

# Galois Theory

Richard Koch

April 2, 2015

# Contents

<b>1</b>	<b>Preliminaries</b>	<b>4</b>
1.1	The Extension Problem; Simple Groups . . . . .	4
1.2	An Isomorphism Lemma . . . . .	5
1.3	Jordan Holder . . . . .	6
1.4	The Symmetric and Alternating Groups . . . . .	8
<b>2</b>	<b>The Quadratic, Cubic, and Quartic Formulas</b>	<b>10</b>
2.1	The Quadratic Formula . . . . .	10
2.2	The Cubic Formula . . . . .	11
2.3	The Quartic Formula . . . . .	15
<b>3</b>	<b>Field Extensions and Root Fields</b>	<b>18</b>
3.1	Motivation for Field Theory . . . . .	18
3.2	Fields . . . . .	20
3.3	An Important Example . . . . .	21
3.4	Extension Fields . . . . .	21
3.5	Algebraic Extensions; Root Fields . . . . .	22
3.6	Irreducible Polynomials over $Q$ . . . . .	23
3.7	The Degree of a Field Extension . . . . .	25
3.8	Existence of Root Fields . . . . .	26
3.9	Isomorphism and Uniqueness . . . . .	28
3.10	Putting It All Together . . . . .	29
<b>4</b>	<b>Splitting Fields</b>	<b>31</b>
4.1	Factoring $P$ . . . . .	31
4.2	The Splitting Field . . . . .	32
4.3	Proof that Splitting Fields Are Unique . . . . .	33
4.4	Uniqueness of Splitting Fields, and $P(X) = X^3 - 2$ . . . . .	35
<b>5</b>	<b>Finite Fields</b>	<b>38</b>

5.1	Finite Fields . . . . .	38
<b>6</b>	<b>Beginning Galois Theory</b>	<b>40</b>
6.1	Motivation for Galois Theory . . . . .	40
6.2	Motivation; Putting the Ideas Together . . . . .	43
6.3	The Galois Group . . . . .	44
<b>7</b>	<b>Galois Extensions and the Fundamental Theorem</b>	<b>46</b>
7.1	Galois Extensions . . . . .	46
7.2	Fundamental Theorem of Galois Theory . . . . .	49
7.3	Important Note . . . . .	51
7.4	An Example . . . . .	51
7.5	Finite Fields Again . . . . .	51
<b>8</b>	<b>Concrete Cases of the Theory</b>	<b>53</b>
8.1	Cyclotomic Fields . . . . .	53
8.2	Galois Group of a Radical Extension . . . . .	54
8.3	Structure of Extensions with Cyclic Galois Group . . . . .	54
<b>9</b>	<b>Solving Polynomials by Radicals</b>	<b>56</b>
9.1	Solving Polynomial Equations with Radicals . . . . .	56
9.2	The Flaw . . . . .	57
9.3	The Fix . . . . .	58
9.4	Galois' Theorem on Solving Via Radicals . . . . .	59
9.5	Solving Generic Equations by Radicals . . . . .	61
9.6	A Polynomial Equation Which Cannot Be Solved by Radicals . . . . .	63
<b>10</b>	<b>Straightedge and Compass Constructions</b>	<b>66</b>
10.1	Constructions With Straightedge and Compass . . . . .	66
10.2	Complex Extensions and Constructions . . . . .	70
10.3	Trisecting Angles; Doubling the Cube . . . . .	71
<b>11</b>	<b>Irrationality and Transcendence of <math>\pi</math> and <math>e</math></b>	<b>73</b>
11.1	Irrationality $e$ and $\pi$ . . . . .	73
11.2	Transcendence of $e$ . . . . .	75
11.3	Intermission: Fundamental Theorem of Symmetric Polynomials . . . . .	77
11.4	Transcendence of $\pi$ . . . . .	78
<b>12</b>	<b>Special Cases</b>	<b>81</b>
12.1	The Discriminant . . . . .	81
12.2	The Cubic Case . . . . .	82
12.3	Cyclotomic Fields . . . . .	84

<i>CONTENTS</i>	3
12.4 Galois Group of $X^p - a$ over $Q$ . . . . .	88
<b>13 Constructing Regular Polygons</b>	<b>91</b>
13.1 Constructing Regular Polygons . . . . .	91
<b>14 Normal and Separable Extensions</b>	<b>95</b>
14.1 Normal Extensions, Separable Extensions, and All That . . . . .	95
<b>15 Galois Theory and Reduction Modulo <math>p</math></b>	<b>100</b>
15.1 Preview . . . . .	100
15.2 $X^5 - X - 1$ . . . . .	101
15.3 A Tricky Point Concerning Straightedge and Compass Constructions . . . . .	103
15.4 Polynomials with Galois Group $S_n$ . . . . .	104
15.5 Every Finite $G$ is a Galois Group . . . . .	105
15.6 Proof of Dedekind's Theorem . . . . .	106

# Chapter 1

## Preliminaries

Before getting to the main subject, we prove some facts about group theory. These results will be used later in our study of generalizations of the quadratic formula.

In chemistry, every compound is built out of atoms. Knowing the atoms is not enough to determine the compound; for instance, graphite and diamond are both pure carbon. Similarly, we will show that every finite group is built out of "simple groups". But knowing the simple groups is not enough to determine the final group they build.

### 1.1 The Extension Problem; Simple Groups

Suppose we want to classify finite groups  $G$ . It is natural to work by induction on the order of  $G$ . If  $H$  is a subgroup of  $G$ , then we already understand  $H$ . If  $H$  is normal, we also already understand  $G/H$ . Thus in the sequence

$$0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0$$

we understand the groups at the ends and need only fill in the middle group.

One choice for  $G$  is  $H \times G/H$ . There are usually others. The problem of constructing such  $G$  is called *the extension problem* in group theory; it is difficult. For example, suppose  $H = Z_4$  and  $G/H = Z_2$ . Then  $G$  is a group of order 8. After some work, one can show that there are three  $G$  which fit in the sequence,  $Z_4 \times Z_2$ ,  $D_4$ , and  $Q$ . Here  $D_4$  is the dihedral group of order four, that is, the group of symmetries of a square, and the group  $Q$  is the group of unit quaternions  $\{\pm 1, \pm i, \pm j, \pm k\}$ . Note that  $\{\pm 1, \pm i\}$  is a normal subgroup isomorphic to  $Z_4$ . Note also that  $D_4$  and  $Q$  are not isomorphic because  $D_4$  has five elements of order two and  $Q$  has only one element of order two.

Luckily, we don't need to solve the extension problem for Galois theory. But suppose we completely understood this problem. What more would be needed to classify finite groups?

Some groups have no non-trivial normal subgroups. We call such groups *simple groups* and we would need to construct them another way. This has been one of the greatest research topics of the twentieth century, and there is now a complete list of all finite simple groups. A knowledge of this list, and a complete solution of the extension problem (which will never happen!) would produce a complete classification of all finite groups.

From the list we only need the *abelian* simple groups:

**Theorem 1** *A finite abelian group is simple if and only if it equals  $Z_p$  for a prime  $p$ .*

*Proof:* All subgroups of an abelian group are normal, so it suffices to list all groups with no non-trivial subgroups. Certainly  $Z_p$  has no non-trivial subgroups, since every subgroup has order dividing  $p$  and thus equals  $\{e\}$  or  $Z_p$ . Conversely if  $G$  has no non-trivial subgroups and  $g \neq e$  is in  $G$ , then the cyclic subgroup generated by  $g$  must be all of  $G$ , so  $G$  is cyclic of some order  $n$ . If  $n$  is not prime then it has non-trivial cyclic subgroups.

## 1.2 An Isomorphism Lemma

We are about to prove the *Jordan Holder theorem*, which says that every finite group is built from a uniquely determined collection of simple groups. In the midst of the proof, we need a little lemma, so we prove it first.

Let  $G$  be a group with subgroups  $A$  and  $B$ . By definition  $AB$  is the set of all elements in  $G$  of the form  $a_1b_1a_2b_2 \dots a_kb_k$  for varying  $k$ . This set is clearly a subgroup of  $G$ . If  $A$  and  $B$  are normal, so is  $AB$  because

$$g(a_1b_1a_2b_2 \dots a_kb_k)g^{-1} = (ga_1g^{-1})(gb_1g^{-1}) \dots (ga_kg^{-1})(gb_kg^{-1})$$

**Lemma 1** *If  $A$  and  $B$  are normal,*

$$AB/B \cong A/(A \cap B)$$

*Proof:* We have a natural group homomorphism

$$A \rightarrow AB \rightarrow AB/B$$

and clearly this map sends  $A \cap B$  to the identity. So it induces

$$A/(A \cap B) \rightarrow AB/B$$

The kernel of this map is  $e$  because if  $a \in A$  maps to  $e \in AB/B$ , then  $a \in A \cap B$ . The map is onto because  $a_i b_i \sim a_i$  in  $AB/B$ , so  $a_1 b_1 \dots a_k b_k \sim a_1 \dots a_k \in AB/B$ .

### 1.3 Jordan Holder

Continuing with the idealistic program of the previous sections, suppose  $G$  is an arbitrary finite group. Find a normal subgroup  $H_1 \subset G$  unequal to  $G$  and as large as possible. It is easy to see that  $G/H_1$  must be simple, since if  $K \subset G/H_1$  is a non-trivial normal subgroup, the inverse image of  $K$  in  $G$  will be a normal subgroup  $\tilde{K}$  with  $H_1 \subset \tilde{K} \subset G$ . If we understood the extension problem, we could construct  $G$  from  $H_1$  and  $G/H_1$ .

Continue the process. Find a normal subgroup  $H_2 \subset H_1$  unequal to  $H_1$  and as large as possible. Then  $H_1/H_2$  is simple. If we understood the extension problem, we could construct  $H_1$  from  $H_2$  and  $H_1/H_2$ .

Continuing in this vein, we eventually construct a *complete composition series*, that is, a chain of subgroups

$$\{e\} = H_n \subset H_{n-1} \subset \dots \subset H_1 \subset G$$

with each  $H_i$  normal in  $H_{i-1}$  and as large as possible, and each  $H_{i-1}/H_i$  simple. The group  $G$  is constructed from the simple groups  $H_{i-1}/H_i$  by a series of group extensions.

Unfortunately, the  $H_i$  are not unique. For example, let  $G = Z_6$ . Let  $Z_2 \subset Z_6$  be the subgroup  $\{0, 3\}$  and let  $Z_3 \subset Z_6$  be the subgroup  $\{0, 2, 4\}$ . Then we obtain two composition series

$$\{e\} \subset Z_2 \subset Z_6$$

$$\{e\} \subset Z_3 \subset Z_6$$

However, both series have “length 2”; the simple quotients in the first case are  $Z_2, Z_3 = Z_6/Z_2$  and the simple quotients in the second case are  $Z_3, Z_2 = Z_6/Z_3$  and thus the same up to order.

Remarkably, this holds in general

**Theorem 2 (Jordan-Holder)** *Any two complete composition series for a finite group have the same length, and their simple quotients are isomorphic up to order.*

*Remark:* This theorem is a generalization of the unique factorization theorem for integers. Indeed, if  $n = p_1^{n_1} \dots p_k^{n_k}$ , it is easy to find a composition series for  $Z_n$  with simple quotients  $Z_{p_i}$ , each repeated  $n_i$  times.

*Remark:* Sometimes progress is made in mathematics by throwing information away until only the crucial information remains. Composition series allow us to throw away the

intricate extension information until only the simple quotient information remains. It is this quotient information which is important in Galois theory.

In the previous section, we listed the three groups of order four obtained by extending  $Z_4$  by  $Z_2$ . Notice that the simple quotients of all three groups are  $Z_2, Z_2, Z_2$ . So in this case, extension information is definitely thrown away.

*Proof:* We prove the theorem by induction on the order of  $G$ ; the result is trivial for groups of order less than or equal to two.

In the induction step, suppose  $G$  has two composition series

$$\dots \subset A_1 \subset A \subset G$$

$$\dots \subset B_1 \subset B \subset G$$

If  $A = B$ , the theorem holds by induction, so suppose  $A \neq B$ . Then  $AB$  is a normal subgroup larger than  $A$ , and so  $AB = G$ . Notice that

$$A \cap B \subset A \subset G$$

$$A \cap B \subset B \subset G$$

By the isomorphism lemma,  $G/A = AB/A = B/A \cap B$  and  $G/B = AB/B = A/A \cap B$ . It follows that the partial sequences above can be refined to composition sequences which are the same except at the beginning, and whose last two composition quotients are interchanged.

$$\dots H_2 \subset A \cap B \subset A \subset G$$

$$\dots H_2 \subset A \cap B \subset B \subset G$$

Compare these to the existing series

$$\dots A_2 \subset A_1 \subset A \subset G$$

$$\dots B_2 \subset B_1 \subset B \subset G$$

By induction, the theorem is true for  $A$  and  $B$ , so the lengths of the  $A_i$  and  $H_i$  series for  $A$  are equal, and the lengths of the  $B_i$  and  $H_i$  series for  $B$  are equal, and thus the lengths of the  $A_i$  and  $B_i$  series are equal. Moreover, the composition quotients for  $A$  are equal up to order, so  $\{H_i/H_{i+1}\} = \{A_i/A_{i+1}\}$  up to order. Similarly, the composition quotients for  $B$  are equal up to order, so  $\{H_i/H_{i+1}\} = \{B_i/B_{i+1}\}$  up to order. Adding  $G/A$  and  $A/A_1$  to the first set, and  $G/B$  and  $B/B_1$  to the second set and applying the isomorphism lemma as above finishes the proof.

*Important Remark:* In the material which follows, we often obtain series

$$\dots \subset G_2 \subset G_1 \subset G$$

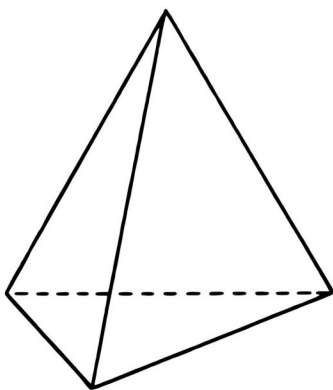


in which the  $G_{i+1}$  are normal in  $G_i$ , but not necessarily maximal with this property. We still call these *composition series*. It is easy to see that every such series can be extended to a complete composition series by adding additional subgroups between the  $G_{i+1}$  and the  $G_i$ .

## 1.4 The Symmetric and Alternating Groups

Let us apply the previous theory to the symmetric group  $S_n$ , which has the alternating group  $A_n$  as a maximal normal subgroup. When  $n = 2$ , the alternating group is trivial and there is only one composition factor,  $Z_2$ . When  $n = 3$ ,  $S_3$  is isomorphic to the dihedral group  $D_3$  and the group has two composition factors,  $Z_2$  and  $Z_3$ .

When  $n = 4$ ,  $S_4$  is the group of all symmetries of a tetrahedron, and  $A_4$  is the group of rotational symmetries of this tetrahedron. The group  $A_4$  of order 12 has a normal subgroup  $Z_2 \times Z_2$  of order 4, namely the group which rotates opposite pairs of lines by 180 degrees about their centers. Thus  $A_4$  has composition factors  $Z_2, Z_2, Z_3$ .



We are going to prove that  $A_n$  is simple for  $n \geq 5$ . This will turn out to be the central reason that equations of degree five and higher cannot be solved by radicals.

It turns out that the group  $A_5$  of order 60 is the smallest non-abelian simple group. This group is the group of rotational symmetries of a dodecahedron, so it is not surprising that the dodecahedron is part of the logo of the MAA.

A few simple observations will make our work easier. Every permutation can be written as a product of cycles:  $\tau = (1, 3)(2, 5, 7, 8)(4, 6)$ . The sign of a cycle with  $k$  entries is  $(-1)^{k-1}$  and the sign of a product of cycles is the product of the signs of the individual cycles. If  $\sigma$  is a permutation, an easily calculation shows that, for instance,

$$\sigma\tau\sigma^{-1} = \sigma \circ (1, 3)(2, 5, 7, 8)(4, 6) \circ \sigma^{-1} = (\sigma(1), \sigma(3)) (\sigma(2), \sigma(5), \sigma(7), \sigma(8)) (\sigma(4), \sigma(6))$$

**Theorem 3** *If  $n \geq 5$ ,  $A_n$  is simple. Thus the composition quotients of  $S_n$  are  $Z_2$  and  $A_n$ .*

*Proof of Theorem:* It is well known that every element of  $S_n$  is a product of transpositions. Similarly, every element of the alternating group is a product of three cycles. Indeed, every element is a product of an even number of transpositions, so it suffices to show that the product of any two transpositions is a three cycle. If the transpositions contain a letter in common, then up to conjugation we have  $(1, 2)(2, 3) = (1, 3, 2)$ . Otherwise up to conjugation we have  $(1, 2)(3, 4) = (3, 2, 4)(1, 3, 2)$ .

Since  $n \geq 5$ , a normal subgroup of  $A_n$  with one three cycle contains all three cycles and consequently is all of  $A_n$ . For instance, if the normal subgroup contains  $(1, 2, 3)$  and  $\sigma$  is a permutation taking  $i_1$  to 1,  $i_2$  to 2, and  $i_3$  to 3, then after conjugation by  $\sigma$  it contains  $(i_1, i_2, i_3) = (\sigma(1), \sigma(2), \sigma(3))$ . We need to make sure that the conjugating  $\sigma$  has even sign, but if not, choose two other indices  $i_4$  and  $i_5$  and multiply  $\sigma$  by the transposition which interchanges these extra indices.

We now prove that a nonzero normal subgroup of  $A_n$  is all of  $A_n$ . Suppose first that this subgroup contains an element  $g$  whose cycle notation has at least one cycle of length greater than three. For example, suppose the element is  $g = (1, 2, \dots, k)(\dots)$ . Then the subgroup contains

$$g^{-1}[(1, 2, 3)^{-1}g(1, 2, 3)] = (2, 3, k)$$

and so all three cycles, and so is all of  $A_n$ .

Suppose next that this subgroup contains an element  $g$  whose cycle notation has at least two cycles of length three. For instance, suppose  $g = (1, 2, 3)(4, 5, 6) \dots$  is in the subgroup. Then

$$g^{-1}[(1, 2, 4)^{-1}g(1, 2, 4)] = (1, 2, 4, 3, 6)$$

and so by the previous step the subgroup is all of  $A_n$ .

Suppose next that the subgroup contains an element  $g$  whose cycle notation has only one cycle of length three and otherwise just transpositions. For instance, suppose  $g = (1, 2, 3) \dots$  is in the subgroup. Then the subgroup contains  $g^2 = (1, 3, 2)$  and so the subgroup is all of  $A_n$ .

Suppose finally that every element of the subgroup is a product of transpositions. Since there must be at least two of these transpositions, we can suppose  $g = (1, 2)(3, 4) \dots$ . Then

$$g[(1, 2, 3)^{-1}g(1, 2, 3)] = (1, 4)(2, 3)$$

is in the subgroup. So

$$(1, 4)(2, 3)[(1, 2, 5)^{-1}(1, 4)(2, 3)(1, 2, 5)] = (1, 2, 3, 4, 5)$$

is in the subgroup and consequently the subgroup is all of  $A_n$ . QED.

## Chapter 2

# The Quadratic, Cubic, and Quartic Formulas

### 2.1 The Quadratic Formula

**Theorem 4 (Quadratic Formula)** *The solutions of  $ax^2 + bx + c = 0$  are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

*Proof:* Introduce new coordinates  $u = x - \lambda$  where  $\lambda$  is a constant to be chosen soon. Substituting in the equation gives

$$a(u + \lambda)^2 + b(u + \lambda) + c = au^2 + (2a\lambda + b)u + (a\lambda^2 + b\lambda + c) = 0$$

Now choose  $\lambda$  to make the coefficient of  $u$  vanish, so  $\lambda = -\frac{b}{2a}$ . The equation becomes

$$au^2 + \left( \frac{b^2}{4a} - \frac{b^2}{2a} + c \right) = 0$$

or

$$u^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

This equation is easily solved:

$$u = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Hence

$$x = u + \lambda = \pm \frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## 2.2 The Cubic Formula

We easily reduce an arbitrary cubic to the special form below, as will be seen shortly.

**Theorem 5 (Cubic Formula)** *The solutions of  $x^3 + Ax + B = 0$  are*

$$x = \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}} - \frac{A}{3\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}}$$

*Remark:* Consider the equation  $ax^3 + bx^2 + cx + d = 0$ . Again introduce new coordinates  $u = x - \lambda$  where  $\lambda$  is a constant. Substituting,

$$a(u + \lambda)^3 + b(u + \lambda)^2 + c(u + \lambda) + d = 0$$

or

$$au^3 + (3a\lambda + b)u^2 + (3a\lambda^2 + 2b\lambda + c)u + (a\lambda^3 + b\lambda^2 + c\lambda + d) = 0$$

Now choose  $\lambda$  to make the coefficient of  $u$  vanish, so  $\lambda = -\frac{b}{3a}$  and the equation becomes

$$au^3 + \frac{3ac - b^2}{3a}u + \frac{2b^3 - 9abc + 27a^2d}{27a^2} = 0$$

Dividing by  $a$  gives

$$u^3 + \frac{3ac - b^2}{3a^2}u + \frac{2b^3 - 9abc + 27a^2d}{27a^3} = 0$$

Naming the coefficients in this equation  $A$  and  $B$  gives  $u^3 + Au + B = 0$ , and it suffices to solve this simpler equation.

All of this was known early in the Renaissance. In Italian, the word “cosa” means “thing.” So “cosa nostra” means “our thing” in the gangster world, and the above equation was known in Renaissance Italy as the equation of the “cube and the cosa.” Many people tried to find the solution; even Omar Khayyam. The solution was discovered around 1500 by Scipione del Ferro. Independently, Niccolo Tartaglia found the solution, and later his solution was published by Gerolamo Cardano in his book *Ars Magna*.

*Proof:* To deduce the solution we introduce a trick. Write

$$x = u + \frac{\alpha}{u}$$

where  $\alpha$  is a nonzero constant to be determined later. We can certainly write  $x$  in this form, since solving for  $u$  amounts to solving

$$u^2 - xu + \alpha = 0$$

and this equation has a nonzero solution. However,  $u$  might be complex even if  $x$  is real.

Substituting in the original equation gives

$$\left(u + \frac{\alpha}{u}\right)^3 + A\left(u + \frac{\alpha}{u}\right) + B = 0$$

or

$$u^3 + (3\alpha + A)u + (3\alpha^2 + A\alpha)\frac{1}{u} + B + \alpha^3\frac{1}{u^3} = 0$$

We now choose

$$\alpha = -\frac{A}{3}$$

and notice that the resulting equation is

$$u^3 + B - \frac{A^3}{27u^3} = 0$$

or

$$(u^3)^2 + Bu^3 - \frac{A^3}{27} = 0$$

but this is a quadratic equation in  $u^3$ . So

$$u^3 = \frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}$$

and

$$u = \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}$$

Therefore

$$x = \sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}} - \frac{A}{3\sqrt[3]{\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2}}}$$

*Remark:* By putting the equations of the previous paragraphs together, it is possible to write a single formula solving the general cubic and thus generalizing the quadratic formula.

If you insist on doing so, go right ahead. In practice, we always simplify as above and solve the easier  $x^3 + Ax + B = 0$ .

*Remark:* In these equations, we are to choose one of the two values of the square root and use it consistently, and we are to choose one of the three values of the cube root and use it consistently. All told, then, there are six possible choices to produce the three solutions of the original cubic equation. So we expect that each root will be given twice by these formulas.

*Remark:* There are some unfortunate features of this solution. Consider what happens when the equation we want to solve is

$$x^3 - 2x - 4 = (x - 2)(x^2 + 2x + 2) = 0$$

In this case  $A = -2$  and  $B = -4$  and

$$u^3 = \frac{4 \pm \sqrt{16 - \frac{32}{27}}}{2} = 2 \pm 10\sqrt{\frac{1}{27}}$$

Taking the positive square root and the real cube root, we obtain

$$x = \sqrt[3]{2 + 10\sqrt{\frac{1}{27}}} + \frac{2}{3} \frac{1}{\sqrt[3]{2 + 10\sqrt{\frac{1}{27}}}}$$

There is no hint here that the solution is an integer. I leave it to the reader to show that this expression equals 2.

