friedmann equation

\[ H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G_N \rho}{3} - \frac{K c^2}{R^2} + \frac{\Lambda}{3} \]

- If we ignore \( \Lambda \), we see that the total energy is revealing the meaning of \( K \)
  - -1, positive energy, open, expanding without bound
  - +1, negative energy, closed, collapses
Friedmann Equation

- For \( K=\Lambda=0 \)
  \[
  \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G_N \rho}{3} = 2 \frac{G_N M}{R^3}
  \]

  \[R = \left( \frac{9G_N M}{2} \right)^{1/3} t^{2/3}\]

- therefore, the value of the Hubble constant today:
  \[H_0^{-1} = \frac{R}{\dot{R}} = 3t_0/2\]

  \[t_0 = \frac{1}{\sqrt{6\pi G_N \rho_0}} = \frac{2}{3} H_0^{-1} = \frac{6.6}{h_0} \text{ Gyr} = 8-11 \text{ Gyr for } h_0 = 0.7 \pm 0.1\]
Friedmann Equation

\[
t_0 = \frac{1}{\sqrt{6\pi G_N \rho_0}} = \frac{2}{3} H_0^{-1} = \frac{6.6}{h_0} \text{ Gyr } = 8-11 \text{ Gyr for } h_0 = 0.7 \pm 0.1
\]

- but this age is a problem for other observations
  - eg. The ages of globular clusters
  - white dwarf cooling rates
  - uranium isotope dating
- these suggest \( t_0 = 10-14 \text{ Gyr}, \) marginally consistent with Hubble's constant

- But, \( K \) and \( \Lambda \) need not be 0, which will change the estimate
  - current thinking based on observations is that \( \Lambda \) is not 0
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