A different problem is revealed by the example below, which shows that complex numbers can occur even if the final solution is real:

*Example:* Consider the equation  $X^3 - 15X - 4 = 0$ . The cubic formula asks us to compute

$$\frac{-B \pm \sqrt{B^2 + \frac{4A^3}{27}}}{2} = \frac{4 \pm \sqrt{16 - \frac{4 \cdot 15^3}{27}}}{2} = 2 \pm \sqrt{4 - 15^3} = 2 \pm \sqrt{-121} = 2 \pm 11i$$

According to the formula, the solution is then

$$X = \sqrt[3]{2 + 11i} + \frac{15}{\sqrt[3]{2 + 11i}} = \sqrt[3]{2 + 11i} + \frac{15\sqrt[3]{2 - 11i}}{\sqrt[3]{2 + 11i}\sqrt[3]{2 - 11i}} = \sqrt[3]{2 + 11i} + \frac{15\sqrt[3]{2 - 11i}}{\sqrt[3]{125}}$$

and so

$$X = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$$

According to the paper *On The Casus Irreducibilis of Solving the Cubic Equation* by Jay Villaneuva at Florida Memorial University, Bombelli noticed in 1550 that  $(2+i)^3 = 2+11i$  and  $(2-i)^3 = 2-11i$  and thus that

$$X = (2+i) + (2-i) = 4$$

which indeeds solves  $X^3 - 15X - 4 = 0$ .

*Remark:* If a cubic has a multiple root, it is easily solved. Otherwise a cubic with real roots either has one real root or three real roots. It turns out that when there is one real root, the expression under the square root sign is positive. The cubic formula then asks for the cube root of some real number, which always exists. So the cubic formula yields the real solution.

But if there are three real roots, then the expression under the square root sign is negative, and the cubic formula forces us to compute the cube root of a complex number. The final formula involves this cube root twice, and miraculously the complex part of the result cancels and we obtain real roots.

The case of the cubic formula when the cubic has three real roots is known as the *casus irreducibilis*. The real reason that mathematicians introduced complex numbers and began to take them seriously is that the cubic formula requires us to calculate cube roots of complex quantities in this case.

We will later prove that in the casus irreducibilis case, there is no expression involving only real roots giving the roots of the cubic. So the use of complex numbers in this case is essential.

**Theorem 6** *If  $x^3 + Ax + B$  has one real root and two imaginary roots,  $B^2 + \frac{4A^3}{27}$  is positive and  $u$  can be chosen to be real. If  $x^3 + Ax + B$  has three distinct real roots,  $B^2 + \frac{4A^3}{27}$  is negative and  $u$  is necessarily complex.*

*Proof:* Call the roots  $x_1, x_2$ , and  $x_3$ . We are going to show that

$$[(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]^2 = -27B^2 - 4A^3.$$

If the roots are real and distinct, it follows that  $27B^2 + 4A^3$  is negative. On the other hand, if  $x_1 = a, x_2 = b + ic, x_3 = b - ic$ , then  $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = [(a - b)^2 + c^2](-2ic)$  and the square of this expression is  $[(a - b)^2 + c^2]^2(-4c^2)$  and so negative, so  $27B^2 + 4A^3$  is positive.

Consider the expression  $(x - x_1)(x - x_2)(x - x_3) = x^3 + Ax + B$ . Differentiate and then set  $x$  successively to  $x_1, x_2, x_3$  to obtain

$$(x_1 - x_2)(x_1 - x_3) = 3x_1^2 + A$$

$$(x_2 - x_1)(x_2 - x_3) = 3x_2^2 + A$$

$$(x_3 - x_1)(x_3 - x_2) = 3x_3^2 + A$$

Multiply left sides and right sides together to obtain

$$[(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]^2 = -(3x_1^2 + A)(3x_2^2 + A)(3x_3^2 + A)$$

The right side of this expression is

$$-27(x_1x_2x_3)^2 - 9A(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 3A^2(x_1^2 + x_2^2 + x_3^2) - A^3$$

and we want to show that it equals  $-(27B^2 + 4A^3)$ .

However  $(x - x_1)(x - x_2)(x - x_3) = x^3 + Ax + B$  and consequently

$$x_1 + x_2 + x_3 = 0$$

$$x_1x_2 + x_1x_3 + x_2x_3 = A$$

$$x_1x_2x_3 = -B$$

Squaring the first of these formulas gives  $x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3) = 0$  and consequently

$$x_1^2 + x_2^2 + x_3^2 = -2A$$

Squaring the second formula gives  $x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + 2(x_1x_2x_3)(x_1 + x_2 + x_3) = A^2$  and consequently

$$x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 = A^2$$

Thus the expression which interests us is

$$\begin{aligned} -27(x_1x_2x_3)^2 - 9A(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) - 3A^2(x_1^2 + x_2^2 + x_3^2) - A^3 &= \\ -27B^2 - 9A(A^2) - 3A^2(-2A) - A^3 &= -27B^2 - 4A^3 \end{aligned}$$

## 2.3 The Quartic Formula

The quartic formula was discovered by Lodovico Ferrari in 1540. Ferrari's formula reduces the quartic to a related cubic, which is then solved using the result of del Ferro and Tartaglia.

Ferrari was a student of Cardano, and he learned of Tartaglia's solution from Cardano. Unfortunately, Tartaglia revealed his solution to Cardano late at night over a glass of wine, and made Cardano swear an oath never to reveal his solution. Although Tartaglia claimed



that he would soon publish the result, he never did. This put Cardano and Ferrari in a bind, because solving a quartic depended on Tartaglia's secret solution of the cubic.

This knot was untangled when Cardano realized that del Ferro, who by that time had died, had independently solved the cubic. So Cardano visited del Ferro's widow and discovered that she saved her husband's papers. Sacred oaths were taken seriously in the Renaissance, but they were also taken literally, so revealing del Ferro's formula was not the same thing as revealing Tartaglia's formula. In this way, Cardano published del Ferro's cubic formula, and Ferrari's quartic result became public.

Rather than using Ferrari's method, we discuss a simple method described in the Wikipedia article on the quartic. Begin with a general quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Divide the equation by  $a$  to obtain a corresponding equation with  $a = 1$ . Set  $x = u - \frac{b}{4}$  and notice that the corresponding equation for  $u$  has the form

$$u^4 + Au^2 + Bu + C = 0$$

Now comes the main trick. We attempt to factor this as a product of two quadratics:

$$u^4 + Au^2 + Bu + C = (u^2 + pu + q)(u^2 + ru + s)$$

If we can find  $p, q, r, s$ , then we can solve the two quadratics by the quadratic formula and find all four roots of the quartic. Multiplying out, we find that we need to solve

$$0 = p + r$$

$$A = s + pr + q$$

$$B = ps + qr$$

$$C = qs$$

The first equation implies that

$$r = -p$$

so it suffices to solve

$$A = s - p^2 + q$$

$$B = ps - pq$$

$$C = qs$$

The first and second equations can be rewritten

$$A + p^2 = s + q$$

$$\frac{B}{p} = s - q$$

$$C = qs$$

We can solve these equations for  $s$  and  $q$  in terms of  $p$ :

$$s = \frac{1}{2} \left( A + p^2 + \frac{B}{p} \right)$$

$$q = \frac{1}{2} \left( A + p^2 - \frac{B}{p} \right)$$

Thus it suffices to find  $p$  satisfying the last equation

$$C = qs = \frac{1}{4} \left( (A + p^2)^2 - \frac{B^2}{p^2} \right)$$

or equivalently

$$p^6 + 2Ap^4 + (A^2 - 4C)p^2 - B^2 = 0$$

This is a cubic equation in  $p^2$ . To solve the reduced quartic, we solve this equation for  $p^2$ , extract a root to find  $p$ , and use the result to compute  $r, s, q$ , and then solve the two resulting quadratics.

*Remark:* From here, it is possible to write down explicit formulas for the roots. I won't bother.

We should worry about edge cases. This happens when all solutions of the cubic are zero. That can only happen if  $A = 0, C = 0$  and  $B = 0$ , but then the quartic is  $u^4 = 0$ , which has trivial solutions. We might worry that the cubic formula would produce a complicated expression which happens to simplify to zero. This would happen if the cubic has *one* zero solution, which happens if  $B = 0$ . But then the quartic is a quadratic equation in  $u^2$ ,  $u^4 + Au^2 + C = 0$ , which can be solved by two applications of the quadratic formula.

## Chapter 3

# Field Extensions and Root Fields

### 3.1 Motivation for Field Theory

We now want to show that there cannot be equivalent formulas for equations of degree five and higher. This, of course, is much harder. To get started, we have to look at our earlier formulas from a more general point of view.

Consider the solution of a cubic. Starting with the coefficients  $A$  and  $B$  of the equation and the standard rational numbers, we use addition, subtraction, multiplication, and division to obtain the expression

$$B^2 + \frac{4A^3}{27}.$$

Then we form the square root of this expression:

$$\sqrt{B^2 + \frac{4A^3}{27}}$$

Using this new number,  $A$ ,  $B$ , and the rationals, we use addition, subtraction, multiplication, and division to obtain a fancier expression:

$$\frac{-B + \sqrt{B^2 + \frac{4A^3}{27}}}{2}$$

Then we form the cube root of this expression:

$$\sqrt[3]{\frac{-B + \sqrt{B^2 + \frac{4A^3}{27}}}{2}}$$

Finally, starting from  $A$ ,  $B$ , the rationals, and the two radicals, we apply addition, subtraction, multiplication, and division to obtain the actual root:

$$x = \sqrt[3]{2 + 10\sqrt{\frac{1}{27}}} + \frac{2}{3} \frac{1}{\sqrt[3]{2 + 10\sqrt{\frac{1}{27}}}}$$

It is convenient to think of this process in the following way. Imagine a large pot of numbers — indeed an infinite pot of numbers. When we begin, this pot contains the positive integers  $1, 2, 3, \dots$  and  $A$  and  $B$ . We want to add numbers to the pot until finally it contains the solutions to the cubic. We can add numbers in five ways: by adding two numbers already in the pot, by multiplying two numbers already in the pot, by subtracting two numbers already in the pot, by dividing two numbers already in the pot, and by adding a radical of a number already in the pot.

It turns out that adding a radical is by far the most important of these operations. So it is convenient to imagine the process as follows. First we add *all* numbers that can be obtained from the positive integers and  $A$  and  $B$  by adding, subtracting, multiplying, and dividing. This adds infinitely many numbers; call the resulting set  $K_0$ . Next compute a radical of *one* of the numbers in the pot and add that. This radical might be a square root, or a cube root, or whatever. After that, add all numbers which can be obtained from numbers now in the pot by adding, subtracting, multiplying, and dividing. Call the resulting set  $K_1$ , so  $K_0 \subset K_1$ .

Continue. Add a radical of a number currently in the pot, and then add all numbers that can be obtained from these by adding, subtracting, multiplying, and dividing. Call the new set  $K_2$ . In this way we obtain a finite chain of sets of numbers:

$$K_0 \subset K_1 \subset K_2 \subset \dots \subset K_n$$

In the end, the roots of our equation should belong to  $K_n$ .

This way of thinking puts the emphasis on constructing the radical, since that's the process that gives the inclusion sign  $\subset$ . The process of adding, subtracting, multiplying, and dividing is hidden in the construction of  $K_i$  after the radical is added. In the following sections, we study these two processes in detail. We'll begin with the process of completing a set using addition, subtraction, multiplication, and division.

## 3.2 Fields

**Definition 1** A field is a set  $K$  with two binary operations  $x, y \rightarrow x + y$  and  $x, y \rightarrow x \cdot y$  satisfying the following axioms:

1.  $x + y = y + x$  for all  $x, y$
2.  $(x + y) + z = x + (y + z)$  for all  $x, y, z$
3. there is an element  $0$  such that  $0 + x = x$  for all  $x$
4. given  $x$ , there is an element  $-x$  such that  $x + (-x) = 0$
5.  $x \cdot y = y \cdot x$  for all  $x, y$
6.  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$  for all  $x, y, z$
7. there is an element  $1 \neq 0$  such that  $1 \cdot x = x$  for all  $x$
8. given  $x \neq 0$ , there is an element  $x^{-1}$  such that  $x \cdot x^{-1} = 1$
9.  $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$  for all  $x, y, z$

*Remark:* It is easy to prove that  $0$  and  $1$  are unique, and that  $(-x)$  and  $x^{-1}$  are uniquely determined by  $x$ .

*Remark:* Clearly the set of rational numbers  $\mathbb{Q}$ , the set of real numbers  $\mathbb{R}$ , and the set of complex numbers  $\mathbb{C}$  are fields. Almost all of the fields in these notes are subsets  $K \subset \mathbb{C}$  with the standard addition and multiplication. Such a subset is a field if it satisfies five conditions:

1.  $0 \in K$  and  $1 \in K$
2.  $x, y \in K$  implies  $x + y \in K$
3.  $x \in K$  implies  $-x \in K$
4.  $x, y \in K$  implies  $x \cdot y \in K$
5.  $x \in K$  and  $x \neq 0$  implies  $x^{-1} \in K$

*Remark:* The set  $Z_n$  of integers modulo  $n$  is a commutative ring, satisfying all field axioms except possibly axiom 8. If  $n = ab$  has a non trivial factorization, then  $Z_n$  is not a field because  $a$  and  $b$  represent nonzero elements of  $Z_n$  whose product is zero. But if  $n = p$  is prime, then  $Z_p$  is a field; indeed if  $0 < a < p$ , then  $a$  and  $p$  are relatively prime, so there exist integers  $A$  and  $B$  with  $Aa + Bp = 1$  and consequently  $A$  represents a multiplicative inverse of  $a$  in  $Z_p$ .

**Theorem 7** *Every field  $K$  has a smallest subfield, which is isomorphic to either  $\mathbb{Q}$  or  $\mathbb{Z}_p$  for some prime  $p$ .*

*Proof:* The intersection  $F$  of all subfields of  $K$  is a field, and thus the smallest subfield.

Define a map  $\varphi : Z \rightarrow F$  by sending the generator  $1 \in Z$  to  $1 \in F$ . This map is necessarily a ring homomorphism, and thus its kernel is an ideal,  $I$ . This ideal is prime, for if  $ab \in I$ , then  $0 = \varphi(ab) = \varphi(a)\varphi(b)$ , so  $\varphi(a) = 0$  or  $\varphi(b) = 0$  and thus  $a \in I$  or  $b \in I$ . We conclude that  $I = (0)$  or else  $I = (p)$  for some prime  $p$ . In the first case  $Z \subset F$  and we rapidly conclude that  $\mathbb{Q} \subset F$ . In the second case  $\mathbb{Z}_p \subset F$ . Since  $F$  is the smallest subfield,  $\mathbb{Q} = F$  or  $\mathbb{Z}_p = F$ .

### 3.3 An Important Example

*Example:* In the previous section we started with  $1 \in K$  and discovered that by adding all sums, additive inverses, products, and quotients we would end up with either  $\mathbb{Q}$  or  $\mathbb{Z}_p$ .

It is instructive to examine a similar example. Suppose we start with  $\mathbb{Q}$  and  $\sqrt{2}$  in a larger field  $K$ , and again add all sums, additive inverses, products, and quotients, to arrive at the smallest subfield of  $K$  containing  $\mathbb{Q}$  and  $\sqrt{2}$ . In the argument, imagine that the larger  $K$  is either  $\mathbb{R}$  or  $\mathbb{C}$ .

If  $a, b \in \mathbb{Q}$ , then surely we will obtain all  $a + b\sqrt{2}$ . Surprisingly, we get nothing more. Indeed the sum of two such elements is another, and the additive inverse of such an element is another. It is only slightly harder to see that the product of two such elements is another, essentially because  $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbb{Q}$ . Finally the quotient of two such elements is another because

$$\frac{a + b\sqrt{2}}{c + d\sqrt{2}} = \frac{(a + b\sqrt{2})(c - d\sqrt{2})}{(c + d\sqrt{2})(c - d\sqrt{2})} = \frac{(ac - 2bd) + (bc - ad)\sqrt{2}}{c^2 - 2d^2} = \left( \frac{ac - 2bd}{c^2 - 2d^2} \right) + \left( \frac{bc - ad}{c^2 - 2d^2} \right) \sqrt{2}$$

Notice that if  $c \neq 0$  or  $d \neq 0$  then  $c^2 - 2d^2 \neq 0$  because  $\sqrt{2}$  is not rational.

### 3.4 Extension Fields

Imagine a very large field like the field of complex numbers, which we will pretend is the set of all possible numbers. Suppose we have a subfield  $K$  of this large field and an element  $a$  of the large field. Define  $K(a)$  to be the set of all numbers which can be formed from  $K$  and  $a$  by adding, subtracting, multiplying, and dividing. We are going to obtain a very

concrete description of such fields  $K(a)$ . Of course if  $a \in K$  then  $K(a)$  is just  $K$ , so assume that  $a \notin K$ .

There are two possibilities. If  $a$  satisfies a non-trivial polynomial  $P(x)$  with coefficients in  $K$ , we say the extension  $K(a)$  is *algebraic over  $K$* . If there is no such polynomial, we say the extension is *transcendental over  $K$* . For instance, if  $K$  is the field of rational numbers  $Q$  and  $a = \sqrt{2}$ , then  $a$  satisfies  $x^2 - 2 = 0$ . But if  $K = Q$  and  $a = \pi$ , then there is no such polynomial.

We'll consider the easier transcendental case first. If  $P(x)$  and  $Q(x)$  are polynomials with coefficients in  $K$  and  $Q(x)$  is not identically zero, then  $P(a)$  and  $Q(a)$  are certainly in the extension field  $K(a)$ . Moreover  $Q(a) \neq 0$  because  $a$  is transcendental. So  $\frac{P(a)}{Q(a)}$  is in this field. On the other hand, the set of all such elements is a field, and so

$$K(a) = \left\{ \frac{P(a)}{Q(a)} \mid P(x) \text{ and } Q(x) \in K[x] \right\}$$

This is not the end of the story, because it is conceivable that polynomials  $P_1, P_2, Q_1$ , and  $Q_2$  exist with

$$\frac{P_1(a)}{Q_1(a)} = \frac{P_2(a)}{Q_2(a)}$$

But if so, then  $P_1(a)Q_2(a) - P_2(a)Q_1(a) = 0$  and so the polynomial  $P_1(x)Q_2(x) - P_2(x)Q_1(x)$  vanishes on  $x = a$ . Since  $a$  is transcendental, this polynomial must be trivial and  $P_1(x)Q_2(x) = P_2(x)Q_1(x)$ .

The quotient field of the integral domain  $K[x]$  is by definition the set of all  $\frac{P(x)}{Q(x)}$  with two such elements identified exactly if  $P_1(x)Q_2(x) = P_2(x)Q_1(x)$ . It is called *the field of rational functions over  $K$  in  $x$* . We have therefore proved

**Theorem 8** *If  $K$  is a subfield of a larger field which contains  $a \notin K$ , and if  $a$  is transcendental over  $K$ , then the field  $K(a)$  is canonically isomorphic to the field of rational functions over  $K$  in  $x$ .*

### 3.5 Algebraic Extensions; Root Fields

We now study the more interesting case of  $K(a)$  when  $a$  is algebraic over  $K$ . It is useful to keep in mind the special case when  $K = Q$  and  $a = \sqrt{2}$  studied at the end of section ten.

Since  $a$  is algebraic, it satisfies a polynomial  $P(x)$  with coefficients in  $K$ . Factoring this polynomial if necessary, we can assume  $P$  is irreducible. Of all such irreducible polynomials

vanishing on  $a$ , select  $P$  to have smallest degree and leading coefficient 1. We call this  $P$  the *minimal polynomial of  $a$* .

The minimal polynomial divides any polynomial over  $K$  which vanishes on  $a$ . Indeed if  $Q(x)$  is nontrivial and vanishes on  $a$ , we can divide  $Q(x)$  by  $P(x)$  to obtain

$$Q(x) = A(x)P(x) + R(x)$$

with  $R(x)$  either zero or else of smaller degree than  $P$ . Since  $R$  vanishes on  $a$  and  $P$  is minimal, we conclude that  $R$  is identically zero and thus that  $P$  divides  $Q$ . If  $P$  and  $Q$  are both minimal, they both have leading coefficient one and must be equal.

Suppose the minimal polynomial has degree  $n$  and consider the set of expressions

$$K(a) = \{ r_0 + r_1a + r_2a^2 + \dots + r_{n-1}a^{n-1} \mid r_i \in K \}$$

This set contains  $K$  and  $a$  and is clearly closed under addition. It is also closed under multiplication, for if we multiply two such expressions, we get another expression of the same form, but with degree possibly larger than  $n - 1$ . However,  $a$  is a root of its minimal polynomial  $P(x) = x^n + k_{n-1}x^{n-1} + \dots + k_0$ , so we can write

$$a^n = -k_0 - k_1a - \dots - k_{n-1}a^{n-1}$$

Thus  $a^n \in K(a)$  and by the same technique all higher powers of  $a$  belong to this set.

Finally, the set  $K(a)$  is also closed under division by the following lemma (justifying our notation):

**Lemma 2** *Let  $K$  be a field, and suppose  $K \subset L$  where  $L$  is an integral domain which is finite dimensional over  $K$ . Then  $L$  contains inverses of all nonzero elements and thus is itself a field.*

*Proof:* Let  $b \in L$ ,  $b \neq 0$ . Consider the map  $L \rightarrow L$  given by multiplication by  $b$ . This is a linear transformation over  $K$ . Since  $L$  is an integral domain, the map is one-to-one. By linear algebra, it is thus onto, so there exists  $c \in L$  with  $bc = 1$ .

*Remark:* Notice that this simple lemma replaces the calculation at the end of section 3.3 showing that inverses exist in that special case.

### 3.6 Irreducible Polynomials over $Q$

In applications of Galois theory, we often assume the ground field is the field  $Q$  of rational numbers. We need an ample supply of irreducible polynomials over this field to give examples of the root fields just introduced. Such polynomials are usually found using two well-known results:



**Theorem 9 (Gauss)** *If a polynomial with integer coefficients factors over  $Q$ , then it factors over the integers.*

**Theorem 10 (Eisenstein)** *Let  $P(X) = a_0X^n + a_1X^{n-1} + \dots + a_n$  be a polynomial with integer coefficients. Fix a prime  $p$  and suppose*

1.  $p$  does not divide  $a_0$
2.  $p$  divides  $a_i$  for  $i > 0$
3.  $p^2$  does not divide  $a_n$

*Then  $P(X)$  is irreducible over the rationals.*

*Proof:* Call a polynomial over the integers *primitive* if no prime divides all of its coefficients. If  $P(X)$  is a nonzero polynomial over the integers, we can factor out integers that divide all coefficients, and eventually write  $P(X) = c_P \tilde{P}$  where  $c_P$  is a positive integer and  $\tilde{P}$  is primitive. This representation is clearly unique.

If  $P(X)$  is a nonzero polynomial over  $Q$ , we can similarly factor out all denominators of individual terms and eventually write  $P(X) = c_P \tilde{P}$  where  $c_P$  is a positive rational and  $\tilde{P}$  is a primitive polynomial over the integers. This representation is again unique.

Now we come to the key point: the product of two primitive polynomials is again primitive. Indeed consider such a product

$$(a_0X^m + a_1X^{m-1} + \dots + a_m) (b_0X^n + b_1X^{n-1} + \dots + b_n)$$

If the result is not primitive, there is a prime  $p$  which divides all terms of the product. Since  $a_0X^m + \dots + a_m$  is primitive, there is a  $j$  such that  $p$  does not divide  $a_j$  but divides  $a_{j+1}, \dots, a_m$ . Note that  $j$  might be  $m$ . Similarly there is a  $k$  such that  $p$  does not divide  $b_k$  but does divide  $b_{k+1}, \dots, b_n$ . Consider the term

$$\dots + a_{j-1}b_{k+1} + a_jb_k + a_{j+1}b_{k-1} + \dots$$

This coefficient in the product is not divisible by  $p$  because every subterm is divisible by  $p$  except  $a_jb_k$ , contradicting our assumption on  $p$ .

We can now prove Gauss' result. Suppose that  $P(X)$  is a polynomial with integer coefficients which factors over the rationals as  $Q(X)R(X)$ . Then

$$c_P \tilde{P} = c_Q \tilde{Q} c_R \tilde{R} = (c_Q c_R) \tilde{Q} \tilde{R}.$$

Since  $\tilde{Q} \tilde{R}$  is primitive and this representation is unique, we find that  $\tilde{P} = \tilde{Q} \tilde{R}$ . Since  $P$  has integer coefficients,  $c_P$  is an integer and  $P = c_P \tilde{P} = c_P \tilde{Q} \tilde{R} = (c_P \tilde{Q}) \tilde{R}$ .

We propose to prove Eisenstein's result using the notation of the proof just given. Suppose  $P$  is not irreducible over the rationals. Then it can be factored as a product of polynomials with integer coefficients:

$$\dots + a_{j-1}b_{k+1} + a_j b_k + a_{j+1}b_{k-1} + \dots$$

If the constant coefficient of the product is divisible by  $p$  but not  $p^2$ , then  $p$  must divide exactly one of  $a_m b_n$ . Say  $p$  divides  $a_m$  and  $p$  does not divide  $b_n$ . Find  $k$  such that  $p$  does not divide  $a_k$  but  $p$  divides  $a_{k+1}, \dots, a_m$ . Such a  $k$  must exist, else  $p$  would divide all  $a_i$  and thus the highest coefficient of the product. One of the coefficients of the product is then

$$a_k b_n + a_{k+1} b_{n-1} + \dots$$

and  $p$  does not divide this term because it divides all subterms except the first. So this must be the highest coefficient of the product, and  $k + n = 0$ . So the second term is constant and we really don't have a factorization.

*Example:* It follows from this result that there are root field extensions of the rational numbers with any degree. Indeed,  $X^n + 2X^{n-1} + \dots + 2$  is irreducible by Eisenstein's result and has degree  $n$ .

### 3.7 The Degree of a Field Extension

At the end of the previous section, we proved a result using linear algebra. This section is about the deeper structure behind that proof.

Suppose  $K \subset L$  is a field extension. By forgetting some of the axioms of  $L$ , we see that  $L$  is a vector space over  $K$ . The dimension of this vector space is called the *degree of  $L$  over  $K$*  and denoted  $[L : K]$ .

**Corollary 1** *If  $K \subset K(a)$  is an extension, then  $[K(a) : K]$  is infinite when  $a$  is transcendental over  $K$ , and finite if  $a$  is algebraic over  $K$ . In this second case,  $[K(a) : K]$  is the degree of the minimal polynomial of  $a$ .*

*Proof:* This is an immediate consequence of previous results.

**Theorem 11** *Suppose  $K \subset L \subset M$ . Then*

$$[M : K] = [M : L][L : K]$$

*Proof:* Suppose  $[M : L]$  has degree  $p$  and let  $m_1, \dots, m_p$  be a basis for  $M$  over  $L$ . Suppose  $[L : K]$  has degree  $q$  and let  $l_1, \dots, l_q$  be a basis of  $L$  over  $K$ . We will prove that  $\{m_i l_j\}$  is a basis of  $M$  over  $K$ . The theorem immediately follows.

We first prove these elements linearly independent. Suppose  $\sum_{i,j} a_{ij} l_i m_j = 0$  with  $a_{ij} \in K$ . Then  $\sum_i (\sum_j a_{ij} l_i) m_j = 0$ . Since the  $m_j$  are linearly independent over  $L$ ,  $(\sum_i a_{ij} l_i) = 0$  for each fixed  $j$ . Since the  $l_i$  are linearly independent over  $K$ , each  $a_{ij} = 0$ .

We now prove these elements generate  $M$  over  $K$ . Let  $m \in L$ . Since the  $m_j$  form a basis of  $M$  over  $L$ , we can write  $m = \sum_j b_j m_j$  for coefficients  $b_j \in L$ . Since the  $l_i$  form a basis of  $L$  over  $K$ , each  $b_j$  can be written  $b_j = \sum_i a_{ij} l_i$ . So  $m = \sum_{i,j} a_{ij} l_i m_j$ .

### 3.8 Existence of Root Fields

This section contains an important new way to look at the results of section 3.5. In that section, we started with a “universal field” like the complex numbers and an element  $a$  in this field, produced a minimal polynomial  $P(x)$ , and constructed a larger extension field  $K(a)$  containing  $K$  and the root  $a$  of  $P$ .

There is another way to think of this result. Suppose we do not have a universal field or a root  $a$ , but instead start with a field  $K$  and an irreducible polynomial  $P$  over  $K$ . We can then construct a new field  $L$ , whose elements are abstract symbols rather than complex numbers, with the properties that  $K \subset L$ , that  $P(x)$  has a root in  $L$ , and that  $L$  is generated by this root and  $K$ . We call this the *abstract root field associated with  $P$* .

The construction is very simple. Start with the set of all polynomials with coefficients in  $K$ . Denote this set by  $K[X]$ . It is a commutative integral domain. Let  $\mathcal{J}$  be the ideal in this ring generated by the polynomial  $P(X)$ . Thus it consists of all polynomials of the form  $P(X)R(X)$  for arbitrary polynomials  $R$ . It is easy to check that  $\mathcal{J}$  is a prime ideal, and therefore the quotient ring  $K[X]/\mathcal{J}$  is a field. This is our new field  $L$ . So

$$L = K[X]/\mathcal{J}$$

It is easy to make this construction look more concrete. If  $A(X) \in K[X]$  is a polynomial, we can divide  $A$  by  $P$  to obtain  $A(X) = P(X)Q(X) + R(X)$  where the degree of  $R$  is smaller than the degree of  $P$ . In the quotient field,  $A$  is equivalent to  $R$  because their difference is in the ideal  $\mathcal{J}$ . So every element in the quotient is equivalent to exactly one polynomial of degree smaller than  $n$ :

$$L = \{ k_0 + k_1 X + k_2 X^2 + \dots + k_{n-1} X^{n-1} \mid k_i \in K \}$$

Notice that this expression is essentially the same as the expression for a general element of the root field  $K(a)$ :

$$L = \{ k_0 + k_1a + k_2a^2 + \dots + k_{n-1}a^{n-1} \mid k_i \in K \}$$

Clearly addition in both fields is “the same”. Remarkably, multiplication is also “the same”. A special case will make this clear. Let us compare the operation of multiplying an arbitrary element of the abstract field by  $X$  and the operation of multiplying an arbitrary element of the root field by  $a$ . We then get two expressions

$$k_0X + k_1X^2 + \dots + k_{n-1}X^n$$

and

$$k_0a + k_1a^2 + \dots + k_{n-1}a^n$$

The trouble is that the first expression contains  $X^n$  and the second contains  $a^n$ . In the abstract case,  $P(X) = X^n + p_1X^{n-1} + \dots + p_n \in \mathcal{J}$  so we can write

$$X^n = -p_n - p_{n-1}X - \dots - p_1X^{n-1}$$

In the second case,  $a$  has minimal polynomial  $P(X)$ , so  $P(a) = a^n + p_1a^{n-1} + \dots + p_n$  and we can write

$$a^n = -p_n - p_{n-1}a - \dots - p_1a^{n-1}$$

These formulas allow us to write the product as a linear combination of  $1, X, \dots, X^{n-1}$  in the first case, and as a linear combination of  $1, a, \dots, a^{n-1}$  in the second case, and we get completely analogous results in the two cases.

It remains to show that the polynomial  $P$  has a root in the abstract case. Indeed  $X \in L$ , so we can insert this element into  $P$  to obtain  $P(X)$ . Since  $P(X) \in \mathcal{J}$ , the result is zero and  $X$  is a root of  $P$ .

So we have constructed an abstract field  $L$ , with  $K \subset L$  such that  $P$  has a root in  $L$ .

Clearly this abstract field is isomorphic to the concrete root field  $K(a)$  we constructed in a previous section. This isomorphism deserves a separate section.

*Optional remark:* Some people find the proof that  $X$  is a root of  $P$  in  $L$  unconvincing. These people have a point because our notation is ambiguous.

To clarify the argument, write the initial polynomial as  $P(Y)$  and let the initial integral domain be  $K[X]$ . Here  $X$  and  $Y$  are two abstract symbols. Introduce the quotient field  $K[X]/\mathcal{J}$ . Distinguish elements of  $K[X]$  and elements of the quotient field by writing

$$\overline{Q(X)}$$

to indicate the element of the quotient field represented by a polynomial  $Q(X) \in K[X]$ .

We want to form  $P(\overline{X})$  by substituting  $\overline{X}$  for  $Y$  in the formula for  $P(Y)$ . Clearly this can be done using representatives, so  $P(X)$  represents the element  $P(\overline{X}) \in L$ .

But  $P(X) \in \mathcal{J}$ , so  $P(X)$  represents zero, and thus

$$P(\overline{X}) = 0$$

(These overline symbols rapidly become distracting, so I urge the reader to adopt the more relaxed treatment in these notes.)

### 3.9 Isomorphism and Uniqueness

If  $K$  and  $L$  are fields, an *isomorphism*  $\varphi : K \rightarrow L$  is a one-to-one and onto map which preserves addition and multiplication. That is

- $\varphi(x + y) = \varphi(x) + \varphi(y)$
- $\varphi(xy) = \varphi(x)\varphi(y)$

It is easy to prove that such a map preserves 0, 1, subtraction, and division.

Suppose  $K$  is a subfield of a universal field, which we suppose is the complex numbers. Suppose  $P(x)$  is a polynomial irreducible over  $K$ . Since the complex numbers are algebraically closed, we can find a complex root  $a$  of  $P(x)$ , and form  $K(a)$ . We can also construct the abstract field  $L$  of the previous section. Comparing sections 12 and 14, it is clear that we have proved

**Theorem 12** *There is a unique isomorphism  $\varphi : L \rightarrow K(a)$  which is the identity on  $K$  and takes  $X \in L$  to  $a$ .*

This theorem has a very important corollary:

**Corollary 2** *Suppose  $K$  is a subfield of the complex numbers,  $P(x)$  is a polynomial irreducible over  $K$ , and  $a$  and  $b$  are complex roots of  $P$ . Then there is a unique isomorphism  $K(a) \rightarrow K(b)$  which is the identity on  $K$  and maps  $a$  to  $b$ .*

For example, consider  $P(x) = x^2 - 2$ , which has  $\pm\sqrt{2}$  as roots. In this case,  $K(\sqrt{2})$  and  $K(-\sqrt{2})$  contain the same elements, but the isomorphism sends  $k_1 + k_2\sqrt{2}$  to  $k_1 - k_2\sqrt{2}$ .

Consider  $P(x) = x^3 - 2$ . If  $\theta = e^{\frac{2\pi i}{3}}$ , the roots are  $a = \sqrt[3]{2}$ ,  $b = \sqrt[3]{2}\theta$ , and  $c = \sqrt[3]{2}\theta^2$ , and the fields  $K(a)$ ,  $K(b)$ ,  $K(c)$  do not contain the same elements. Geometrically as subsets of the plane, these fields look different; the first is a subset of the real numbers, while the other two are spread out across the entire plane. But as abstract fields, they are isomorphic. The reader might like to check this directly.

### 3.10 Putting It All Together

It is useful to reread sections ten through fifteen, because they illustrate two ways of thinking about fields which we will employ often. In the first few sections, we thought of fields contained in the complex numbers. Our polynomials often had rational coefficients, and we were particularly interested in fields  $Q \subset K \subset C$  such that  $K$  could be obtained from  $Q$  by adding a finite number of algebraic elements.

But we might also be interested in extensions of  $Z_p$ . In this case, we haven't produced an analogue of the complex numbers. It is possible to prove that any field can be extended to an algebraically closed field, but the proof involves the axiom of choice and can be very abstract. We prefer to avoid this approach, and section fourteen allows us to do that.

Here is a simple example to make this concrete. Consider the field  $Z_2$ . It turns out that this field is contained in larger finite fields, and the number of elements in each such field is a power of two. Let us construct the simplest, a field with four elements.

To use the technique of section fourteen, we need to start with an irreducible polynomial  $P(x)$ . Notice that when this  $P$  is quadratic, the resulting  $L = \{ r_0 + r_1a \}$  where the  $r_i \in Z_2$ , and thus has four elements. So we must find an irreducible monic polynomial of degree two.

The complete list of such polynomials is

- $x^2$
- $x^2 + 1 = (x + 1)^2$
- $x^2 + x = x(x + 1)$
- $x^2 + x + 1$

The first three are reducible, but the last isn't. Let  $P(x) = x^2 + x + 1$ . Then we obtain a field with four elements, represented by the four polynomials  $0$ ,  $1$ ,  $x$ , and  $1 + x$ . It is trivial to write down an addition table. In the multiplication table,  $0$  times anything is zero and  $1$  times anything is that thing, and the only non-trivial products are

- $x \circ x = x^2 = x + 1$

- $x \circ 1 + x = x + x^2 = 1$
- $1 + x \circ 1 + x = x^2 + 1 = x$

For example,  $x \circ x = x^2$ , but since  $x^2 + x + 1 = 0$  we have  $x^2 = -x - 1$  and since  $-1 = 1$  in  $Z_2$ , this equals  $x + 1$ .

## Chapter 4

# Splitting Fields

### 4.1 Factoring $P$

Recall that if a polynomial  $P(x)$  has a root  $a$ , then it can be factored as  $P(x) = (x-a)Q(x)$ . Indeed, dividing by  $x-a$  gives  $P(x) = (x-a)Q(x) + R(x)$  where  $R$  has smaller degree than  $x-a$  and thus is a constant. Setting  $x=a$ , we discover that the constant is zero.

In the previous sections, we started with an irreducible  $P$  over  $K$ , and constructed a field  $K(a)$  containing a root  $a$  of  $P$ . Thus  $P$  factors over  $K(a)$ , but the nature of this factorization is unpredictable. In some cases,  $P$  suddenly factors completely, while in others the term  $x-a$  factors out and the remaining polynomial  $Q(x)$  is irreducible.

For example, consider  $x^3 - 2$  over  $\mathbb{Q}$ . One extension field is generated by  $\sqrt[3]{2}$ , but the two remaining roots are complex and thus not in this field. So  $P(x)$  factors in  $\mathbb{Q}(\sqrt[3]{2})$  into

$$(x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}x + \sqrt[3]{2}^2)$$

On the other hand, consider  $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$ . Ignoring the trivial factor  $x-1$ , we let  $P(x) = x^4 + x^3 + x^2 + x + 1$ . Later on we prove that this polynomial is irreducible. Notice that the complex roots of this polynomial are  $\theta = e^{\frac{2\pi i}{5}}, \theta^2, \theta^3$ , and  $\theta^4$ . Thus  $P$  factors completely in  $\mathbb{Q}(\theta)$  as

$$(x - \theta)(x - \theta^2)(x - \theta^3)(x - \theta^4)$$



## 4.2 The Splitting Field

**Definition 2** Let  $K$  be a field and  $P(x)$  a polynomial over  $K$ , not necessarily irreducible. An extension field  $L$  of  $K$  is called a splitting field of  $P(x)$  over  $K$  if  $P(x)$  factors completely into linear terms over  $L$ , so  $P(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$  and if  $L$  is the smallest subfield of  $L$  containing  $K$  and the  $\alpha_i$ .

Suppose as before that  $K$  is a subfield of the complex numbers and  $P(x)$  is an irreducible polynomial over  $K$ . Let  $a_1, \dots, a_n$  be the complex roots of  $P$ , which exist by the fundamental theorem of algebra. Then  $K(a_1, a_2, \dots, a_n)$  is clearly a splitting field, and is contained in  $C$ .

Such a field exists because it can be defined as the intersection of all subfields containing these elements. Clearly this  $K(a_1, a_2, \dots, a_n)$  is unique.

**Theorem 13** Every element of  $L = K(a_1, a_2, \dots, a_n)$  is algebraic over  $K$ . In particular, if  $P(x)$  is irreducible of degree  $n$ , then

$$n \leq [L : K] \leq n!$$

*Proof:* We can construct  $K(a_1, \dots, a_n)$  as a chain of root fields, each of finite degree over  $K$ :

$$K \subset K(a_1) \subset K(a_1, a_2) \subset \dots \subset K(a_1, a_2, \dots, a_n)$$

so  $K(a_1, \dots, a_n)$  has finite degree over  $K$ . It immediately follows that each element is algebraic.

If  $P(x)$  is irreducible,  $[K(a_1) : K] = n$ , so the first inequality is clear.

Factor  $P(x)$  over  $K(a_1)$  and write  $P(x) = (x - a_1)Q(x)R(x)$  where  $a_2$  is a root of the irreducible  $Q(x)$  over  $K(a_1)$ . The degree of  $Q(x)$  is at most  $n - 1$ , so  $[K(a_1, a_2) : K(a_1)]$  is at most  $n - 1$ . So  $[K(a_1, a_2) : K] = [K(a_1, a_2) : K(a_1)][K(a_1) : K]$  is at most  $n(n - 1)$ .

Continue.

*Remark:* We have much less control over the splitting field than we had over the root field. For the root field, we know the exact degree over  $K$  and the exact structure. For the splitting field, we only know the degree within broad bounds, and we do not know the algebraic structure, or a basis over  $K$ .

We can also construct the splitting field in the situation when there is no large containing field like the complex numbers. Let  $Q_1(x)$  be an irreducible factor of  $P(x)$  over  $K$ , and construct the root field of  $Q_1$  as in section fourteen. Call it  $K_1$ . It contains a root of  $Q_1(X)$

and thus a root of  $P(X)$ . Earlier we called this root  $\bar{X}$ . But now the actual form of the elements of  $K_1$  is irrelevant, so just call the root  $a_1$ .

Factor  $P(x)$  over  $K_1$ . One term is  $x - a_1$ , and there may be other linear factors corresponding to other roots of  $P(x)$  which are already in  $K_1$ . If all factors are linear, we are done. Otherwise let  $Q_2(x)$  be an irreducible factor of  $P(x)$  over  $K_1$ . Construct the root field of  $Q_2(x)$  over  $K_1$  and call it  $K_2$ . Then  $K_2$  contains an additional root of  $P(x)$  not in  $K_1$ . Call it  $a_2$ .

Continue in this manner until a field is constructed containing all roots of the original  $P(x)$ .

**Theorem 14** *The splitting field is unique up to isomorphism.*

*Proof:* This is obvious in the situation when we have an enclosing field like  $C$ . But a proof is required if we use the abstract construction to obtain a splitting field, because the order in which we add roots isn't clear.

Our proof will gradually construct the required isomorphism. We start with two roots  $\theta_1$  and  $\theta_2$ . By previous results, the root fields  $K(\theta_1)$  and  $K(\theta_2)$  are isomorphic. We extend this isomorphism to larger and larger subfields until we get an isomorphism between the complete splitting fields.

Our construction still gives something interesting when there is an enclosing field  $C$ , namely an isomorphism of the splitting field to itself mapping  $\theta_1$  to  $\theta_2$ . An isomorphism from a field to itself is called an *automorphism*. Automorphisms are the central tool in Galois theory, and our proof of the above theorem will become the main tool used to construct them.

Because of the importance of this proof, we will give it in an unusual way. We give the general proof in the next section. In the following section, we go through the proof again, this time studying the special case  $P(X) = X^3 - 2$ , to make certain the central ideas are clear.

### 4.3 Proof that Splitting Fields Are Unique

Suppose we have two splitting fields  $L_1$  and  $L_2$  of a polynomial  $P(x)$  over  $K$ . If  $P(x)$  factors into linear terms over  $K$ , then  $L_1 = L_2 = K$  and we are done.

Otherwise select an irreducible factor  $Q(x)$  of  $P(x)$  over  $K$  of degree at least two. Both  $L_1$  and  $L_2$  contain all roots of  $P(x)$ ; choose one such root in each, calling them  $a_1$  and  $b_1$ . Let  $K(a_1) \subset L_1$  and  $K(b_1) \subset L_2$  be the root fields, that is, the smallest subfields containing  $K$  and either  $a_1$  or  $b_1$ . By previous results, these fields are isomorphic by an isomorphism

which is the identity on  $K$  and takes  $a_1$  to  $b_1$ , because both fields are isomorphic to the “abstract root field” constructed in section fourteen. Let  $\varphi : K(a_1) \rightarrow K(b_1)$  be such an isomorphism.

Factor  $P(x)$  over  $K(a_1)$ . The factors are polynomials with coefficients in  $K(a_1)$ , and consequently  $\varphi$  can be extended to this polynomial product and gives a corresponding factorization of  $P(x)$  over  $K(b_1)$ . These factors are still irreducible, for otherwise we could find an additional factorization over  $K(b_1)$  and then pull it back via  $\varphi^{-1}$  to a factorization over  $K(a_1)$ .

If all of the factors are linear, then all roots of  $P(x)$  belong to  $K(a_1)$  and  $K(b_1)$ , so  $L_1 = K(a_1)$  and  $L_2 = K(b_1)$  and they are isomorphic.

Otherwise let  $Q_1(x)$  be an irreducible non-linear factor of  $P(x)$  over  $K(a_1)$  and let  $Q_2(x)$  be the corresponding irreducible non-linear factor of  $P(x)$  over  $K(b_1)$ . Since  $Q_1$  is a factor of  $P$  and  $P$  factors completely in  $L_1$ , there is a root  $a_2$  of  $Q_1$  in  $L_1$ . Similarly there is a root  $b_2$  of  $Q_2$  in  $L_2$ . Form the root fields  $K(a_1) \subset K(a_1, a_2) \subset L_1$  and  $K(b_1) \subset K(b_1, b_2) \subset L_2$ .

We now claim that we can extend  $\varphi$  to an isomorphism  $\varphi : K(a_1, a_2) \rightarrow K(b_1, b_2)$  which is the identity on  $K$  and takes  $a_1$  to  $b_1$  and  $a_2$  to  $b_2$ . If so, the rest of the proof will be obvious, for we can again factor  $P(x)$  over  $K(a_1, a_2)$  and proceed as before. Eventually, we will have isomorphic  $K(a_1, a_2, \dots, a_k)$  and  $K(b_1, b_2, \dots, b_k)$  over which  $P$  factors into linear terms. It will follow that the first is  $L_1$  and the second is  $L_2$  and they are isomorphic.

So we need a lemma. But this lemma will be applied again and again in later sections, so we promote it to a theorem. QED modulo:

**Theorem 15** *Let  $\varphi : K_1 \rightarrow K_2$  be an isomorphism, and let  $P_1(X)$  be irreducible over  $K_1$  and  $P_2(X)$  be the image of this polynomial over  $K_2$ , obviously still irreducible.*

*Suppose  $K_1 \subset L_1$  and suppose  $a \in L_1$  is a root of  $P_1(X)$ . Suppose  $K_2 \subset L_2$  and  $b \in L_2$  is a root of  $P_2(X)$ . Form  $K_1 \subset K_1(a) \subset L_1$  and  $K_2 \subset K_2(b) \subset L_2$ .*

*Then  $\varphi$  extends to an isomorphism  $K_1(a) \rightarrow K_2(b)$  mapping  $a$  to  $b$ .*

*Proof:* A typical element of  $K_1(a)$  is  $k_0 + k_1a + k_2a^2 + \dots + k_{n-1}a^{n-1}$  where the  $k_i \in K_1$  and  $n$  is the degree of  $P_1(X)$ . A typical element of  $K_2(b)$  is  $k_0 + k_1b + k_2b^2 + \dots + k_{n-1}b^{n-1}$  where the  $k_i \in K_2$  and  $n$  is the degree of  $P_2(X)$ . Define our extended isomorphism to be

$$k_0 + k_1a + \dots + k_{n-1}a^{n-1} \rightarrow \varphi(k_0) + \varphi(k_1)b + \dots + \varphi(k_{n-1})b^{n-1}$$

This map is clearly one-to-one and onto, so it suffices to prove that it respects multiplication. But the multiplication rule in  $K_1(a)$  is derived from the polynomial  $P_1(X) = X^n + r_1X^{n-1} + \dots + r_n$  by repeated applications of the rule

$$a^n = -r_1a^{n-1} - \dots - r_n$$

and the multiplication rule in  $K_2(X)$  is derived from the polynomial  $P_2(X) = \varphi(P_1(X)) = X^n + \varphi(r_1)X^{n-1} + \dots + \varphi(r_n)$  by repeated applications of the rule

$$b^n = -\varphi(r_1)b^{n-1} - \dots - \varphi(r_n)$$

and our extended isomorphism maps the first of these to the second. QED.

**Corollary 3** *Let  $L$  be the splitting field of an irreducible polynomial  $P(X)$ , and let  $\theta_1$  and  $\theta_2$  be roots of  $P$  in  $L$ . Then there is an automorphism  $\varphi$  of  $L$  mapping  $\theta_1$  to  $\theta_2$ .*

*Proof:* This follows from the proof of theorem 14, since we began the proof by constructing an isomorphism of root fields  $K(\theta_1) \rightarrow K(\theta_2)$  and ended by extending it to an automorphism of  $L$ .

#### 4.4 Uniqueness of Splitting Fields, and $P(X) = X^3 - 2$

In this section we repeat the proof in the previous section, this time using the specific example  $P(X) = X^3 - 2$  over the base field  $Q$ .

It is convenient to simplify the notation before starting the proof. Let  $\alpha = \sqrt[3]{2}$  and  $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ . Here  $\alpha$  is real. The roots of our equation are  $\theta_1, \theta_2, \theta_3 = \alpha, \omega\alpha$ , and  $\omega^2\alpha$ .

Let's try to guess a basis for the splitting field. This field must contain  $\alpha, \omega\alpha$ , and  $\omega^2\alpha$ . It must contain quotients, so it also contains  $\omega$  and  $\omega^2$ . It must be closed under multiplication, so it contains powers of  $\alpha$ . Since  $\alpha^3 = 2$ , we only need  $\alpha$  and  $\alpha^2$ . This leads to an initial guess that a basis is  $\{1, \omega, \omega^2, \alpha, \omega\alpha, \omega^2\alpha, \alpha^2, \omega\alpha^2, \omega^2\alpha^2\}$ .

However,  $\omega^3 - 1 = 0 = (\omega - 1)(\omega^2 + \omega + 1)$  and so  $\omega^2 = -1 - \omega$ . Removing dependencies, a better guess is that a basis is  $\{1, \alpha, \alpha^2, \omega, \omega\alpha, \omega\alpha^2\}$ . This indeed turns out to be a basis. Without giving details, we could prove it by showing that these elements are linearly independent, and that the set of linear combinations is closed under multiplication by each basis element. Division then takes care of itself by an earlier argument.

We won't use this guess because it follows from our more general considerations below.

Our  $P$  is already irreducible over  $Q$ . Let us arbitrarily select  $\theta_1 = \alpha$  and  $\theta_2 = \omega\alpha$ . Form the root fields

$$\begin{aligned} Q(\theta_1) &= \{ q_0 + q_1\alpha + q_2\alpha^2 \} \\ Q(\theta_2) &= \{ q_1 + q_1\omega\alpha + q_2\omega^2\alpha^2 \} \end{aligned}$$

They are isomorphic by a map  $\varphi$  fixing rational numbers and sending  $\alpha$  to  $\omega\alpha$ .

Now factor  $P(X)$  over  $Q(\theta_1)$ :

$$X^3 - 2 = (X - \alpha)(X^2 + \alpha X + \alpha^2)$$

The splitting field  $L_1$  can be obtained from  $Q(\theta_1)$  by adding a root of the quadratic term, which is irreducible over  $Q(\theta_1)$  because its roots are complex and  $Q(\theta_1)$  only contains real numbers. Once we add one such root, the polynomial  $P(X)$  splits completely over  $L_1$ .

Since we know all roots of  $X^3 - 2$ , it is easy to guess the factorization of the quadratic term:

$$X^2 + \alpha X + \alpha^2 = (X - \omega\alpha)(X - \omega^2\alpha)$$

This is easily checked by direct multiplication.

Let us arbitrarily pick one of the roots of the quadratic term, say  $\omega\alpha$ . We can then write down the full splitting field starting with  $Q(\theta_1)$  using our standard root construction:

$$Q(\alpha, \omega\alpha) = \{ (q_0 + q_1\alpha + q_2\alpha^2) + (q_3 + q_4\alpha + q_5\alpha^2)\omega\alpha \}$$

We now repeat this construction over  $Q(\theta_2)$ . Notice that our factorization of  $X^3 - 2$  does not make sense over  $Q(\theta_2)$ ; for instance example,  $\alpha \notin Q(\theta_2)$ . But  $\varphi$  maps  $Q(\theta_1)$  to  $Q(\theta_2)$ , so it maps polynomials over  $Q(\theta_1)$  to polynomials over  $Q(\theta_2)$ . Consequently, the corresponding factorization over  $Q(\theta_2)$  is

$$X^3 - 2 = (X - \varphi(\alpha))(X^2 + \varphi(\alpha)X + \varphi(\alpha^2)) = (X - \omega\alpha)(X^2 + \omega\alpha + \omega^2\alpha^2)$$

Therefore, another way to get the full splitting field is to start with  $Q(\theta_2)$  and add a root of  $X^2 + \omega\alpha + \omega^2\alpha^2$ .

It is easy to find the desired roots without resorting to the quadratic formula because we know all the roots of  $X^3 - 2$ ; the roots of  $X^2 + \alpha X + \alpha^2$  over  $Q(\theta_2)$  are  $\alpha$  and  $\omega^2\alpha$ , as is easily checked. Let us arbitrarily pick the root  $\omega^2\alpha$ . Then our standard root construction gives another version of the splitting field

$$Q(\omega\alpha, \omega^2\alpha) = \{ (q_0 + q_1\omega\alpha + q_2\omega^2\alpha^2) + (q_3 + q_4\omega\alpha + q_5\omega^2\alpha^2)\omega^2\alpha \}$$

The map  $\varphi$  was originally chosen to map  $\alpha \rightarrow \omega\alpha$ . We extend it to also map the root  $\omega\alpha$  of  $X^2 + \alpha X + \alpha^2$  to the root  $\omega^2\alpha$  of  $X^2 + \omega\alpha + \omega^2\alpha^2$ . In the end, we obtain an isomorphism between our two representations of the splitting field of  $X^3 - 2$ :

$$(q_0 + q_1\alpha + q_2\alpha^2) + (q_3 + q_4\alpha + q_5\alpha^2)\omega\alpha \rightarrow (q_0 + q_1\omega\alpha + q_2\omega^2\alpha^2) + (q_3 + q_4\omega\alpha + q_5\omega^2\alpha^2)\omega^2\alpha$$

This isomorphism was our goal. But the previous formula gives more. It is easy to see directly that our two forms of the splitting field can be rewritten to have the basis we

guessed originally:  $\{ 1, \alpha, \alpha^2, \omega, \omega\alpha, \omega\alpha^2 \}$ . After doing this rewritting, our isomorphism becomes an automorphism of this field mapping  $\alpha$  to  $\omega\alpha$ .

Indeed, we can rewrite the isomorphism to get

$$q_0 + q_1\alpha + q_2\alpha^2 + q_3\omega\alpha + q_4\omega\alpha^2 + q_52\omega \rightarrow q_0 + q_1\omega\alpha + q_2(-\alpha^2 - \omega\alpha^2) + q_3(-\alpha - \omega\alpha) + q_4\alpha^2 + q_52\omega$$

and so our basis vectors map as follows

- $1 \rightarrow 1$
- $\alpha \rightarrow \omega\alpha$
- $\alpha^2 \rightarrow -\alpha^2 - \omega\alpha^2 = \omega^2\alpha$
- $\omega \rightarrow \omega$
- $\omega\alpha \rightarrow -\alpha - \omega\alpha = \omega^2\alpha$
- $\omega\alpha^2 \rightarrow \alpha^2$

Notice that this isomorphism permutes the roots of  $X^3 - 2$ ; indeed

$$\alpha \rightarrow \omega\alpha \rightarrow \omega^2\alpha \rightarrow \alpha$$

*Final Remark:* Our proof that splitting fields are unique ends up finding a non-trivial isomorphism of the splitting field  $L$  for  $X^3 - 2$ . We made several random choices during the argument. Changing these choices would yield additional automorphisms of  $L$ . Indeed, a careful inventory of all the choices would give the complete automorphism group of  $L$ . We do this more generally in later sections.

## Chapter 5

# Finite Fields

### 5.1 Finite Fields

At this point, our field theory consists of three central ideas: the degree of an extension, the root field of an irreducible polynomial, and the splitting field of an arbitrary polynomial. In this intermission, we use these results to obtain a complete theory of finite fields.

If  $F$  is a finite field, then  $F$  contains a smallest subfield  $Z_p \subset F$ . Then  $F$  is a vector space over  $Z_p$  of some finite dimension  $n$ . It immediately follows that  $F$  contains  $p^n$  elements since every element has the form  $a_1f_1 + \dots + a_nf_n$  for coefficients  $a_i \in Z_p$  and a basis  $f_1, \dots, f_n$ . So

**Theorem 16** *Every finite field has order  $p^n$  for some prime  $p$  and positive integer  $n$ .*

*Remark:* Thus there might be fields of orders 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, but certainly cannot be fields of orders 6, 10, 12, 14, 15.

**Theorem 17** *If  $p$  is a prime and  $n$  is a positive integer, there is a field of order  $p^n$ . This field is unique up to isomorphism.*

*Proof:* The nonzero elements of a field  $F$  of order  $p^n$  form a multiplicative subgroup of order  $p^n - 1$ , so every element except the identity satisfies  $x^{p^n-1} - 1 = 0$ . Multiplying by  $x$ , every element of the field would satisfy  $P(x) = x^{p^n} - x = 0$ .

Let  $F$  be a splitting field for this polynomial over  $Z_p$ . Factor the polynomial. If two terms are equal, then  $P(x) = (x - a)^2Q(x) = x^{p^n} - x$ . Formally differentiating both sides,

$$2(x - a)Q(x) + (x - a)^2Q'(x) = p^n x^{p^n-1} - 1 = -1$$

Substituting  $x = a$  gives  $0 = -1$ , which is impossible. So all roots are distinct, and there are  $p^n$  roots.

On the other hand, the roots themselves form a field, and so the splitting field contains only roots and thus is a field with  $p^n$  elements. Indeed 0 and 1 are roots. If  $a$  and  $b$  are roots, then  $(ab)^{p^n} = a^{p^n} b^{p^n} = ab$ , so products of roots are roots. Also  $(a + b)^p = a^p + b^p$  because all remaining binomial coefficients are divisible by  $p$ . So  $(a + b)^{p^2} = ((a + b)^p)^p = (a^p + b^p)^p = a^{p^2} + b^{p^2}$ , and so forth, so  $(a + b)^{p^n} = a^{p^n} + b^{p^n} = a + b$ . So sums of roots are roots. If  $a$  is a root, then  $(-a)^{p^n} = (-1)^{p^n} a^{p^n} = (-1)^{p^n} a = -a$  if  $p$  is odd, and  $-1 = 1$  if  $p$  is even, so additive inverses of roots are roots. So the set of all roots is a finite integral domain, and thus automatically closed under inverses and a field.

In short, the splitting field is a finite field of order  $p^n$ . Since the splitting field is unique up to isomorphism, a field of order  $p^n$  is unique up to isomorphism.

**Theorem 18** *The multiplicative group of a finite field is cyclic and thus has a generator. More generally, any finite multiplicative subgroup of a field is cyclic.*

*Example:* Consider the field of order four described on page 20. The nonzero elements are  $x, x^2 = x + 1$ , and  $x^3 = x^2 + x = 1$ .

*Proof:* Many proofs are known. Here is one I like. Consider first the special case  $Z_2 + Z_4$ . We claim this cannot be a multiplicative subgroup of a field. For if it were, then every element would satisfy  $x^4 - 1 = 0$ , but this polynomial has at most 4 roots, not eight.

In general suppose  $G$  is a finite subgroup of a field. Any finite abelian group is a sum of groups of the form  $Z_{p^k}$ . Let  $p_1, \dots, p_s$  be the primes for  $G$  and let their highest order subgroups have orders  $p_1^{k_1}, \dots, p_s^{k_s}$ . Let  $m = p_1^{k_1} \cdot \dots \cdot p_s^{k_s}$ . Note that every element of  $G$  satisfies  $x^m = 1$  and there are at most  $m$  roots of this polynomial. Consequently the  $Z_{p_i^{s_i}}$  are the only groups involved, and there are no other summands with  $p_i$  of the same or lower order. But the sum of cyclic groups of prime power order, for *different* primes, is itself cyclic. QED.



## Chapter 6

# Beginning Galois Theory

### 6.1 Motivation for Galois Theory

Ruffini is the first mathematician to prove that the quadratic formula cannot be generalized to equations of degree five or higher, but his proof had a gap. This gap was filled by Abel, who gave the first complete proof. Galois gave a deeper proof by bringing ideas hidden in Abel's argument to the foreground. I don't know if Galois knew about Abel's or Ruffini's work.

Galois succeeded in finding a necessary and sufficient condition that a polynomial  $P(x)$  be solvable by radicals. To do this, he had to find a deep property that is true of some polynomials but not others. In our previous work, with each irreducible  $P(x)$  we associated two fields, the root field  $K(\theta)$  and the splitting field  $K(\theta_1, \dots, \theta_n)$ . All root fields had the same concrete structure, so it is unlikely that they reveal the required deep property of  $P$ . But splitting fields are not all the same. Sometimes adding one root gives all roots, while other times adding a root still leaves us far from the complete field. It turns out that the idea we need is hidden in the construction of the splitting field.

At the end of the previous chapter, we constructed the splitting field  $L$  of  $X^3 - 2$  and in the process found an automorphism of this field mapping the root  $\alpha$  to the root  $\omega\alpha$ . At several points in the argument, we made arbitrary choices. Changing these choices would have led to different automorphisms. In this particular case, there are six possible automorphisms.

If  $K \subset L$ , the automorphisms of  $L$  fixing each point of  $K$  form a group  $G$ , now called the *Galois group*. As it turns out, this group codifies the secrets of the splitting field. Galois proved that knowledge of the group completely determines whether or not the polynomial can be solved with radicals.

The mathematicians Ruffini, Abel, and Galois did not immediately introduce the Galois group. They were led to it by other considerations. It is useful to try to reconstruct their line of reasoning and we'll do that next.

Suppose we have an irreducible polynomial  $P(X)$  with rational coefficients. By the fundamental theorem of algebra, this polynomial can be factored over the complex numbers. Let  $\theta_1, \theta_2, \dots, \theta_n$  be its roots. We can calculate these numerically. For instance, perhaps  $\theta_1 = 0.43895 + 2.71834 i$ .

We only know approximate values for the roots. But the  $\theta_i$  are far from arbitrary complex numbers because  $\theta_1 + \dots + \theta_n$  and  $\theta_1\theta_2 \dots \theta_n$  are rational. More generally

$$(X - \theta_1)(X - \theta_2) \dots (X - \theta_n) = X^n - \left(\sum_i \theta_i\right)X^{n-1} + \left(\sum_{i < j} \theta_i\theta_j\right)X^{n-2} + \dots \pm (\theta_1\theta_2 \dots \theta_n)$$

and all of these coefficients are rational.

We can construct a large number of additional relations as follows. A polynomial  $S(X_1, \dots, X_n)$  is said to be *symmetric* if any permutation of the  $X_i$  gives the same polynomial. For instance, the following polynomials, known as the *elementary symmetric polynomials*, are symmetric:

- $\sigma_1 = X_1 + X_2 + \dots + X_n$
- $\sigma_2 = X_1X_2 + X_1X_3 + \dots + X_{n-1}X_n$
- $\sigma_i = \dots$
- $\sigma_n = X_1X_2 \dots X_n$

A key theorem about symmetric polynomials asserts that any symmetric polynomial can be written uniquely as a polynomial in the  $\sigma_i$ :

$$S(X_1, \dots, X_n) = Q(\sigma_1, \dots, \sigma_n)$$

If  $S$  has rational coefficients, so does  $Q$ . If  $S$  has integer coefficients, so does  $Q$ .

It follows that when  $S$  is *any* symmetric polynomial with rational coefficients, the expression  $S(\theta_1, \dots, \theta_n)$  is rational. This gives a gigantic number of relations satisfied by the roots of  $P$ . For example,  $\theta_1^2 + \theta_2^2 + \dots + \theta_n^2$  is rational.

For some polynomials, there are additional relations that arise for a completely different reason. Consider, for example,  $X^7 - 1$ . Factoring, we have

$$X^7 - 1 = (X - 1)(X^6 + X^5 + X^4 + X^3 + X^2 + X + 1)$$

We'll later show that the second factor is irreducible. The roots of the second factor are generated by  $\theta = e^{\frac{2\pi i}{7}}$  and equal  $\theta, \theta^2, \theta^3, \theta^4, \theta^5$  and  $\theta^6$ .

These roots satisfy new relations, including

- $\theta_1^2 - \theta_2 = 0$
- $\theta_1^3 - \theta_3 = 0$
- $\theta_1^4 - \theta_4 = 0$
- $\theta_1^5 - \theta_5 = 0$
- $\theta_1^6 - \theta_6 = 0$

The new relations don't come from symmetric polynomials because they aren't invariant under all permutations of the roots. For instance, if we leave  $\theta_1$  fixed and interchange  $\theta_2$  and  $\theta_3$ , the resulting equations are no longer all true.

On the other hand, some permutations leave these relations unchanged. Indeed, we can map  $\theta_1$  to any desired root, but if we want to retain the previous relations, then the remaining roots must map as in the rows of the following table:

$\theta_1$	$\theta_2$	$\theta_3$	$\theta_4$	$\theta_5$	$\theta_6$
$\theta_2$	$\theta_4$	$\theta_6$	$\theta_1$	$\theta_3$	$\theta_5$
$\theta_3$	$\theta_6$	$\theta_2$	$\theta_5$	$\theta_1$	$\theta_4$
$\theta_4$	$\theta_1$	$\theta_5$	$\theta_2$	$\theta_6$	$\theta_3$
$\theta_5$	$\theta_3$	$\theta_1$	$\theta_6$	$\theta_4$	$\theta_2$
$\theta_6$	$\theta_5$	$\theta_4$	$\theta_3$	$\theta_2$	$\theta_1$

For example, if we map  $\theta_1 \rightarrow \theta_2$  and we want the equation  $\theta_1^2 = \theta_2$  to be invariant, then the equation will become  $\theta_2^2 = ?$  where  $?$  is the image of  $\theta_2$ , so we must map  $\theta_2 \rightarrow \theta_4$ .

These symmetries of the roots are *transitive*; any root can be taken to any other root by a symmetry. It is tempting to say “of course, since these are seventh roots of unity, and these occur at the vertices of a regular 7-gon, which can be rotated at will.” *However!* When we factored, we removed 1 as a root, so geometrically there are no symmetries. The above symmetries are *algebraic*.

Each map from the top line to another line gives a permutation of the roots which preserves the equations listed earlier. Notice that the resulting six permutations form a group. The permutation  $\theta_1 \rightarrow \theta_2$  represented by the second line of the matrix has order three in the group, for applying it three times will send  $\theta_1 \rightarrow \theta_2 \rightarrow \theta_4 \rightarrow \theta_1$ . On the other hand, the permutation sending  $\theta_1 \rightarrow \theta_3$  has order six, since applying it six times sends  $\theta_1 \rightarrow \theta_3 \rightarrow \theta_2 \rightarrow \theta_6 \rightarrow \theta_4 \rightarrow \theta_5 \rightarrow \theta_1$ .

So surprisingly, the algebraic symmetries of the roots of  $X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$  form the group  $Z_6$ , not at all like the full group  $S_6$  of all permutations of the roots. Indeed  $Z_6$  has order 6, while  $S_6$  has order 720.

## 6.2 Motivation; Putting the Ideas Together

How, then, are the notions of splitting fields, relations among roots, and automorphisms related?

Suppose  $P(X)$  has rational coefficients. The splitting field  $L$  consists of all sums and products of the roots  $\theta_i$ . Therefore if  $Q(X_1, X_2, \dots, X_n)$  is a polynomial with rational coefficients,  $Q(\theta_1, \theta_2, \dots, \theta_n)$  is an element of  $L$ . Just as the root field is modeled by polynomials in  $Q[X]$ , the splitting field is modeled by polynomials in  $Q[X_1, X_2, \dots, X_n]$ .

In the root field case, the root  $\theta$  satisfies  $P(\theta) = 0$ , so the actual root field is

$$Q[X]/(\text{multiples of } P(X))$$

Similarly let us define  $\mathcal{J}$  to be the set of all polynomials  $R(X_1, X_2, \dots, X_n)$  with rational coefficients which satisfy  $R(\theta_1, \theta_2, \dots, \theta_n) = 0$ . This is the set of all *relations* satisfied by the roots. It is easy to see that  $\mathcal{J}$  is an ideal in the polynomial ring, and the splitting field is

$$L = \frac{Q[X_1, \dots, X_n]}{\mathcal{J}} = \frac{Q[X_1, \dots, X_n]}{(\text{ideal of relations } R)}$$

Among these relations are all symmetric polynomials. Galois' initial idea is that equations solvable by radicals will have many, many more relations than just these. Studying the relations is difficult because there are so many. Galois' second idea is that it is better to study the symmetry group  $G$  of the relations. By definition, a symmetry of the set of relations is a permutation  $\sigma$  of the  $X_i$  such that whenever  $R(X_1, \dots, X_n)$  is a relation,  $R(X_{\sigma(1)}, \dots, X_{\sigma(n)})$  is also a relation. As we get more relations, we get fewer symmetries. Often the group  $G$  is the full set  $S_n$  of all permutations, but for  $X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ , it is much smaller, just  $Z_6$ . Studying  $G$  is much easier than studying the set of relations because  $G$  is a finite group.

A permutation of the roots which leaves the relations invariant obviously induces an automorphism of the full  $L$ . Conversely, each automorphism of  $L$  must send a root of  $P$  to another root and thus must permute the roots; to induce an automorphism, this permutation must preserve the relations.

In this way, we are led from the construction of the splitting field to the relations among the roots, and thus to the automorphisms of the splitting field.

### 6.3 The Galois Group

**Definition 3** Let  $K \subset L$  be fields and suppose  $[L : K]$  is finite. The Galois group  $G$  of this extension is the set of all automorphisms of  $L$  which leave all elements of  $K$  individually fixed.

*Remark:* Unfortunately, this group may give no information. For example, consider  $Q \subset Q(\sqrt[3]{2})$ . An automorphism must map the polynomial  $X^3 - 2$  back to itself, and thus map each root of this polynomial to another root. Since the field only contains the real root, the only automorphism is the identity map.

**Theorem 19** Let  $P(X)$  be a polynomial over  $K$ , not necessarily irreducible. Let  $L$  be a splitting field of  $P$ .

- Every element of the Galois group of  $K \subset L$  permutes the roots of  $P$
- This permutation completely determines the element of  $G$
- If  $P(X)$  is irreducible over  $K$  and  $\theta_i$  and  $\theta_j$  are two roots, there is an element  $\sigma$  in the Galois group of the splitting field  $L$  such that  $\sigma(\theta_i) = \theta_j$

*Proof:* If  $P(\theta) = 0$  and  $\sigma$  is a Galois automorphism, then  $\sigma$  fixes the coefficients of  $P$  and therefore  $P(\sigma(\theta)) = 0$ . Since the roots generate the splitting field, the second assertion is clear.

The third assertion was proved as theorem 13 in section 18 on splitting fields.

*Remark:* The last theorem of the previous section was the first of a string of theorems showing that non-trivial automorphisms exist. We now come to a somewhat difficult, but extremely important theorem which guarantees the existence of many more automorphisms:

**Theorem 20** Let  $P(X)$  be a polynomial over  $K$ , not necessarily irreducible. Let  $L$  be a splitting field for  $P(X)$ . Suppose that  $P(X)$  does not have multiple roots in  $L$ . Then

$$|G| \geq [L : K]$$

and thus the Galois group has many elements.

*Proof:* Factor  $P$  and let  $P_1(X)$  be one of the irreducible factors. This polynomial factors completely in  $L$ ; call its roots  $a_1, \dots, a_{n_1}$ . The root fields  $K(a_1), \dots, K(a_{n_1})$  inside  $L$  are isomorphic by a previous result. Let  $\varphi_i : K(a_1) \rightarrow K(a_i)$  be such isomorphisms over  $K$ . In particular,  $\varphi_1$  is the identity map from  $K(a_1)$  to itself. We are going to extend these isomorphisms to automorphisms of  $L$ . We know that this will give  $[K(a_1) : K]$  automorphisms because all the roots of  $P_1$  are distinct.

Factor  $P(X)$  over  $K(a_1)$ . If  $P$  factors completely into linear factors, then  $K(a_1) = L$  and consequently  $\varphi_i : L \rightarrow L$ . This gives the same number of automorphisms as  $[L : K] = [K(a_1) : K]$  and we are done.

Otherwise factor  $P(X)$  over  $K(a_1)$  as  $Q_1Q_2 \dots Q_s$  and let  $P_2(X)$  be one of the nonlinear factors. We have isomorphisms  $\varphi_i : K(a_1) \rightarrow K(a_i)$ ; in particular  $\varphi_1$  is the identity. Applying these isomorphisms, we discover that  $P(X)$  also factors as

$$\varphi_i(Q_1)\varphi_i(Q_2) \dots \varphi_i(Q_s)$$

over  $K(a_i)$ . In particular,  $\varphi_i(P_2(X))$  is one of the factors over  $K(a_i)$ . For each  $i$ , including  $i = 1$ , let  $b_{ij}$  be all roots of  $\varphi_i(P_2(X))$  in  $K(a_i)$ . The extension lemma on page 24 implies that we can uniquely extend the  $\varphi_i$  to isomorphisms  $\varphi_{ij} : K(a_1, b_{11}) \rightarrow K(a_i, b_{ij})$ . Notice that the map  $K(a_1, b_{11})$  to itself is the identity.

Now factor  $P(X)$  over  $K(a_1, b_1)$ . Suppose first that all factors are linear. Then all  $K(a_i, b_{ij})$  are  $L$  and each  $\varphi_{ij} : K(a_1, b_1) \rightarrow K(a_i, b_{ij})$  is an automorphism  $L \rightarrow L$ . These automorphisms are all different. Indeed, they take  $a_1$  to  $a_i$  and the  $a_i$  are all different, so two equal automorphisms must be associated with the same  $i$ . But then the automorphisms map  $b_1$  to elements  $b_{ij}$  for fixed  $i$ , and these are roots of  $\varphi_{ij}(P_2(X))$ , hence roots of  $P(X)$ , and so all different. The total number of automorphisms is the number of  $a_i$  multiplied by the number of  $b_{ij}$ , which equals the degree of  $P_1(X)$  multiplied by the degree of  $P_2(X)$ , or  $[K(a_1) : K][K(a_1, b_1) : K(a_1)] = [L, K]$ .

If the factorization  $P(X) = R_1(X)R_2(X) \dots R_p(X)$  of  $P(X)$  over  $K(a_1, b_1)$  has non-linear terms, fix one and call it  $P_3(X)$ . Applying the isomorphism  $\varphi_{ij} : K(a_1, b_1) \rightarrow K(a_i, b_{ij})$ , we discover that  $P(X)$  also factors as

$$\varphi_{ij}(R_1)\varphi_{ij}(R_2) \dots \varphi_{ij}(R_p)$$

over  $K(a_i, b_{ij})$ . For each  $i$  and  $j$ , let  $c_{ijk}$  be all roots of  $\varphi_{ij}(P_3(X))$  over  $K(a_i, b_{ij})$ .

Then for fixed  $i$  we have an isomorphism carrying  $a_1$  to  $a_i$ , and once  $i$  is fixed we have a fixed extended isomorphism carrying  $b_2$  to  $b_{ij}$  and for fixed  $i$  and  $j$  we have an isomorphism defined on  $K(a_1, b_1, c_1)$  carrying  $c_1$  to  $c_{ijk}$ . These isomorphisms extend to automorphisms of  $L$ . If two are equal, they have the same  $i$  since the  $a_i$  are distinct. They then have the same  $j$  since for fixed  $i$  the  $b_{ij}$  are distinct, and they have the same  $k$  since for fixed  $i$  and  $j$  the  $c_{ijk}$  are distinct. Counting as before, we have at least  $\deg(P_1(X))\deg(P_2(X))\deg(P_3(X))$  automorphisms, and this last number is  $[K(a_1, b_1, c_1) : K(a_1, b_1)] [K(a_1, b_1) : K(a_1)] [K(a_1) : K]$ . If we stop at this stage, this number is  $[L : K]$  and we have enough automorphisms.

Otherwise continue step by step in the same way. QED.

*Remark:* In the next chapter we will prove that for any finite extension  $K \subset L$ , we have  $|G| \leq [L : K]$ . Combining this with the previous theorem then gives the exact number of automorphisms for splitting fields of polynomials with no repeated roots.

## Chapter 7

# Galois Extensions and the Fundamental Theorem

### 7.1 Galois Extensions

**Definition 4** We say  $K \subset L$  is a Galois extension if the only elements of  $L$  left fixed by every element of the Galois group are elements of  $K$ .

*Remark:* The Galois theory developed in the next section works only for Galois extensions. So the remaining results in this section are crucial, showing that many important field extensions are Galois extensions.

The next developments come from the paper *On the Characterization of Galois Extensions* by Meinolf Geck, American Mathematical Monthly, August-September, 2014.

**Lemma 3** The Galois group of a finite extension  $K \subset L$  is finite.

*Proof:* Let  $\alpha_1, \dots, \alpha_n$  be a basis for  $L$  over  $K$ . An automorphism is completely determined by its action on the  $\alpha_i$ , so it suffices to prove that each  $\alpha_i$  can have only finitely many images. Let  $P(X)$  be the minimal polynomial of  $\alpha_i$  over  $K$  and notice that the image of  $\alpha_i$  must also be a root of this minimal polynomial, which has finitely many roots.

**Lemma 4** If  $K \subset L$  is a finite extension, then  $L$  is not the union of a finite number of strict subfields  $K \subset M \subset L$ .

*Proof:* If  $K$  is infinite, an even stronger statement is true: a finite dimensional vector space over an infinite field cannot be a union of a finite number of proper subspaces. To prove

this, pick a basis for  $L$ , and write the elements of  $L$  as  $n$ -tuples. Consider the set  $S$  of elements  $(1, t, t^2, \dots, t^{n-1}) \in L$ . There are infinitely many such elements.

We can enlarge each subspace  $M$  to a subspace of codimension one, and thus to a subspace defined by an equation  $a_0x_0 + a_1x_1 + \dots + a_{n-1}x_{n-1} = 0$ . The element  $(1, t, \dots, t^{n-1})$  belongs to this subspace only if  $a_0 + a_1t + \dots + a_{n-1}t^{n-1} = 0$ , and this polynomial equation has only finitely many roots. So only finitely many elements of the set  $S$  belong to the finite union of subspaces.

If  $K$  is finite, we need to restrict to subfields  $M \subset L$ , but then everything in sight is a finite field. So the multiplicative group  $L^*$  of nonzero elements of  $L$  is cyclic. Let  $g \in L$  be a generator. This  $g$  cannot belong to any strict subfield  $M \subset L$ . QED.

**Corollary 4** *For an arbitrary finite extension  $K \subset L$ , there is an element  $\theta \in L$  moved by every nontrivial element of the Galois group.*

*Proof:* For each nontrivial  $g \in G$ , the set  $M_g$  of fixed points of  $g$  is a proper subfield of  $L$  containing  $K$ . By the lemma, the union of the subfields is not the entire field.

**Corollary 5** *We have  $|G| \leq [L : K]$ . If equality holds, there exists  $\theta \in L$  such that  $L = K(\theta)$ , and  $L$  is a splitting field of the minimal polynomial of  $\theta$ , which has no multiple roots.*

*Proof:* Choose  $\theta$  as in the previous corollary and let  $P(X)$  be its minimal polynomial over  $K$ . The Galois group must map  $\theta$  to other roots of  $P(X)$ . Since no element of the Galois group except the identity leaves  $\theta$  fixed, the number of distinct roots of  $P(X)$  must be at least  $|G|$ , so the degree of  $P$  is at least  $|G|$ . Therefore

$$|G| \leq \deg(P) = [K(\theta) : K] \leq [L : K]$$

If equality holds, then  $|G| = \deg(P)$  and  $K(\theta) = L$ . In particular, every root of  $P$  must equal  $\sigma(\theta)$  for some  $\sigma \in G$ , so  $P$  splits and its roots are distinct.

**Theorem 21** *Let  $K \subset L$  be a field extension of finite degree. The following are equivalent:*

- $|G| = [L : K]$
- $K \subset L$  is a Galois extension
- $L$  is a splitting field over  $K$  for a polynomial  $P(X)$  without multiple roots.
- $L = K(\theta)$  where  $\theta$  is a root of a polynomial  $P(X)$  which splits completely in  $L$  and does not have multiple roots.

*If the characteristic of the ground field is zero, we can add a fifth item*

- $L$  is a splitting field over  $K$  for an arbitrary polynomial  $P(X)$



*Proof of theorem:* First a) implies b), for let  $M$  be the set of all elements of  $L$  fixed by every element of  $G$ . Clearly  $M$  is a subfield and  $K \subset M \subset L$ . Hence  $G$  is also the Galois group of  $L$  over  $M$ , and by corollary 5 we have  $|G| \leq [L : M] \leq [L : K]$ . The left and right sides of this equation are equal by assumption, so  $M = K$ .

Second, b) implies c) as follows. Find a basis  $x_1, \dots, x_n$  for  $L$  over  $K$ . For each  $i$  satisfying  $1 \leq i \leq n$  and each  $\sigma \in G$  we can form  $\sigma(x_i) \in L$ . The set of all these elements is a set which may have duplicates; let  $S$  be the same set with duplicates removed.

If we act on the  $\sigma(x_i)$  on the left by a fixed  $\tau \in G$ , we just rearrange the elements. If a particular element occurs  $r$  times originally, the image of this element will occur  $r$  times. Consequently we can remove duplicates before or after multiplication by  $\tau$  and get the same result. So the set  $S$  is invariant under the action of any fixed  $\tau$ .

Form the polynomial

$$P(X) = \prod_S (X - \sigma(x_i))$$

The coefficients of this polynomial are left fixed by applying any  $\tau$ . For example, the first coefficient is the sum of elements in  $S$  and the set  $S$  is invariant under multiplication by  $\tau$ . Since  $K \subset L$  is a Galois extension and the coefficients of  $P$  are invariant under the Galois group, all of these coefficients belong to  $K$ .

Since  $G$  contains the identity, the set  $S$  contains all  $x_i$ , so the field generated by the roots of  $P$  is  $L$ . Since  $P$  splits completely, it is a splitting field. By construction,  $P$  has no repeated roots.

Third, c) implies a). We already know that  $|G| \leq [L : K]$  from Corollary 5. By theorem 20 in section 6.3, c) implies that  $|G| \geq [L : K]$ . Therefore  $|G| = [L : K]$ .

Condition d) implies condition c). This condition implies conditions a) and b), and condition a) implies condition d) by corollary 5.

Condition c) implies condition e). Conversely, if the ground field has characteristic zero, condition e) implies condition c) as follows.

If the characteristic of the ground field is zero, suppose  $K \subset L$  is a splitting field of an arbitrary  $P(X)$ . Factor  $P(X)$  over  $K$  into irreducible polynomials  $P_1(X)P_2(X)\dots P_j(X)$ . If an irreducible factor occurs more than once, we can remove the redundant term and get a new polynomial  $\tilde{P}$  with the same roots and the same splitting field. So we can assume there are no redundancies.

Fix  $i$  and consider  $P_i(X)$ . If  $\theta$  is a root in a splitting field  $L$ , then  $P_i$  is the minimal polynomial of  $\theta$ . If  $\theta$  is a multiple root, we can write  $P_i(X) = (X - \theta)^2 Q(X)$  over  $L$ . Taking formal derivatives,  $P_i'(X) = 2(X - \theta)Q(X) + (X - \theta)^2 Q'(X)$  and consequently  $\theta$  is a root

of  $P'_i(X)$ . Writing  $P_i(X) = X^k + a_1X^{k-1} + \dots$ , we have  $P'_i(X) = kX^{k-1} + \dots$ . In a field of characteristic zero,  $k \neq 0$  and so  $P'_i(X)$  has smaller degree than  $P_i$ , a contradiction.

A root cannot be a root of two terms  $P_i$  and  $P_j$ , for otherwise they would both be minimal polynomials of the root, and thus equal up to a constant, so one of them would have been removed when we removed redundant terms. QED.

## 7.2 Fundamental Theorem of Galois Theory

Let  $K \subset L$  be a fixed Galois extension with Galois group  $G$ . Suppose that  $K_1$  is a field,  $K \subset K_1 \subset L$ . Associate this field with a subgroup of  $G$ , namely

$$\{g \in G \mid g \text{ fixes each element of } K_1\}$$

Conversely let  $H$  be a subgroup of  $G$ ,  $\{e\} \subset H \subset G$ . Associate this subgroup with a subfield of  $L$ , namely

$$\{l \in L \mid h(l) = l \text{ for all } h \in H\}$$

**Theorem 22 (The Fundamental Theorem of Galois Theory)** *The maps just defined are inverse to each other and set up a one-to-one correspondence between the set of all subgroups of  $G$  and the set of all subfields of  $L$  containing  $K$ .*

*Remark:* This is astonishing, since fields are complicated and hard to pin down, while subgroups of a finite group can be enumerated mechanically.

*Remark:* We gather here a few extra features which are part of the theorem and will be proved as we prove the theorem.

- The correspondence is order reversing. The identity subgroup corresponds to  $L$  and the entire  $G$  corresponds to  $K$ .
- Suppose  $K \subset M \subset L$  is a subfield of  $L$  corresponding to a subgroup  $H$  of  $G$ . Then the extension  $M \subset L$  is a Galois extension, and its Galois group is  $H$ . The extension  $K \subset M$  is a Galois extension if and only if  $H$  is normal, and in that case its Galois group is  $G/H$ .

*Proof of Fundamental Theorem:* Consider the map

$$\{\text{fields}\} \rightarrow \{\text{groups}\} \rightarrow \{\text{fields}\}$$

Suppose we start with a subfield  $M$  on the left. Since  $K \subset L$  is a Galois extension, it is a splitting field for a polynomial  $P(X)$  over  $K$  without multiple roots in  $L$ . This  $P(X)$

certainly has coefficients in  $M$ , so  $L$  is the splitting field of  $P$  over  $M$  and there are no multiple roots. Hence  $M \subset L$  is Galois.

The Galois group of  $M \subset L$  is the set of all automorphisms of  $L$  which fix  $M$ . Such an automorphism certainly fixes  $K$ , so it belongs to the Galois group of  $K \subset L$ . Thus it is  $\{ \sigma \in G \mid \sigma \text{ fixes } M \}$ .

This is exactly the group  $H$  assigned to  $M$  by the Galois correspondence. The corresponding subfield attached to this group is the set of all elements of  $K$  fixed by the elements of  $H$ . Since  $M \subset L$  is Galois, this set is  $M$  itself.

Consider the map

$$\{\text{groups}\} \rightarrow \{\text{fields}\} \rightarrow \{\text{groups}\}$$

Start with a subgroup  $H$  and consider the set  $M$  of all elements fixed by  $H$ . This  $M \subset L$  is a Galois extension as above. Its Galois group is the set  $N$  of all automorphisms of  $L$  fixing  $M$ , and this is the group assigned to  $M$  on the right of the above sequence. So  $|N| = [L : M]$ . Notice that  $H \subset N$ .

By corollary 5 in section 7.1, there is a  $\theta \in L$  such that  $L = M(\theta)$ . Define

$$P_1(X) = \prod_{\sigma \in H} (X - \sigma(\theta))$$

By the standard argument, the coefficients of this polynomial are invariant under  $H$  and thus belong to  $M$ . Since  $\theta$  is a root of  $P_1(X)$ , the minimal polynomial of  $\theta$  over  $M$  divides  $P_1(X)$  and thus has degree at most  $|H|$ . But  $[L : M]$  equals the degree of this minimal polynomial, so  $[L : M] \leq |H|$ . Therefore  $|N| \leq |H|$ . Since  $H \subset N$ , we have  $H = N$ . QED.

**Theorem 23** *If  $K \subset L$  is a Galois extension and  $K \subset M \subset L$  corresponds to a subgroup  $H \subset G$ , then  $K \subset M$  is Galois if and only if  $H$  is normal in  $G$ , and in that case the Galois group of  $K \subset M$  is  $G/H$ .*

*Proof:* Let  $\sigma \in G$ . If  $K \subset M \subset L$  corresponds to  $H \subset G$ , then  $\sigma(M)$  corresponds to  $\sigma H \sigma^{-1}$ , since an element of this group maps  $\sigma(M)$  to  $M$ , and then leaves  $M$  fixed, and then maps it back to  $\sigma(M)$ . It follows that  $H$  is normal in  $G$  if and only if  $\sigma(M) \subset M$  for all  $\sigma \in G$ .

Suppose  $H$  is normal, and thus  $\sigma(M) \subset M$ . It follows that every automorphism in  $G$  induces an automorphism of  $M$ . Two automorphisms  $\sigma_1$  and  $\sigma_2$  induce the same automorphism of  $M$  if and only if  $\sigma_1^{-1}\sigma_2$  is the identity on  $M$ , or equivalently if and only if  $\sigma_1^{-1}\sigma_2 \in H$ , so the group  $G/H$  acts effectively on  $M$ . Notice that  $|G| = |H||G/H|$  and  $[L : K] = [L : M][M : K]$ . Since  $K \subset L$  and  $M \subset L$  are Galois,  $|G| = [L : K]$  and  $|H| = [L : M]$ . It follows that  $|G/H| = [M : K]$ , so  $M$  is Galois with Galois group  $G/H$ .

Conversely if  $K \subset M$  is Galois, then there is a  $\theta \in M$  such that  $M = K(\theta)$  where  $\theta$  is a root of a polynomial  $P(X)$  which splits completely. If  $\sigma \in G$ , then  $\sigma$  maps  $\theta$  to other roots of  $P$ , and thus maps  $M$  to itself, so  $H$  is normal. QED.

### 7.3 Important Note

The Galois theory developed above works for all fields, including fields of positive characteristic. The only cautionary note is that we cannot apply the theorem to a splitting field in characteristic  $p > 0$  unless we know that it is the splitting field of a polynomial with no multiple roots.

### 7.4 An Example

Consider the polynomial  $P(X) = X^3 - 2$  studied in section 4.4. It is irreducible over  $Q$  by Eisenstein's theorem. The splitting field of  $P(X)$  is  $Q(\alpha, \omega\alpha, \omega^2\alpha)$ . Since it is a splitting field,  $Q \subset L$  is Galois and the Galois group is a transitive permutation group on the roots and hence  $Z_3$  or  $D_3$ . But complex conjugation is clearly an automorphism of degree 2, so  $G = D_3$ .

The subgroups of  $D_3$  are  $\{e\}$ , three copies of  $Z_2$  formed by the three reflections of the equilateral triangle,  $Z_3$ , and  $D_3$ . The Galois correspondence is order reversing, and the corresponding subfields of  $L$  are  $L$ , the three root fields  $Q(\alpha)$ ,  $Q(\omega\alpha)$ , and  $Q(\omega^2\alpha)$ , a field corresponding to  $Z_3$ , and  $Q$  itself.

What subfield corresponds to  $Z_3$ ? Notice that the cube roots of unity belong to  $L$ , and thus  $M = Q(\theta)$  is a subfield with splitting polynomial  $X^3 - 1 = (X - 1)(X^2 + X + 1)$ . Hence  $[Q(\theta) : Q] = 2$ . This field is a Galois extension of  $Q$ , so it must correspond to a normal subgroup of  $D_3$  and thus to  $Z_3$ . It's Galois group is  $D_3/Z_3 = Z_2$ .

### 7.5 Finite Fields Again

In section 16 we constructed a field with four elements. This construction works in general. Suppose we want a field  $L$  with  $p^n$  elements. Find an irreducible polynomial over  $Z_p$  of degree  $n$ . Then our field will be the root field of this polynomial.

We know that an irreducible polynomial of degree  $n$  exists for the following reason. We proved that there is a field  $L$  with  $p^n$  elements. The group  $L^*$  is cyclic with generator  $\gamma$ . So  $L = Z_p(\gamma)$ . The minimal polynomial  $P(X)$  of  $\gamma$  has degree  $n$ .

Since  $\gamma$  satisfies  $X^{p^n} - X = 0$ ,  $P(X)$  is a factor of this polynomial.

*Example:* Suppose we want a field with 27 elements. We must find an irreducible polynomial over  $Z_3$  with degree 3. It suffices to find a cubic with no roots in  $Z_3$ . Randomly trying examples,  $X^3 + 2X + 2$  works. So a typical element of the field is  $a_0 + a_1X + a_2X^2$  and  $X^3 = -2X - 2 = X + 1$ .

**Definition 5** *The field with  $p^n$  elements is often denoted  $GF(p^n)$ .*

**Definition 6** *Let  $F = GF(p^n)$ . The Frobenius automorphism is the map  $\sigma : F \rightarrow F$  given by  $a \rightarrow a^p$ .*

**Theorem 24** *The extension  $Z_p \subset GF(p^n)$  is Galois with Galois group  $Z_n$  generated by the Frobenius automorphism.*

*Proof:* The extension is a splitting field for  $X^{p^n} - X$ . The roots of this polynomial are distinct, indeed exactly the elements of  $GF(p^n)$ . The Frobenius automorphism satisfies  $\sigma(a + b) = (a + b)^p = a^p + b^p = \sigma(a) + \sigma(b)$  because all remaining coefficients in the expansion are divisible by  $p$ . It trivially preserves multiplication and takes nonzero elements to nonzero elements. It fixes each element of  $Z_p$ . Since it is a linear transformation over  $Z_p$  which is one-to-one, it is also onto.

We show that  $\sigma$  has order  $n$  in the full Galois group. Certainly its order divides this number, because every element  $a \in GF(p^n)$  satisfies  $a^{p^n} = a$ . If  $\sigma$  has smaller order  $d$ , then every element of  $GF(p^n)$  would satisfy  $X^{p^d} - X = 0$ , but this equation has at most  $p^d$  roots.

Finally, we show that every automorphism of  $GF(p^n)$  is a power of  $\sigma$ . An automorphism of  $GF(p^n)$  must fix 1, hence  $Z_p$ . Therefore the number of automorphisms is  $[GF(p^n) : Z_p] = n$ , and powers of  $\sigma$  give this many automorphisms. QED.

*Remark:* The fundamental theorem of Galois theory then reveals the lattice structure for the subfields of  $GF(p^n)$ . The subgroups of  $Z_n$  are all abelian and have the form {all powers of  $\sigma^d$ } for  $d|n$ . The order of such a subgroup is  $\frac{n}{d}$ . The corresponding subfield is all elements satisfying  $\sigma^d(a) = a^{p^d} = a$ , but these elements exactly form  $GF(p^d)$ . It follows that the Galois group of  $GF(p^n)$  over  $GF(p^d)$  is  $Z_{n/d}$ , and generated by  $\sigma^d$ .

The finite subfields of  $GF(p^n)$  are  $GF(p^d)$  for  $d|n$ .

## Chapter 8

# Concrete Cases of the Theory

### 8.1 Cyclotomic Fields

We now want to prove Galois' theorem giving a necessary and sufficient condition that an equation be solvable in radicals. This proof requires partial calculation of a small number of specific Galois groups. In this chapter, we only carry calculations far enough to deduce what we will need in our discussion of solvability. A later chapter will completely compute the Galois groups discussed here.

**Theorem 25** *Let  $n$  be a positive integer and consider  $P(X) = X^n - 1$ . Suppose the ground field  $K$  has characteristic zero or else that its characteristic does not divide  $n$ . Then the splitting field  $L$  of  $P(X)$  is a Galois extension, and its Galois group is abelian.*

*Proof:* Some caution is required, because the polynomial  $P$  is reducible. For example,  $X^4 - 1 = (x^2 - 1)(X^2 + 1) = (X - 1)(X + 1)(X^2 + 1)$ .

Notice that  $P$  cannot have multiple roots by the standard argument, since its formal derivative  $nX^{n-1}$  is not zero. So  $K \subset L$  is a Galois extension. Every root of  $P$  is an  $n$ th root of unity, and there are  $n$  distinct roots. So the roots are exactly the  $n$ th roots of unity. This is a finite subgroup of a field, hence cyclic.

Let  $\omega$  be a generator of the group of  $n$ th units. If  $\sigma$  belongs to the Galois group,  $\sigma(\omega) = \omega^i$  for some fixed  $i$ ,  $1 \leq i \leq n$ . Then  $\sigma(\omega^j) = \sigma(\omega)^j = \omega^{ij}$ . It follows that  $i$  completely determines the automorphism. The map  $\sigma \rightarrow i$  sends the Galois group of  $K \subset L$  in a one-to-one manner into  $Z_n$ . If  $\sigma \rightarrow i$  and  $\tau \rightarrow j$ , then  $\tau(\sigma(\omega)) = \tau(\omega^i) = \tau(\omega)^i = \omega^{ij}$ , so our map is a group homomorphism into the multiplicative group  $Z_n^*$  of invertible elements in  $Z_n$ , which is abelian. Since  $G$  is injected into an abelian group, it is abelian. The map

in question need not be onto  $Z_n^*$ , and certainly isn't when  $K$  already contains some roots of unity. QED.

## 8.2 Galois Group of a Radical Extension

**Theorem 26** *Let  $K$  be a field containing all  $n$ th roots of unity. If the characteristic is a positive prime  $p$ , suppose that  $p$  does not divide  $n$ . Suppose  $a \in K$ , and let  $L = K(\sqrt[n]{a})$ . Then  $K \subset L$  is a Galois extension, and its Galois group is cyclic.*

*Proof:* Since the roots of unity are solutions to  $X^n - 1$  and  $p$  does not divide  $n$ , the standard argument via differentiation shows that the roots of this polynomial are distinct. Thus the  $n$ th roots of unity in  $K$  form a group of order  $n$ . Let  $\omega$  be a primitive  $n$ th root of unity.

Then the roots of  $X^n - a$  are  $\omega^k \sqrt[n]{a}$ , so  $L$  is a splitting field for  $F$  and thus a Galois extension. Notice that the  $\omega^k \sqrt[n]{a}$  need not be linearly independent over  $K$ , and  $X^n - a$  need not be irreducible.

Each element of the Galois group sends the root  $\sqrt[n]{a}$  to some other root  $\omega^k \sqrt[n]{a}$ . Then  $\omega^j \sqrt[n]{a}$  must map to  $\omega^{j+k} \sqrt[n]{a}$ , so the initial  $k$  determines the automorphism. Notice also that composition corresponds to addition of exponents, since  $\sqrt[n]{a} \rightarrow \omega^k \sqrt[n]{a} \rightarrow \omega^{k+l} \sqrt[n]{a}$ .

Hence the set of all  $k$  in  $0 \leq k < n$  corresponding to automorphisms of  $L$  is a subgroup of  $Z_n$ . All subgroups of  $Z_n$  are cyclic of some order  $d$  dividing  $n$ . It follows that the Galois group of  $K \subset L$  is cyclic. QED.

## 8.3 Structure of Extensions with Cyclic Galois Group

Next we'd like to go backward and understand the structure of a Galois Extension if we only know that the Galois group is cyclic. The following theorem is a sort of converse of the previous result.

**Theorem 27** *Let  $K$  be a field containing all  $n$ th roots of unity. If the characteristic is a positive prime  $p$ , suppose that  $p$  does not divide  $n$ . Let  $K \subset L$  be a Galois extension with Galois group cyclic of order  $n$ . Then there exists a nonzero  $a \in L$  such that  $L$  is a splitting field for  $F(X) = X^n - a$  over  $K$ , and this polynomial has distinct roots.*

*Proof:* Let  $\sigma$  generate the Galois group.

**Lemma 5** *The maps  $1, \sigma, \sigma^2, \dots, \sigma^{n-1} : L \rightarrow L$  are linearly independent over  $L$ , so it is impossible to choose  $a_i$  not all zero in  $L$  with  $\sum a_i \sigma^i$  identically zero on  $L$ .*

*Proof:* Suppose we have  $n$  distinct automorphisms  $\sigma_i$  and ignore the assumption that these are the only automorphisms. We prove the maps independent by induction on  $n$ . To prove the induction step, suppose  $\sigma_1, \dots, \sigma_{n-1}$  independent. Let  $\sum_{i=1}^n a_i \sigma_i = 0$ . Since  $\sigma_1 \neq \sigma_n$ , we can find  $k_0$  with  $\sigma_1(k_0) \neq \sigma_n(k_0)$ . Then  $\sum a_i \sigma_i(kk_0) = 0$  so that

$$\sum a_i \sigma_i(k) \sigma_i(k_0) = 0$$

$$\sum a_i \sigma_i(k) \sigma_n(k_0) = 0$$

Subtracting,

$$\sum a_i \sigma_i(k) (\sigma_i(k_0) - \sigma_n(k_0)) = 0$$

The last term vanishes, so this is a dependence relation among the first  $n-1$  terms. Hence all coefficients vanish. In particular,  $a_1 (\sigma_1(k_0) - \sigma_n(k_0)) = 0$ . Since  $\sigma_1(k_0) \neq \sigma_n(k_0)$ , we have  $a_1 = 0$ .

Repeat the argument using a different  $k_0$  chosen so  $\sigma_2(k_0) \neq \sigma_n(k_0)$ . This time we conclude that  $a_2 = 0$ . Etc. Eventually all  $a_i$  vanish for  $i < n$ . We conclude that  $a_n \sigma_n = 0$ , and so  $a_n = 0$ . QED.

*Proof of theorem:* Let  $\omega$  be a primitive  $n$ th root of unit and consider

$$1 + \omega\sigma + \omega^2\sigma^2 + \dots + \omega^{n-1}\sigma^{n-1}$$

This expression is not identically zero, so we can find  $\beta \in L$  making it nonzero:

$$\theta = \beta + \omega\sigma(\beta) + \omega^2\sigma^2(\beta) + \dots + \omega^{n-1}\sigma^{n-1}(\beta) \neq 0$$

Applying  $\omega\sigma$  to both sides gives

$$\omega\sigma(\theta) = \omega\sigma(\beta) + \omega^2\sigma^2(\beta) + \dots + \omega^n\sigma^n(\beta)$$

The last term is  $\beta$ , so

$$\omega\sigma(\theta) = \beta + \omega\sigma(\beta) + \omega^2\sigma^2(\beta) + \dots + \omega^{n-1}\sigma^{n-1}(\beta) = \theta$$

So  $\sigma(\theta) = \omega^{-1}\theta$  and  $\sigma(\theta^n) = \theta^n$ . It follows that  $\theta^n \in K$ . Let  $a = \theta^n$ .

The element  $\theta$  is then  $\sqrt[n]{a}$ . Consider the polynomial  $F(X) = X^n - a$ . The roots of this polynomial are  $\omega^i \sqrt[n]{a}$ , which are distinct. Thus  $K(\theta)$  is a splitting field for  $F(X)$  over  $K$ , and the roots of this polynomial in  $K(\theta)$  are distinct. QED.



## Chapter 9

# Solving Polynomials by Radicals

### 9.1 Solving Polynomial Equations with Radicals

In this entire chapter, we assume that  $P(X)$  is irreducible over a field of characteristic zero. Often this ground field is  $Q$ .

**Definition 7** *Suppose  $P(X)$  is an irreducible polynomial over a field  $K$  of characteristic zero. We say this polynomial can be solved by radicals if there are fields*

$$K \subset K_1 \subset K_2 \subset \dots \subset K_m$$

*such that each extension  $K_i$  is a root field over  $K_{i-1}$  of  $X^n - a_i$  for some  $a_i \in K_{i-1}$ , and the splitting field of  $P(X)$  over  $K$  is inside  $K_m$ .*

**Definition 8** *A finite group  $G$  is said to be solvable if each of its Jordan-Holder simple quotients is abelian, and hence cyclic of prime order  $p$ .*

*Remark:* Galois proved the beautiful theorem that an irreducible polynomial over a field of characteristic zero can be solved by radicals if and only if its Galois group is solvable. In particular, a polynomial with a Galois group equal to  $S_n$  cannot be solved by radicals if  $n \geq 5$ .

We will give a very easy proof of half of this theorem: if the polynomial can be solved by radicals, then the Galois group is solvable. Unfortunately, our proof is wrong, as we'll explain in the next section. It turns out that our error is exactly the gap in Ruffini's original proof that there is no formula solving all equations of degree five by radicals.

**Theorem 28 (Proof below incorrect)** *If  $P(X)$  is an irreducible polynomial over a base field of characteristic zero, and  $P$  can be solved by radicals, then the Galois group of  $P$  is solvable.*

*Proof:* By taking the least common multiple of the degrees of the successive roots, we can find a common  $N$  such that each radical extension adds an  $N$ th root. Instead of extending to a root field, let us extend to the full splitting field of  $X^N - a$ . Consequently we have a tower

$$K = K_1 \subset K_2 \subset \dots \subset K_m$$

of Galois extensions, each a splitting field of  $X^N - a_i$ . Without loss of generality, we can suppose that the first extension  $K_1 \subset K_2$  is a splitting field of  $X^N - 1$ , so that all subsequent fields contain all  $N$ th roots of unity. By assumption the splitting field  $L$  of  $P(X)$  is contained in  $K_n$ . Apply the fundamental theorem of Galois theory to obtain a corresponding reverse tower of Galois groups

$$G_1 \supset G_2 \supset \dots \supset G_m = \{e\}$$

By the previous sections on the Galois group of a cyclotomic extension and the Galois group of a radical extension, we know that each intermediate composition factor is abelian. Refine to a complete composition series, which must still have abelian composition factors.

By assumption  $K \subset L \subset K_m$ , so  $G_m \supset G \supset \{e\}$  where  $G$  is the Galois group of  $L$ . Refine this series to a complete composition series, and then apply the Jordan-Holder theorem. It follows that the composition quotients for  $G$  are all abelian. QED.

## 9.2 The Flaw

In the previous proof, we applied the fundamental theorem of Galois theory to the Galois extension  $K_1 \subset K_m$  generated by a tower of splitting fields, and thus a tower of Galois extensions

$$K_1 \subset K_2 \subset \dots \subset K_m$$

But why is  $K_1 \subset K_m$  Galois?

It is tempting to guess that there is a simple lemma stating that if  $K \subset L \subset M$  and each extension is Galois, then  $K \subset M$  is Galois. But this lemma is false.

Consider, for example, the extensions  $Q \subset Q(\sqrt{2}) \subset Q(\sqrt[4]{2})$ . The first is Galois because it is a splitting field of  $X^2 - 2$ . The second is Galois because it is the splitting field of  $X^2 - \sqrt{2}$ . However the full extension is not Galois because  $X^4 - 2$  is irreducible with roots  $\pm\sqrt[4]{2}$  and  $\pm i\sqrt[4]{2}$ , so its splitting field contains complex numbers.

*Remark:* Looking back at the previous section, we notice that the proof only requires knowing that  $K \subset K_m$  is Galois. We don't really need to know that  $K_{i-1} \subset K_i$  is Galois. So in the argument given in the previous section, we can assume that each  $K_{i-1} \subset K_i$  is merely a root field of  $X^N - a_i$ , a more natural assumption.

### 9.3 The Fix

The following lemmas allow us to fix the argument.

**Lemma 6** *Suppose  $K \subset L$  is a Galois extension. If  $F(X)$  is an irreducible polynomial over  $K$  and  $F$  has a root in  $L$ , then  $F$  factors completely in  $L$ .*

*Proof:* Let  $\theta \in L$  be a root, and form  $G(X) = \prod_{\sigma \in G} (X - \sigma(\theta))$ . By the standard arguments,  $G(X)$  has coefficients in  $K$ . Since  $F(X)$  is a minimal polynomial for  $\theta$  and  $\theta$  is a root of  $G(X)$ ,  $F(X)$  divides  $G(X)$ . Consequently,  $F(X)$  splits completely. QED.

**Lemma 7** *Suppose we can find a tower of extensions*

$$K \subset K_1 \subset K_2 \dots \subset K_m$$

*such that each  $K_i$  is obtained from  $K_{i-1}$  as a root field of a polynomial  $X^n - a$ , where  $a \in K_{i-1}$ . The exponents are allowed to vary from extension to extension. Then there is a splitting field  $M$  over  $K$  containing all of these fields, such that  $M$  can be obtained as a (possibly different) tower of radical extensions. If all of the original extensions satisfy  $X^n - a = 0$  for a common  $n$ , we can assume this true for the new tower for  $M$ .*

*Remark:* In this lemma, there is no hypothesis that the first extension adds roots of unity, and no requirement that each radical extension be Galois over its predecessor.

*Proof:* The tower of extensions has the form

$$K \subset K_1 = K(\alpha_1) \subset K_2 = K(\alpha_1, \alpha_2) \subset \dots \subset K_m = K(\alpha_1, \alpha_2, \dots, \alpha_m)$$

where  $\alpha_i^{k_i} \in K_{i-1}$ . For each  $i$ , let  $L_i(X)$  be the minimal polynomial of  $\alpha_i$  over  $K$ . Let  $M$  equal the splitting field of  $L_1(X)L_2(X)\dots L_m(X)$ . Then  $M$  is Galois over  $K$ . Let  $G = \{\sigma_1, \dots, \sigma_t\}$  be the Galois group.

Each  $L_i$  is irreducible over  $K$  with a root  $\alpha_i \in M$ . It follows that each  $L_i$  factors completely over  $M$ . Let  $\alpha_i = u_{i1}, u_{i2}, \dots, u_{ik_i}$  be the roots of  $L_i$ . For each  $j$ , there is an isomorphism of root fields  $\sigma : K(\alpha_i) \rightarrow K(u_{ij})$ . This isomorphism can be extended to an automorphism of  $M$ . Hence the roots of  $L_i$  for a fixed  $i$  all have the form  $\sigma(\alpha_i)$  for some  $\sigma$  in the Galois group. Conversely, automorphisms of  $L$  map roots of  $L_i$  to other roots of  $L_i$ . So the roots of  $L_i$  are precisely the  $\sigma(\alpha_i)$ .

The field  $M$  is generated by the roots of its splitting polynomial and thus by all

$$\{\sigma(\alpha_i) \mid \sigma \in G, i = 1, 2, \dots, m\}$$

Let  $B_1$  be the subfield of  $M$  generated by  $\{\sigma(\alpha_1) \mid \sigma \in G\}$ . Notice that

$$K \subset K(\sigma_1(\alpha_1)) \subset K(\sigma_1(\alpha_1), \sigma_2(\alpha_1)) \dots \subset K(\sigma_1(\alpha_1), \sigma_2(\alpha_1), \dots, \sigma_t(\alpha_1)) = B_1$$

Each of these extensions is a radical extension because  $\alpha_1^n = A \in K$  implies  $\sigma_i(\alpha_1)^n = \sigma_i(\alpha_1^n) = \sigma_i(A) = A$ . Some of our inclusions may be equalities, and there is no claim that  $n$  is the smallest exponent making  $\sigma_i(\alpha_1)^n \in K$ .

Now assume by induction that a radical extension  $B_i$  was constructed generated by all

$$\{\sigma(\alpha_j) \mid 1 \leq j \leq i \text{ and } \sigma \in G\}$$

Let

$$B_{i+1} = B_i(\{\sigma(\alpha_{i+1}) \mid \sigma \in G\})$$

To finish the proof, we need only prove that  $B_{i+1}$  is a radical extension of  $B_i$ . Notice that  $\alpha_{i+1}^n$  is in the field generated by  $\alpha_1, \dots, \alpha_i$ . Thus we can find a polynomial with coefficients in  $K$  such that  $\alpha_{i+1}^n = P(\alpha_1, \dots, \alpha_i)$ . Applying  $\sigma \in G$ , we have  $\sigma(\alpha_{i+1})^n = \sigma(\alpha_{i+1}^n) = \sigma P(\alpha_1, \dots, \alpha_i) = P(\sigma(\alpha_1), \dots, \sigma(\alpha_i))$ . This element is in  $B_i$  since the  $\sigma(\alpha_j)$  are in  $B_i$  and  $B_i$  is a field. So  $B_{i+1}$  is a radical extension of  $B_i$ . QED, and whew!

## 9.4 Galois' Theorem on Solving Via Radicals

**Theorem 29 (Galois)** *Let  $P(X)$  be an irreducible polynomial over a field  $K$  of characteristic zero. If one root of  $P(X)$  can be written in terms of radicals, then every root of  $P(X)$  can be written that way.*

*Let  $K \subset L$  be the splitting field of  $P$ . Then  $K \subset L$  is a Galois extension. The Galois group of this extension is solvable if and only if  $P(X) = 0$  is solvable by radicals.*

*Proof:* If one root of the equation is solvable by radicals, we can find a tower of extensions

$$K \subset K_1 \subset K_2 \dots \subset K_m$$

such that each  $K_i$  is obtained from  $K_{i-1}$  as a root field of a polynomial  $X^k - a$ , where  $a \in K_{i-1}$ , such that  $K \subset L \subset K_m$ . By finding the least common multiple of the various  $k$ , we can find a common  $n$  and assume that all extensions are root fields of a polynomial of the form  $X^n - a$ . To simplify matters further, we can add to the start of the chain an

extension  $K \subset \tilde{K} \subset K_1$  where  $\tilde{K}$  is a splitting field of  $X^n - 1$ . That is, we can assume that all  $n$ th roots of unity belong to the ground field.

We now apply the main theorem of the previous section. This theorem allows us to extend  $K_m$  to a field  $M$  such that  $K \subset M$  is a Galois extension, and each intermediate field in the chain is a radical extension generated by a root of  $X^n - a$  for our common  $n$ . Some care is required here because the first extension in our chain,  $K \subset \tilde{K}$ , need not itself be a radical extension. But it can clearly be written as a sequence of radical extensions. So in the construction of  $M$ , this sequence can be left alone, and only later extensions will require the addition of additional radical extensions to finally give  $M$ .

Let  $G$  be the Galois group of  $K \subset M$ . By the fundamental theorem of Galois theory, the sequence of radical extensions gives rise to a sequence of reverse inclusions

$$G \supset G_1 \supset G_2 \dots \supset G_{n-1} \supset \{e\}$$

The sequence on the left,  $G \supset G_1 \supset \{e\}$  corresponds to the field extensions  $K \subset \tilde{K} \subset M$ . Since  $K \subset \tilde{K}$  is a splitting field and thus a Galois extension,  $G_1$  is a normal subgroup of  $G$  and the quotient group  $G/G_1$  is the Galois group of the cyclotomic extension  $K \subset \tilde{K}$ . By the main theorem in section 8.1, this group is abelian.

The remaining  $G_i \supset G_{i+1} \supset M$  correspond to field extensions  $K_i \subset K_{i+1} \subset M$ . Each of these field extensions is a radical extension obtained by adding a root of  $X^n - a = 0$ . Since  $K_i$  contains all roots of unity, this extension completely splits the polynomial, and thus is a Galois extension. So  $G_{i+1}$  is a normal subgroup of  $G_i$  with quotient group the Galois group of  $K_i \subset K_{i+1}$ . By the main theorem in section 8.2, this Galois group is cyclic.

It follows that  $G$  has a composition series for which all composition quotients are abelian. Therefore a maximal composition series will have composition quotients  $Z_p$  for primes  $p$ . In particular,  $G$  is solvable.

Now we bring the polynomial  $P(X)$  into play. If this polynomial has one root which can be expressed via radicals, then there is a radical extension as above which contains one root of  $P(X)$ . Since the final  $K \subset M$  is a splitting field,  $P(X)$  splits completely and thus all roots can be expressed by radicals.

The sequence  $K \subset L \subset M$  corresponds to a sequence  $G \supset H \supset \{e\}$  where  $H$  is the Galois group for  $F(X)$ . Extend this to a complete composition sequence for  $G$ . Since  $G$  is solvable, all composition quotients are abelian. In particular, all composition quotients for  $H \supset \{e\}$  are abelian, so  $H$  is solvable.

*Proof in the converse direction:* Suppose the splitting field  $L$  of  $F(X)$  has a Galois group  $H$  which is solvable. Then we can find a sequence

$$H \supset H_1 \supset H_2 \supset \dots \supset \{e\}$$

where each  $H_i \supset H_{i+1}$  is a normal subgroup and each quotient group is  $Z_p$  for some prime  $p$ . By the fundamental theorem of Galois theory, this corresponds to a sequence of fields

$$K \subset K_1 \subset K_2 \subset \dots \subset L$$

Choose  $n$  a multiple of each  $[K_{i+1}, K_i]$  and let  $L \subset \tilde{L}$  be a splitting field for  $X^n - 1$ . The set of  $n$ th roots of unity in  $\tilde{L}$  is cyclic of order  $n$ . Let  $\omega$  be a generator. Note that  $\omega$  could actually be in  $L$  or even in  $K$ . Extend each  $K_i$  by  $\omega$ , obtaining

$$K \subset K(\omega) \subset K_1(\omega) \subset K_2(\omega) \subset \dots \subset L(\omega)$$

We claim each of these extensions is a radical extension. In that case,  $L$  is inside such an extension, and thus  $F$  is solvable by radicals.

Certainly  $K \subset K(\omega)$  comes from a sequence of radical extensions.

Since  $H_i \supset H_{i+1} \supset \{e\}$  is normal,  $K_i \subset K_{i+1} \subset L$  is Galois. So  $K_i \subset K_{i+1}$  is a Galois extension with cyclic Galois group. We now claim that  $K_{i-1}(\omega) \subset K_i(\omega)$  is Galois with Galois group a possibly different cyclic group. If this is true, then since  $K_{i-1}(\omega)$  contains all  $n$ th roots of unity, the main theorem in section 8.3 asserts that the extension is radical, and we will be done.

Since  $K_{i-1} \subset K_i$  is Galois, we can apply theorem 21 of section 7.1 to find a polynomial  $G(X)$  over  $K_{i-1}$  such that  $K_i = K_{i-1}(\theta)$  for a root  $\theta$  of  $G$ , and such that the polynomial splits completely. Clearly  $K_{i-1}(\omega) \subset K_i(\omega)$  is a splitting field of  $G(X)$ , and so it is a Galois extension. However,  $G(X)$  may not be irreducible over  $K_{i-1}(\omega)$ . Factor it and let  $G_1(X)$  be an irreducible factor with root  $\theta$ .

Let  $\sigma : K_i(\omega) \rightarrow K_i(\omega)$  be an automorphism over  $K_{i-1}(\omega)$ . Then  $\sigma$  is determined by  $\sigma(\theta)$ , and  $\sigma(\theta)$  is another root of  $G_1(X)$ . This other root is a root of  $G(X)$ , so  $\sigma$  comes from an automorphism of  $K_i$  over  $K_{i-1}$ . The conclusion is that the automorphism group of  $K_i(\omega)$  is a subgroup of the automorphism group of  $K_i$ . Since the automorphism group of  $K_i$  is cyclic, the automorphism group of  $K_i(\omega)$  is also cyclic, possibly of a different order. QED.

## 9.5 Solving Generic Equations by Radicals

Let  $a_1, \dots, a_n$  be arbitrary symbols. Form the field  $K = Q(a_1, \dots, a_n)$  consisting of all  $\frac{P}{Q}$  where  $P$  and  $Q$  are polynomials in the symbols.

The equation  $P(X) = X^n + a_1X^{n-1} + \dots + a_n = 0$  is called the *generic equation of degree*

$n$ . The quadratic formula gives a solution of the generic equation of degree 2:

$$\text{Solution of } X^2 + a_1X + a_2: \quad \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Similarly the cubic and quartic formulas give solutions of the generic equations of degree 3 and 4.

**Theorem 30** *The generic equation of degree  $n$  cannot be solved by radicals if  $n \geq 5$*

*Proof:* This follows from Galois' theorem and earlier results on the symmetric group, once we prove that the Galois group of the generic equation of degree  $n$  is the symmetric group of order  $n$ .

Let  $\lambda_1, \dots, \lambda_n$  be new arbitrary symbols, and form the field  $Q(\lambda_1, \dots, \lambda_n)$ . Formally write

$$(X - \lambda_1)(X - \lambda_2) \dots (X - \lambda_n) = X^n + a_1X^{n-1} + \dots + a_n$$

This holds if  $a_i = (-1)^i \sigma_i(\lambda_1, \dots, \lambda_n)$  where the  $\sigma_i$  are the elementary symmetric functions. Recall the definitions:

$$\begin{aligned} \sigma_1(X_1, \dots, X_n) &= X_1 + X_2 + \dots + X_n \\ \sigma_2(X_1, \dots, X_n) &= X_1X_2 + X_1X_3 + \dots + X_{n-1}X_n \\ &\dots = \dots \\ \sigma_n(X_1, \dots, X_n) &= X_1X_2 \dots X_n \end{aligned}$$

Redefine the original field  $Q(a_1, \dots, a_n)$  to be

$$Q(-\sigma_1(\lambda_1, \dots, \lambda_n), \dots, \pm\sigma_n(\lambda_1, \dots, \lambda_n))$$

so  $Q(a_1, a_2, \dots, a_n) \subset Q(\lambda_1, \lambda_2, \dots, \lambda_n)$ . Clearly the field  $Q(\lambda_1, \dots, \lambda_n)$  is a splitting field of the polynomial  $P(X) = (X - \lambda_1) \dots (X - \lambda_n) = X^n + a_1X^{n-1} + \dots + a_n$ . The elements of the Galois group permute these roots. Since the elementary symmetric functions are preserved by all permutations, all permutations work and the Galois group is the full symmetric group.

However, there is something else to prove. We have identified the transcendental extension  $Q(a_1, \dots, a_n)$  with the subfield of  $Q(\lambda_1, \dots, \lambda_n)$  generated by the elementary symmetric polynomials. This is only legal if the  $\sigma_i$  are independent. So we must prove that whenever  $P$  is a polynomial such that  $P(\sigma_1, \dots, \sigma_n) = 0$ , we have  $P = 0$ .

**Lemma 8** *The elementary symmetric functions are algebraically independent; if  $P$  is a polynomial and  $P(\sigma_1, \dots, \sigma_n) = 0$ , then  $P$  is identically zero.*

*Proof:* Each elementary symmetric function involves  $X_1, X_2, \dots, X_n$ . We prove the theorem by induction on  $n$ . The result is clear when  $n = 1$ .

Assume the result for  $n-1$  variables. To prove the induction step, suppose that  $P(Y_1, \dots, Y_n)$  is a polynomial of minimal total degree satisfied by the elementary symmetric functions  $\lambda_1, \dots, \lambda_n$ , each a function of  $X_1, \dots, X_n$ . Write  $P = p_0(Y_1, \dots, Y_{n-1}) + p_1(Y_1, \dots, Y_{n-1})Y_n + \dots + p_k(Y_1, \dots, Y_{n-1})Y_n^k$ . If  $p_0$  is zero, then  $P = Y_n \tilde{Q}(Y_1, \dots, Y_n)$  for a polynomial  $\tilde{Q}$  of smaller degree and  $\sigma_n \tilde{Q}(\sigma_1, \dots, \sigma_n) = 0$ . Since  $\sigma_n = X_1 \dots X_n$  in the integral domain  $Q[X_1, \dots, X_n]$ , we have  $\tilde{Q}(\sigma_1, \dots, \sigma_n) = 0$ , contradicting the assumption that  $P$  has smallest degree. So we can assume  $p_0(\sigma_1, \dots, \sigma_{n-1}) \neq 0$ .

We have

$$p_0(\sigma_1, \dots, \sigma_{n-1}) + p_1(\sigma_1, \dots, \sigma_{n-1})\sigma_n + \dots = 0$$

Each  $\sigma_i$  is a function of  $X_1, \dots, X_n$ . Substitute 0 for  $X_n$ . Then  $\sigma_n(X_1, \dots, X_{n-1}, 0) = 0$ , and we obtain  $p_0(\sigma_1, \dots, \sigma_{n-1}) = 0$ . However  $\sigma_i(X_1, X_2, \dots, X_{n-1}, 0)$  are exactly the elementary symmetric functions in the first  $n-1$  variables, so we get the contradiction that  $p_0 = 0$  by induction on  $n$ . QED.

## 9.6 A Polynomial Equation Which Cannot Be Solved by Radicals

We now know that there is no general formula solving an equation of degree  $n \geq 5$ . It is possible, however, that a solution can always be found involving radicals, but varying from equation to equation. To rule this out, we will find a quintic with integer coefficients whose solutions cannot be written in terms of radicals.

We will prove that  $P(x) = x^5 - 4x + 2$  is not solvable by radicals. Note that  $P(x)$  is irreducible over the rationals by Eisenstein's theorem, using  $p = 2$ .

**Lemma 9 (Cauchy)** *If a prime  $p$  divides the order of a group  $G$ , then  $G$  has an element of order  $p$ .*

*Proof:* Consider the set  $X$  of all  $p$ -tuples  $(g_1, g_2, \dots, g_p)$  whose product is the identity. The first  $p-1$  elements of such a tuple can be arbitrary, and these determine  $g_p$ . So  $X$  has  $|G|^{p-1}$  elements. This number is divisible by  $p$ .

We can cyclically permute the elements in a particular  $p$ -tuple. Suppose we first return to the starting point after  $k$  steps. Clearly  $k$  divides  $p$ , so either  $k = 1$  or  $k = p$ . If  $k = 1$  then all terms are equal; otherwise all terms in the cyclic permutation are different. Since



$p$  divides the number of such terms and  $p$  divides the size of  $X$ ,  $p$  must divide the number of  $(g, g, \dots, g)$  in  $X$ . Each represents an element of order  $p$  except  $(e, e, \dots, e)$ . QED.

*Calculation of the Galois Group:* Below is a plot of  $P(X)$ :

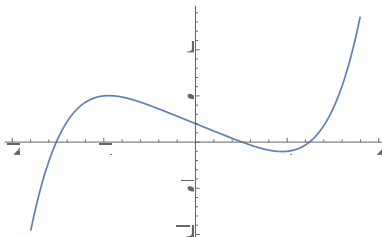


Figure 9.1: example caption

This plot shows that the polynomial has three real roots between  $-2$  and  $2$ . It has no other real roots, for  $x > 2$  implies that  $P(x) = x(x^4 - 4) + 2 > 2$  and  $x < -2$  implies that  $P(x) = x(x^4 - 4) + 2 \leq 12x + 2 \leq -10$ .

Consequently,  $P$  has two complex roots which are not real. They must be conjugates. The splitting field  $L$  is generated by the three real roots, and the two conjugate complex roots, so complex conjugation is an automorphism of  $Q \subset L$ . As a permutation, it is a two-cycle.

The extension  $Q \subset L$  contains a root field  $Q(\theta)$  for  $P$ . Since  $P$  is irreducible, the order of the root field is 5. It follows that  $[L : K]$  is divisible by 5. Since  $[L : K]$  is also the order of the Galois group, the order of the group is divisible by 5. Then by Cauchy's lemma, the Galois group contains an element  $\sigma$  of order five. Imagine this element written in cyclic notation. It must be a five cycle, else it would have order relatively prime to 5.

To complete the argument, we show that any subgroup  $G$  of  $S_5$  containing a five cycle and a two cycle must be the entire group.

The 2-cycle and the 5-cycle have two elements in common. Number the roots so one common element is 1 and the 5-cycle is  $(1\ 2\ 3\ 4\ 5)$ . The square of this element is  $(1\ 3\ 5\ 2\ 4)$ , its cube is  $(1\ 4\ 2\ 5\ 3)$ , etc. All of these elements except the identity are also 5-cycles whose powers give all these 5-cycles. The second element of one of these elements is the second element of the 2-cycle. By renumbering, we can suppose the 5-cycle is  $(1\ 2\ 3\ 4\ 5)$  and the 2-cycle is  $(1\ 2)$ .

The set of all two cycles generates the full symmetric group  $S_n$ . To see this, write an arbitrary permutation in cycle notation and consider one of the cycles. By renumbering, it equals  $(1\ 2\ \dots\ k)$  and

$$(1\ 2\ \dots\ k) = (1\ k) \dots (1\ 3)(1\ 2)$$

So it suffices to show that our group contains all two cycles.

But  $\sigma\tau\sigma^{-1}$  applies the permutation  $\sigma$  to the elements of  $\tau$ . For instance  $(12345)(12)(12345)^{-1} = (23)$ .

It follows that our group contains  $(1\ 2)$ ,  $(2\ 3)$ ,  $(3\ 4)$ ,  $(4\ 5)$ , and  $(5\ 1)$ .

Also  $(1\ 3) = (2\ 3)(1\ 2)(2\ 3)$ ,  $(1\ 4) = (3\ 4)(1\ 3)(3\ 4)$ , and  $(1\ 5) = (4\ 5)(1\ 4)(4\ 5)$ . Also  $(2\ 4) = (3\ 4)(2\ 3)(3\ 4)$ ,  $(2\ 5) = (4\ 5)(2\ 4)(4\ 5)$ . Also  $(3\ 5) = (4\ 5)(3\ 4)(4\ 5)$ .

So our group contains all 2-cycles and equals  $S_5$ . This group is not solvable, so  $x^5 - 4x + 2$  cannot be solved with radicals.

## Chapter 10

# Straightedge and Compass Constructions

### 10.1 Constructions With Straightedge and Compass

Euclidean geometry is based on straightedge and compass constructions. The fundamental postulates immediately reveal this, since they proclaim the existence of lines through two points, and of circles with given center and radius. The very first theorem in Euclid asserts that equilateral triangles exist, proving this by a construction. Draw a base, draw circles with radius the base through the two endpoints of the base, and select the intersection of these circles as a third point.

It is well-known that many important constructions cannot be done with straightedge and compass; Galois theory provides exactly the tools needed to show this.

Our first step is to describe a straightedge and compass construction with sufficient rigor.

**Definition 9** *A construction problem is a finite set of points in the plane, called the **givens**, and another finite set of points in the plane, called the **unknowns**. Additional points may be added to the givens using one of the three methods listed below. The goal is to do this over and over until finally the unknowns are among the givens.*

- *If  $P_1$  and  $Q_1$  are distinct points already known, and if  $P_2$  and  $Q_2$  are distinct points already known, draw the line  $L_1$  through  $P_1$  and  $Q_1$  and draw the line  $L_2$  through  $P_2$  and  $Q_2$ . If these lines are different and not parallel, their intersection point can be added to the givens.*

- Suppose  $P_1$  and  $P_2$  are distinct points already known. Suppose  $P_3$  is a known point. Suppose  $P_4$  and  $P_5$  are distinct points already known. Draw the circle  $C$  through  $P_3$  with radius the distance from  $P_1$  to  $P_2$ . Draw the line  $L$  through  $P_4$  and  $P_5$ . This line and circle intersect in from 0 to 2 points. These intersection points can be added to the givens.
- Suppose  $P_1$  and  $P_2$  are distinct points already known. Suppose  $P_3$  is a known point. Suppose  $P_4$  and  $P_5$  are distinct points already known, and  $P_6$  is a known point. Draw the circle through  $P_3$  with radius the distance from  $P_1$  to  $P_2$ . Draw the circle through  $P_4$  with radius the distance from  $P_4$  to  $P_5$ . If the two circles are not equal, they may intersect in from zero through two points. In this case, add any intersection points to the givens.

*Remark:* At first it is not clear that these rules capture what is allowed during a construction. Consider the problem of bisecting an angle. We are actually given two intersecting lines, and we are asked to construct a third line.

However, we easily translate to the new language. We could describe this problem by giving the vertex  $P_1$  and a second point  $P_2$  on the first line. Once we know the second line, we could intersect it with the circle with center  $P_1$  through  $P_2$ , so we can give the second line by giving the point  $P_3$  where this circle intersects the second line. The angle bisector will then determine a final point on the circle, which becomes our unknown  $P_4$ .

Select coordinates so  $P_1 = (0, 0)$  and  $P_2 = (1, 0)$ . Then  $P_3 = (\cos \theta, \sin \theta)$ . These three points are our givens. The unknown is  $(\cos \frac{\theta}{2}, \sin \frac{\theta}{2})$ .

### Theorem 31

- If only one point is given, it is impossible to construct other points.
- If at least two points  $P_1$  and  $P_2$  are given, we can choose orthonormal coordinates so  $P_1 = (0, 0)$  and  $P_2 = (1, 0)$ . If  $P_3 = (a, b)$  is a third point, we can construct  $(a, 0)$  and  $(b, 0)$ . Conversely if these points are given, we can construct  $(a, b)$ .

*Proof:* The first item, and the first line of the second option, are obvious. So assume  $P_1 = (0, 0)$  and  $P_2 = (1, 0)$ . Construct a perpendicular to the line through these points at  $P_1$ . If  $(a, b)$  is given, construct the perpendiculars from these points to the two coordinate lines, getting  $(a, 0)$  and  $(0, b)$ . Using a circle with vertex the origin, construct  $(b, 0)$  from  $(0, b)$ .

Conversely, given  $(a, 0)$  and  $(b, 0)$ , draw the perpendicular at  $(0, 0)$  as before. Using a circle through the origin, convert  $(b, 0)$  to  $(0, b)$ . Construct perpendiculars to  $(a, 0)$  and  $(0, b)$ . These perpendiculars intersect at  $(a, b)$ . QED

*Remark:* Using this theorem, the general construction problem can be reformulated. Without loss of generality, we can take the givens to be finitely many real numbers, including

in particular 0 and 1, and we can take the unknowns to be finitely many additional real numbers.

**Theorem 32** *Given a known finite set of given real numbers, additional unknowns can be constructed using addition, subtraction, multiplication, division, and taking square roots of givens. Conversely, any constructible real can be formed by a finite combination of these operations.*

*Proof:* Adding and subtracting reals by compass is easy. Notice that multiplication only makes sense if we have a scale, for  $a^2$  will be larger than  $a$  if  $a > 1$  but smaller than  $a$  if  $a < 1$ . So the technique for multiplying must involve the segment of length 1. Once this is known, a construction is easy.

One is shown below. The idea of this construction is to form similar triangles and notice that  $\frac{a}{1} = \frac{ab}{b}$ . The reader can easily provide details.

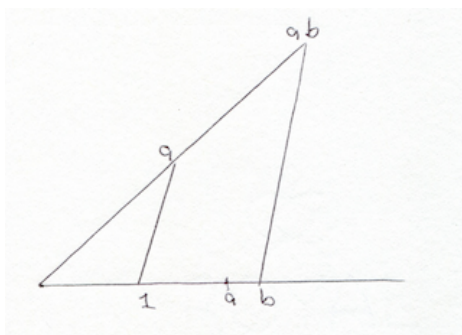


Figure 10.1: Multiply

The same diagram can be used to divide, using the similarity condition  $\frac{a}{b} = \frac{a}{1}$ .

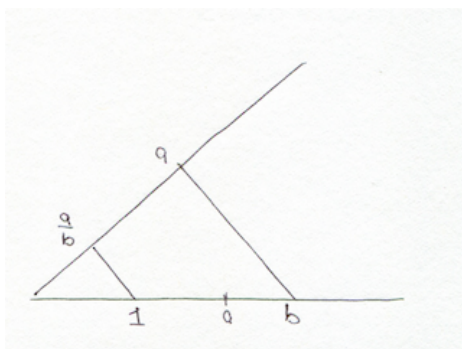


Figure 10.2: Divide

Calculating square roots also requires a unit. Below is one possible construction from <http://math.stackexchange.com>. Let the horizontal distance  $AC = 1$  and draw the vertical line  $AD$ . Call its length  $L$ . By similar triangles,  $\frac{a}{L} = \frac{L}{1}$  and so  $L = \sqrt{a}$ .

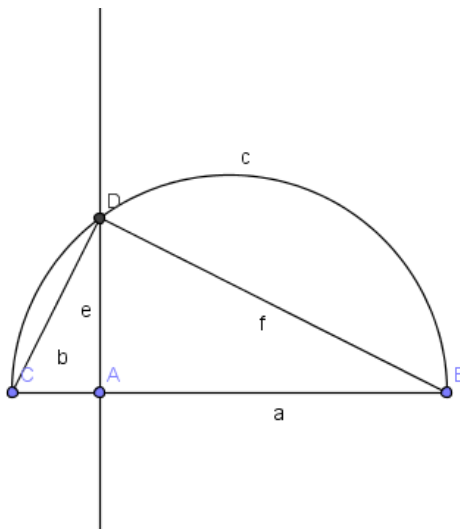


Figure 10.3: Square Root

Conversely, consider the operation of adding the intersection of two lines, when the coordinates of the given points and points constructed so far generate a field  $K$ . Each line is determined by two points with coordinates in  $K$ , so the equations of the lines have the form  $ax + by + c = 0$  and  $dx + ey + f = 0$  where  $a, b, c, d, e, f \in K$ . Ordinary high school algebra can be used to find the intersection point, and this algebra only requires addition, subtraction, multiplication, and division of the coordinates. So the coordinates of the intersection point are still in  $K$ .

Consider the operation of intersecting a line with a circle. Leaving the special case of a vertical line to the reader, we can assume that the line is given by  $y = ax + b$  where  $a, b \in K$ . The equation of the circle is  $(x - p)^2 + (y - q)^2 = r^2$  where  $r$  is the radius. This radius can be computed using the Pythagorean theorem applied to two points with coordinates in  $K$ , so  $p, q, r^2 \in K$ . To solve, we replace  $y$  by  $ax + b$ , obtaining  $(x - p)^2 + (ax + b - q)^2 = r^2$ . This is a quadratic equation in  $x$ , and solving for  $x$  only requires a square root. Once  $x$  is known,  $y$  can be found by solving the same formula, possibly requiring another square root. (There is no need to worry about cases where there is no solution, because if the line and circle do not intersect, we don't generate new points.)

Finally, the operation of intersecting two circles involves solving two equations  $(x - p)^2 + (y - q)^2 = r^2$  and  $(x - s)^2 + (y - t)^2 = u^2$  simultaneously. If we subtract one equation from the other, the terms  $x^2$  and  $y^2$  cancel out and we get a linear equation in  $x$  and  $y$ . Points

on both circles lie on this line, so we are back to the previous case. QED

*Remark:* Summarizing, a straightedge and compass construction begins with *gives*; numbers which can be constructed from them using addition, subtraction, multiplication, and division. This gives a field  $Q \subset K_1$ . We can then construct the extension  $K_2$  formed from  $K_1$  by adding the square root of a positive number in  $K_1$ , and then extending to the field this root generates. Call this  $K_2$ . In general

**Theorem 33**

- If a set of *gives* generates a field  $Q \subset K_1$  and a set of *unknowns* generates a field  $Q \subset L$ , then we can construct the *unknowns* from the *gives* if and only if there is a sequence of extensions

$$Q \subset K_1 \subset K_2 \subset \dots \subset K_n$$

where each  $K_i$  is formed from  $K_{i-1}$  by adding a square root of a positive number in  $K_{i-1}$ , such that  $L \subset K_n$ .

- In particular, the dimension of  $L$  over  $K_1$  must be a power of 2.

*Proof:* This is obvious from the discussion above.

## 10.2 Complex Extensions and Constructions

In the previous section, we reduced straightedge and compass constructions to the existence of a chain of real extensions  $Q \subset K_1 \subset \dots \subset K_n$ . The results remain unchanged, however, if we work in the plane and use complex extensions.

We could then assume that the *gives* form a set of points in the plane. Label two of them as 0 and 1 and use this to identify the plane with the set of complex numbers. The *unknowns* form a larger subset of  $C$ . Using straightedge and compass constructions, we can add, subtract, multiply, and divide complex numbers. We can also find complex square roots, essentially because we can form real square roots and can bisect angles. Therefore

**Theorem 34**

- If a set of *gives* generates a field  $Q \subset K_1$  and a set of *unknowns* generates a field  $Q \subset L$ , then we can construct the *unknowns* from the *gives* if and only if there is a sequence of extensions

$$Q \subset K_1 \subset K_2 \subset \dots \subset K_n$$

where each  $K_i$  is formed from  $K_{i-1}$  by adding a square root of a complex number in  $K_{i-1}$ , such that  $L \subset K_n$ .

- In particular, the dimension of  $L$  over  $K_1$  must be a power of 2.

### 10.3 Trisecting Angles; Doubling the Cube

**Theorem 35** *It is not possible in general to trisect an angle using straightedge and compass. In particular, given two starting points  $(0, 0)$  and  $(1, 0)$ , we can construct a 60-degree angle at the origin, but cannot construct a 20-degree angle at the origin.*

*Proof:* Construct an equilateral triangle at the origin using Euclid's first theorem. Intersect this triangle with the circle centered at the origin and with radius 1, to get the point  $(\cos 60^\circ, \sin 60^\circ) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ . The unknown is the point  $(\cos 20^\circ, \sin 20^\circ) = (\cos \theta, \sin \theta)$ . By de Moivre,

$$(\cos \theta + i \sin \theta)^3 = (\cos 3\theta + i \sin 3\theta) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Comparing real parts,  $\cos^3 \theta - 3 \cos \theta \sin^2 \theta = \frac{1}{2}$ . So  $\cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta) = \frac{1}{2}$ . If  $X = \cos \theta$ , then

$$4X^3 - 3X - \frac{1}{2} = 0$$

But  $P(X) = 4x^3 - 3x - \frac{1}{2}$  is irreducible over  $Q$  since it does not have a rational root. Indeed if  $\frac{a}{b}$  is written in lowest terms and solves the equation, then  $8a^3 - 6ab^2 - b^3 = 0$ . If  $p$  divides  $a$ , then  $p$  divides  $b$ , which is disallowed, so we can assume  $a = 1$ . Thus  $8 - 6b^2 - b^3 = 0$ . Since 2 divides  $8 - 6b^2$ , 2 divides  $b$ , so  $b = 2B$ . Then  $8 - 24B^2 - 8B^3 = 0$  and so  $1 - 3B^2 - B^3 = 0$ . If  $p$  divides  $B$ , then  $p$  divides 1, so  $B$  is  $\pm 1$ , but neither of these works.

So  $Q \subset Q(\cos 20^\circ)$  has degree 3 and that is not a power of 2. QED.

*Warning:* It does not follow that trisection is always impossible. For example, a  $45^\circ$  angle is easily trisected, because a  $15^\circ$  angle can be constructed by making an equilateral triangle and bisecting a vertex angle twice.

*Remark:* In an article on the web by J. J. O'Connor and E. F. Robertson, [http://www-history.mcs.st-and.ac.uk/PrintHT/Doubling\\_the\\_cube.html](http://www-history.mcs.st-and.ac.uk/PrintHT/Doubling_the_cube.html), we read two accounts of the so-called Delian problem of ancient Greek mythology. Theon of Smyra quotes Eratosthenes, as translated by Heath:

Eratosthenes, in his work entitled *Platonicus* relates that, when the god proclaimed to the Delians through the oracle that, in order to get rid of a plague, they should construct an altar double that of the existing one, their craftsmen fell into great perplexity in their efforts to discover how a solid could be made the double of a similar solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the god wanted an altar of double the size, but that he wished, in setting them the task, to shame the Greeks for their neglect of mathematics and their contempt of geometry.



Eutocius writes

Eratosthenes to King Ptolemy, greetings. The story goes that one of the ancient tragic poets represented Minos having a tomb built for Glaucus, and that when Minos found that the tomb measured a hundred feet on every side, he said "Too small is the tomb you have marked out as the royal resting place. Let it be twice as large. Without spoiling the form, quickly double each side of the tomb". This was clearly a mistake. For if the sides are doubled the surface is multiplied fourfold and the volume eightfold.

The Greek geometers took this ancient mythology to have the following meaning. Given the base of a cube, construct using straightedge and compass the base of a larger cube whose volume is twice the volume of the original cube.

**Theorem 36** *It is impossible to double a cube using straightedge and compass.*

*Proof:* If we take the endpoints of the base of the original cube to be  $(0, 0)$  and  $(1, 0)$ , then it is required that we construct  $(a, 0)$  where  $a^3 = 2$ . Thus we are to form  $Q(\sqrt[3]{2})$ . But the minimal polynomial of this number is  $X^3 - 2$ , which is irreducible by Eisenstein. So  $[Q(\sqrt[3]{2} : Q) = 3$ , and the construction cannot be done. QED.

The most famous of all Greek construction problems is the problem of squaring the circle. Given the radius of a circle, the problem is to construct the side of a square with the same area. If the center of the circle is  $(0, 0)$  and the circle goes through  $(1, 0)$ , we asked to construct a line of length  $\sqrt{\pi}$ , and so  $(\sqrt{\pi}, 0)$ .

**Theorem 37** *It is impossible to square a circle using straightedge and compass.*

*Proof:* All constructed points are algebraic over  $Q$ . But  $\pi$  is not algebraic, and therefore  $\sqrt{\pi}$  is not algebraic. This is proved in the next chapter.

## Chapter 11

# Irrationality and Transcendence of $\pi$ and $e$

To complete the proof that it is impossible to square a circle, we must prove that  $\pi$  is transcendental. The proof uses  $e^{i\pi} = -1$ , so we study  $e$  and  $\pi$  simultaneously.

### 11.1 Irrationality $e$ and $\pi$

**Theorem 38** *The number  $e$  is irrational*

*Proof:* Write

$$e = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!} \left( 1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots \right)$$

Therefore

$$n!e = n! \left( 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} \right) + R$$

where  $0 < R < \frac{1}{n+1}(e-1)$ . If  $e = \frac{a}{b}$  and  $n$  is larger than  $b$ , then  $n!e$  is an integer. The number  $n! \left( 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} \right)$  is an integer, so  $R$  is an integer. But as  $n$  increases, the estimate on  $R$  shows that it is positive and bounded by a series of numbers approaching 0. Such a series of  $R$ 's cannot all be integers. QED.

*Remark:* The following beautiful proof is essentially due to Niven. It is not the original proof of the theorem, which was by Lambert using continued fractions.

**Theorem 39**  *$\pi$  is irrational*

*Proof:* Consider

$$I_n = \int_0^\pi \frac{x^n(\pi-x)^n}{n!} \sin x \, dx$$

These integrals can be computed using integration by parts, and a short calculation shows that

- $I_0 = 2$
- $I_1 = 4$
- $I_2 = 24 - 2\pi^2$
- $I_3 = 240 - 24\pi^2$
- $I_4 = 3360 - 360\pi^2 + 2\pi^4$
- $I_5 = 60480 - 6720\pi^2 + 60\pi^4$

We claim that this pattern continues;  $I_n$  is a polynomial in  $\pi$  of degree at most  $n$ , with integer coefficients. An easy calculation gives the results in the table for  $n = 0, 1$ .

We now work by induction, and integrate by parts twice. Let  $G(x) = \frac{x^n(\pi-x)^n}{n!}$ . The boundary terms from integrating by parts will vanish because  $F(x)$  and  $F'(x)$  vanish at the limits of integration. So

$$I_n = \int G[x] \frac{d}{dx}(-\cos x) \, dx = \int G'(x) \cos x \, dx = \int G'(x) \frac{d}{dx} \sin x \, dx = - \int G''(x) \sin x \, dx$$

Moreover

$$G'(x) = \frac{x^{n-1}(\pi-x)^n}{(n-1)!} - \frac{x^n(\pi-x)^{n-1}}{(n-1)!}$$

and

$$G''(x) = \frac{x^{n-2}(\pi-x)^n}{(n-2)!} - 2n \frac{x^{n-1}(\pi-x)^{n-1}}{(n-1)!} + \frac{x^n(\pi-x)^{n-2}}{(n-2)!}$$

This expression equals

$$G''(x) = \left( (\pi-x)^2 + x^2 \right) \frac{x^{n-2}(\pi-x)^{n-2}}{(n-2)!} - 2n \frac{x^{n-1}(\pi-x)^{n-1}}{(n-1)!}$$

Notice that

$$(\pi-x)^2 + x^2 = \pi^2 - 2\pi x + x^2 + x^2 = \pi^2 - 2x(\pi-x)$$

It follows that

$$I_n = -\pi^2 I_{n-2} + 2(n-1)I_{n-1} + 2nI_{n-1} = (4n-2)I_{n-1} - \pi^2 I_{n-2}$$

and the lemma follows by induction.

Suppose  $\pi = \frac{a}{b}$ . Then  $b^n I_n$  is an integer, because  $b^n$  clears the denominators of all powers of  $\frac{a}{b}$  of degree less than or equal to  $n$ . From the definition,  $b^n I_n > 0$ . However, by symmetry, the integrand is largest at  $\frac{\pi}{2}$  when it is  $\left(\frac{\pi}{2}\right)^{2n} \frac{1}{n!}$ . Let  $A = \frac{\pi^2}{4}$ .

$$0 < b^n I_n \leq \pi \left(\frac{A}{1}\right) \left(\frac{A}{2}\right) \cdots \left(\frac{A}{n}\right)$$

The right side of this expression goes to zero as  $n$  goes to infinity, so  $b^n I_n$  cannot always be a positive integer. QED

## 11.2 Transcendence of $e$

The proofs of this section come from a 1970 course by Ian Stewart.

**Theorem 40 (Hermite)**  $e$  is transcendental

*Proof:* Suppose

$$a_0 + a_1 e + a_2 e^2 + \cdots + a_n e^n = 0$$

with the  $a_i$  integers.

The proof starts with a simple calculation from freshman calculus. For  $p$  an arbitrary positive integer, define

$$f(x) = \frac{x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p}{(p-1)!}$$

Define

$$F(x) = f(x) + f'(x) + f''(x) + \cdots$$

where the terms eventually vanish because  $f$  is a polynomial. Then

$$\frac{d}{dx} (e^{-x} F(x)) = e^{-x} (F'(x) - F(x)) = -e^{-x} f(x)$$

and therefore

$$a_j \int_0^j e^{-x} f(x) dx = a_j \left[ -e^{-x} F(x) \right]_0^j = a_j F[0] - a_j e^{-j} F[j]$$

Multiply by  $e^j$  and sum from  $j = 0$  to  $j = n$  to get

$$\sum_{j=0}^n a_j e^j \int_0^j e^{-x} f(x) dx = \sum_{j=0}^n a_j e^j F[0] - \sum_{j=0}^n a_j F[j]$$

$$= - \sum_{j=0}^n \sum_i a_j f^{(i)}(j)$$

In the end, we obtain the following result:

$$\sum_{j=0}^n a_j e^j \int_0^j e^{-x} f(x) dx = - \sum_{j=0}^n \sum_i a_j f^{(i)}(j)$$

This equation will lead to a contradiction as follows. The function  $f(x)$  depends on an integer  $p$ . We will prove that as  $p$  goes to infinity, the left side of the equation goes to zero. We will then show that  $f^{(i)}(j)$  is always an integer, and thus the right side is always an integer. Finally we restrict to the case that  $p$  is a prime larger than  $n$ . We'll show that  $f^{(i)}(j)$  is divisible by  $p$  except when  $j = 0$  and  $i = p - 1$ . We'll also show that the term with  $j = 0$  and  $i = p - 1$  is not divisible by  $p$ . It follows that the right side is a nonzero integer for all large primes, and we obtain a contradiction. The theorem follows.

Here are the details. First we estimate the left side of the equation. The term  $\sum_{j=0}^n a_j e^j$  is a constant, so we need only estimate the integral. We can be fairly sloppy. Over the interval  $[0, j] \subset [0, n]$  we have  $|x - k| < n$  and consequently the integrand is at most  $n$  to the power  $(p - 1) + np$ . The term  $e^{-x}$  is bounded by 1. The length of the interval gives an extra  $n$ . so the total integral is smaller than  $\frac{n^{1+(p-1)+np}}{(p-1)!} = \frac{n^{(n+1)p}}{(p-1)!}$ . The term  $n^{n+1}$  is a constant  $C$  and this term is  $\frac{C^p}{(p-1)!}$  which equals

$$C \frac{C}{1} \frac{C}{2} \dots \frac{C}{p}$$

As  $p$  increases, this term goes to zero.

Now the details for the right side. From now on, assume  $p$  prime. If  $1 \leq j \leq n$ , notice that the  $f^{(i)}(j)$  will vanish unless the  $(x - j)^p$  term was differentiated exactly  $p$  times, and in that case the  $(p - 1)!$  in the denominator will be canceled and there will be an extra  $p$  in the numerator. When  $(x - j)^p$  is differentiated  $p$  times, the other terms may also have been differentiated, but these will only contribute integer multiples of an integer already divisible by  $p$ .

The term  $f^{(i)}(0)$  will vanish unless the term  $x^{(p-1)}$  is differentiated exactly  $p - 1$  times. In that case the  $(p - 1)!$  in the denominator will be cancelled, but there will be no  $p$  in the numerator. Of course the remaining terms may also have been differentiated, but if a remaining term is differentiated even once, we'll get an extra  $p$  and the term will be divisible by  $p$ . The only exception is when  $x^{p-1}$  is differentiated  $p - 1$  times and no other term is differentiated. In this case, the value at 0 is not divisible by  $p$  because  $p > n$ . QED.

### 11.3 Intermission: Fundamental Theorem of Symmetric Polynomials

We alluded to the following theorem earlier, but now we need it in the proof of the transcendence of  $\pi$ .

**Theorem 41** *Let  $P(X_1, \dots, X_n)$  be a polynomial over a field  $K$ . Suppose  $P$  is symmetric in the sense that  $P(X_{\tau(1)}, \dots, X_{\tau(n)}) = P(X_1, \dots, X_n)$  for all permutations  $\tau \in S_n$ . Then  $P$  can be written as a polynomial in the elementary symmetric functions:*

$$P(X_1, \dots, X_n) = Q(\sigma_1(X_1, \dots, X_n), \dots, \sigma_n(X_1, \dots, X_n))$$

for some polynomial  $Q$ . If  $P$  has integer coefficients, so does  $Q$ .

*Preliminary Remark:* When  $K$  is a finite field, a polynomial can vanish identically on  $K \times \dots \times K$  and yet not be the zero polynomial. This theorem is about polynomials, not about functions on  $K \times \dots \times K$ . When we claim that two polynomials are equal, we are claiming that they have the same coefficients, not just that they take the same values.

*Proof:* We prove both parts at the same time by assuming that our polynomials have coefficients in a commutative ring with unit  $A$ . Define the *weight* of a polynomial to be the maximum of the weights of the monomials which occur in it, and define the weight of  $X_1^{k_1} X_2^{k_2} \dots X_n^{k_n}$  to be  $k_1 + 2k_2 + \dots + nk_n$ .

We will prove a slightly stronger theorem, namely that  $Q$  exists and has weight at most the degree of  $P$ . This result will be proved by a double induction, first on the number of variables  $n$ , and then on the degree  $d$  of  $P$ .

The theorem is obvious if  $n = 1$  because then there is only one variable and  $Q = P$ . So assume the theorem is true for all polynomials in  $n - 1$  variables. We prove it true when there are  $n$  variables. The result is obvious if the degree of  $P$  is zero, so we can assume it true when the degree of a polynomial is smaller than  $d$ . Suppose our  $P$  has degree  $d$ .

Then  $P(X_1, \dots, X_{n-1}, 0)$  can be written as a polynomial  $G$  in the  $n - 1$  elementary symmetric functions in  $X_1, \dots, X_{n-1}$ . It is easy to see that these are exactly the first  $n - 1$  elementary symmetric functions in  $X_1, \dots, X_n$ , evaluated when  $X_n = 0$ . Form

$$P_1(X_1, \dots, X_n) = P(X_1, \dots, X_n) - G(\sigma_1(X_1, \dots, X_n), \dots, \sigma_{n-1}(X_1, \dots, X_n))$$

Since the degree of  $P(X_1, \dots, X_{n-1}, 0)$  is at most  $d$ , the weight of  $G$  is at most  $d$ , and therefore the degree of  $G(\sigma_1(X_1, \dots, X_n), \dots, \sigma_{n-1}(X_1, \dots, X_n))$  is at most  $d$ .

It now suffices to prove the result for  $P_1$ , which vanishes when  $X_n = 0$ . Since we are dealing with polynomials rather than functions on the ground field, this statement actually means

that every monomial in the polynomial contains  $X_n$ . By invariance under the symmetric group, every monomial contains all  $X_i$ . It follows that

$$P_1(X_1, \dots, X_n) = \sigma_n(X_1, \dots, X_n)P_2(X_1, \dots, X_n)$$

The polynomial  $P_2$  must be symmetric, and it must have degree  $d - n$ . So by induction,  $P_2(X_1, \dots, X_n) = G(\sigma_1, \dots, \sigma_n)$  where the weight of  $G$  is at most  $d - n$ . It follows that  $P_2 = \sigma_n G(\sigma_1, \dots, \sigma_n)$  and the weight of the polynomial is at most  $d$ . QED.

## 11.4 Transcendence of $\pi$

**Theorem 42 (Lindemann)**  $\pi$  is transcendental

*Proof:* If  $\pi$  is algebraic, so is  $i\pi$  and  $e^{i\pi} + 1 = 0$ . Let  $\alpha_1, \dots, \alpha_n$  be the roots of an irreducible polynomial over  $\mathbb{Q}$  satisfied by  $i\pi$ . Then

$$(e^{\alpha_1} + 1)(e^{\alpha_2} + 1) \dots (e^{\alpha_n} + 1) = 0$$

Rewrite this in the form

$$e^{\beta_1} + \dots + e^{\beta_r} + k = 0$$

where  $k$  is a positive integer and the  $\beta_i$  are partial sums, like  $\alpha_3 + \alpha_7 + \alpha_8$ , of the  $\alpha_1, \alpha_2, \dots, \alpha_n$ . The 1's in our formula correspond to those subsums that add to zero.

We claim there is a polynomial  $P(X) = c_0 + c_1X + \dots + c_rX^r$  with integer coefficients such that the roots of  $P$  are exactly the nonzero  $\beta_i$ .

Indeed, the polynomial  $Q(X)$  has integer coefficients and roots  $\alpha_i$ . Call this  $Q_1(X)$ .

Define

$$Q_2(X) = \prod_{i < j} (X - (\alpha_i + \alpha_j))$$

The coefficients of this polynomial are clearly symmetric polynomials in the  $\alpha_i$  with integer coefficients. By the fundamental theorem of symmetric polynomials, each coefficient can be written as a polynomial in the elementary symmetric functions of the  $\alpha_i$  with integer coefficients. These elementary symmetric functions of the  $\alpha_i$  are exactly the coefficients of the original  $Q(X)$ , and so each is an integer. It follows that the coefficients of  $Q_2(X)$  are integers.

Define

$$Q_3(X) = \prod_{i < j < k} (X - (\alpha_i + \alpha_j + \alpha_k))$$

By the previous argument, this polynomial has integer coefficients.

Continue. In the end, let  $P(X) = Q_1(X)Q_2(X)\dots Q_n(X)$ . This polynomial has integer coefficients and its roots are all sums of subsets of the  $\alpha_i$ . Factor out as many  $X$  as possible, and redefine  $P(X)$  to be the remaining polynomial, which now has no zero roots.

We are now in a position to apply the ideas used in the proof that  $e$  is transcendental. Let  $r$  be the degree of  $P(X)$  and let  $c$  be the coefficient of the highest power of  $X$  in  $P(X)$ . If  $p$  is a positive integer, define

$$f(x) = c^{rp-1}x^{p-1}\frac{P(X)^p}{(p-1)!}$$

Define

$$F(x) = f(x) + f'(x) + f''(x) + \dots$$

where the terms eventually vanish because  $f$  is a polynomial. Then

$$\frac{d}{dx}(e^{-x}F(x)) = e^{-x}(F'(x) - F(x)) = -e^{-x}f(x)$$

and therefore

$$\int_0^y e^{-x}f(x) dx = \left[-e^{-x}F(x)\right]_0^y = F[0] - e^{-y}F[y]$$

We now diverge slightly from the algebra in the case of  $e$ . In the integral on the left, substitute  $x = y\lambda$ . Then we get  $y\int_0^1 e^{-\lambda y}f(\lambda y)d\lambda = F[0] - e^{-y}F(y)$ . Multiplying both sides by  $e^y$  gives

$$y\int_0^1 e^{y(1-\lambda)}f(\lambda y) d\lambda = e^yF(0) - F(y)$$

Substitute  $\beta_i$  for  $y$  and sum. Recall that  $\sum e^{\beta_i} = -k$ . So

$$\sum \beta_i \int_0^1 e^{\beta_i(1-\lambda)}f(\lambda\beta_i) d\lambda = -kF(0) - \sum F(\beta_i)$$

As before, we come to the moment when we can derive a contradiction. We claim that for all large primes  $p$ , the right hand side is a nonzero integer. But an easy calculation shows that the left side goes to zero as  $p$  goes to infinity.

Let us study the left side first. The only term in this expression depending on  $p$  is  $f(\lambda\beta_i)$ . Note that

$$f(x) = c^{r-1}P(X)\frac{(c^r x P(x))^{p-1}}{(p-1)!}$$

We need to evaluate this on a bounded set of the complex plane containing  $\lambda\beta_i$  where  $0 \leq \lambda \leq 1$ . On this set,  $x$  and  $P(X)$  are bounded. Clearly, then, the absolutely value of the expression is bounded by

$$B_1 \frac{B_2^{p-1}}{(p-1)!} = B_1 \frac{B_2}{1} \frac{B_2}{2} \dots \frac{B_2}{p-1}$$



and this expression goes to zero as  $p$  goes to infinity.

Finally, consider the right hand side. We claim  $-kF(0)$  is an integer not divisible by  $p$  for  $p$  large enough, and we claim that  $\sum F(\beta_i)$  is an integer divisible by  $p$ . If these claims hold, then for all sufficiently large  $p$ , the right hand side is an integer not divisible by  $p$ , and thus a nonzero integer, and we have a contradiction.

Consider first  $kF(0) = k(f(0) + f'(0) + f''(0) + \dots)$ . The only terms that matter at 0 are terms when  $x^{p-1}$  was differentiated exactly  $p - 1$  times. For exactly one of these terms,  $P(X)^p$  was not differentiated. For the remaining terms,  $P(X)^p$  was differentiated at least once. Differentiating  $x^{p-1}$  for  $p - 1$  times removed the  $(p - 1)!$  in the denominator; this term is not divisible by  $p$  for large  $p$  since  $P(0) = c_0$  is not divisible by  $p$ . The remaining terms involve at least one  $pP(X)^{p-1}$  and thus are divisible by  $p$ .

Therefore the full term  $-kF(0)$  is an integer which is not divisible by  $p$ . To finish the argument, it suffices to show that  $\sum F(\beta_i)$  is an integer which *is* divisible by  $p$ .

Since  $P(\beta_i) = 0$ ,  $f^{(i)}(\beta_i)$  can vanish only if the term  $P(X)^p$  were differentiated at least  $p$  times. Each time a derivative is taken, many other algebraic terms form, but unless a term in the derivative has no  $P(X)$  remaining, it will be zero at  $\beta_i$ . Consequently, the only terms in the derivatives which matter contain  $p!$ , which cancels out the  $(p - 1)!$  in the denominator and leaves an extra  $p$ . What is left after all of this differentiation is a polynomial  $Q(X)$ , possibly of very large degree, with integer coefficients. But  $\sum Q(\beta_i)$  will then be an expression in terms of the form  $\beta_1^k \beta_2^k \dots \beta_b^k$ . These are symmetric in the  $\beta_i$  and thus can be expressed as polynomials with integer coefficients in the elementary symmetric functions, and thus polynomials with integer coefficients in the coefficients of  $P(X)$ . So we get integers divisible by  $p$ .

# Chapter 12

## Special Cases

In this chapter, we obtain much more explicit results about the Galois groups of special extensions.

In particular, we obtain results on cyclotomic extensions. The results are required in the following chapter, when we return to straightedge and compass constructions and discuss regular polygons.

### 12.1 The Discriminant

**Definition 10** Let  $P(X) = a_0X^n + a_1X^{n-1} + \dots + a_n$  be a polynomial with roots  $\lambda_1, \lambda_2, \dots, \lambda_n$ . The discriminant of the polynomial is the expression

$$D = a_0^{2n-2} \prod_{i < j} (\lambda_j - \lambda_i)^2$$

*Remark:* This expression is invariant under all permutations of the roots, and thus expressible in terms of the elementary symmetric functions. These elementary functions give the coefficients of  $P(X)$ , so the discriminant can be written as a polynomial in the coefficients of  $P(X)$ . For example, the discriminant of  $aX^2 + bX + C$  is

$$a^2 \left( \frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right)^2 = a^2 \left( \frac{\sqrt{b^2 - 4ac}}{a} \right)^2 = b^2 - 4ac$$

The discriminant of  $aX^3 + bX^2 + cX + d$  is

$$b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

In particular the discriminant of  $X^3 + aX + b$  is

$$-4a^3 - 27b^2$$

The discriminant of  $ax^4 + bx^3 + cx^2 + dx + e$  is

$$\begin{aligned} & 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e \\ & - 80abc^2de + 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2 \end{aligned}$$

**Theorem 43** *Let  $K \subset L$  be a splitting field for an irreducible  $P(X)$  without multiple roots over a field whose characteristic is not 2. Recall that the Galois group permutes the roots of  $P$  and thus is a subgroup of the symmetric group  $S_n$ . The Galois group is a subgroup of the alternating group  $A_n$  if and only if the discriminant of  $P$  has a square root in  $K$ .*

*Proof:* Since  $P$  does not have repeated roots,  $D \neq 0$ .

The expression

$$\sqrt{D} = a_0^{n-1} \prod_{i < j} (\lambda_j - \lambda_i)$$

belongs to the splitting field and has square  $D$ . It belongs to  $K$  if and only if it is invariant under every element of the Galois group  $G$ . If  $\tau$  is a permutation of the roots, we clearly have  $\tau(\sqrt{D}) = \text{sgn}(\tau)\sqrt{D}$ , so  $\sqrt{D} \in K$  if and only if every element of the Galois group has sign equal one and thus is in  $A_n$ .

## 12.2 The Cubic Case

The Galois group of a cubic is a transitive subgroup of  $S_3$ . The full  $S_3$  is just the dihedral group  $D_3$ , the symmetries of an equilateral triangle. This group has order 6, and contains three reflections and three rotations of 0, 120, and 240 degrees. A transitive subgroup must contain either 3 or 6 elements. If the group has three elements, these elements must be the rotations, because otherwise the group would contain two reflections, and any two reflections generate the full group. So the Galois group can only be  $Z_3$  or  $D_3$ .

**Theorem 44** *Let  $P(X)$  be an irreducible cubic over a field of characteristic zero. Then  $D \neq 0$ . If  $D$  is a perfect square in  $K$ , the Galois group is  $Z_3$ . If  $D$  is not a perfect square, the Galois group is  $D_3$ .*

*Proof:* This is a special case of the result in the previous section.

**Theorem 45** *Let  $P(X)$  be a polynomial with real coefficients. If  $D < 0$ ,  $P(X)$  has one real root and two conjugate complex roots. If  $D = 0$ ,  $P(X)$  has a repeated real root, and in particular is not irreducible. If  $D > 0$ ,  $P(X)$  has three real roots.*

*Proof:* It is clear that  $D = 0$  if and only if  $P(X)$  has a repeated root. In that case,  $P'$  divides  $P$ . If  $P$  has three real roots, each  $(\lambda_j - \lambda_i)$  is real and nonzero, and their product squared is positive. If  $P$  has one real root and two complex roots, the complex roots must be conjugate, so the three terms  $(\lambda_j - \lambda_i)$  contain one purely imaginary term, and two conjugate terms  $e + fi$  and  $e - fi$  with  $f \neq 0$ . The square of the purely imaginary term is negative, and the square of  $(e + fi)(e - fi) = e^2 + f^2$  is positive, so  $D < 0$ .

*Remark:* Turn back to the solution of a cubic of the form  $X^3 + aX + b$  on page 11. Notice that this solution contains  $\sqrt{b^2 + \frac{4a^3}{27}} = \frac{1}{3\sqrt{3}}\sqrt{4a^3 + 27b^2} = \frac{1}{3\sqrt{3}}\sqrt{-D}$ .

If  $D < 0$ , then our cubic has one real root and two complex roots. In this case the square root in the formula is real, and the formula involves a cube root of a real number, which always exists as a real number. So we can compute the real root without using complex numbers. In this case the Galois group is  $D_3$ .

But if  $D > 0$ , then there are three real roots, but the square root in the formula is a pure imaginary. We thus must use trigonometric functions to find the cube root. Miraculously, the complex parts of the two cube roots cancel out and the formula gives three real roots. The Galois group can be either  $Z_3$  or  $D_3$ .

*Remark:* To prove the next result, we need

**Lemma 10** *Let  $K$  be a field of characteristic zero,  $a \in K$ , and  $p$  a prime. Then  $P(X) = X^p - a$  is irreducible over  $K$  if and only if  $a$  is not the  $p$ th power of an element in  $K$ .*

*Proof:* If  $a = b^p$ , then  $X^p - a = X^p - b^p$  has a root  $b$  and therefore factors as  $P(X) = (X - b)S(X)$ .

Conversely suppose  $P(X) = X^p - a = R(X)S(X)$  over  $K$  and the degree of  $R(X)$  is  $r < p$ . Let  $L$  be the splitting field of  $P(X)$ . Then both  $P$  and  $R$  factor completely in  $L$  and roots of  $R$  are roots of  $P$ . So we can write  $R(X) = (X - \lambda_1)\dots(X - \lambda_r)$ . Since the coefficients of this polynomial are in  $K$ ,  $\omega = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_r \in K$ . We have  $\omega^p = \lambda_1^p \cdot \dots \cdot \lambda_r^p = a \cdot a \cdot \dots \cdot a = a^r$ .

Since  $r$  and  $p$  are relatively prime, we can find integers  $A$  and  $B$  with  $1 = Ar + Bp$ . Then  $a = a^{Ar} a^{Bp} = (a^r)^A a^{Bp} = (\omega^p)^A a^{Bp} = (\omega^A a^B)^p$ . It follows that  $\sqrt[p]{a} = \omega^A a^B \in K$ . QED.

**Theorem 46 (Casus Irreducibilis)** *If  $F(X)$  is an irreducible cubic with real coefficients and  $D > 0$ , then the complex roots which occur in the cubic formula are unavoidable. It is impossible to write any root of  $F(X)$  as a combination of real roots of various orders.*

*Proof* Let  $K$  be the base field; often  $K = \mathbb{Q}$ . Consider the extension  $K \subset K(\sqrt{D})$  where  $D$  is the discriminant of  $P(X)$ . Since  $D > 0$ , the left side still contains only real numbers.

Since  $P(X)$  is irreducible of degree 3 over  $K$ , any root field of  $P$  has degree 3, and thus  $K(\sqrt{D})$  does not contain any root of  $P(X)$ .

If the theorem is false, we can find a series of extensions

$$K(\sqrt{D}) \subset K_1 \subset \dots \subset K_n$$

such that each extension is obtained by adding one real radical to the previous field, and such that one root of  $P(X)$  belongs to  $K_n$ . We can suppose each radical extension adds a  $p$ th root for  $p$  a prime, since any root is a composition of such roots.

Clearly we can suppose that no root of  $P(X)$  is in  $K_{n-1}$  by backing up the tower until we come to the transition point.

Let  $K_{n-1} \subset L$  be the splitting field of  $P(X)$  over  $K_{n-1}$ . This is a Galois extension. Since the discriminant  $D$  is a square in  $K_{n-1}$ , the Galois group is  $Z_3$ , so  $[L, K_{n-1}] = 3$ . But any root field  $K_{n-1} \subset K_{n-1}(\lambda) \subset L$  also has degree 3. So any root extension over  $K_{n-1}$  contains all the roots of  $P$ . Since  $K_n$  contains a root, we must have  $K_{n-1} \subset L \subset K_n$ .

But  $[K_n : K_{n-1}] = p$  by the previous lemma. Since  $[K_n : K_{n-1}] = [K_n : L][L : K_{n-1}]$ , we conclude that  $p = 3$  and  $L = K_n$ . But then  $K_n$  is a Galois extension of  $K_{n-1}$ , and must contain all roots of  $X^3 - a = 0$ , which are  $\sqrt[3]{a}$ ,  $\sqrt[3]{a}\omega$  and  $\sqrt[3]{a}\omega^2$  for  $\omega = e^{\frac{2\pi i}{3}}$ . Since  $\omega$  is not real, this is a contradiction. QED.

## 12.3 Cyclotomic Fields

We begin with cyclotomic extensions, that is, extensions of the form  $Q \subset Q(e^{\frac{2\pi i}{n}})$ , and thus with the polynomial  $P(X) = X^n - 1$ . We restrict to a ground field  $Q$  and thus in particular to the characteristic zero case.

A common high school exercise factors this polynomial as  $(X - 1)(X^{n-1} + X^{n-2} + \dots + 1)$ , but we can often continue factoring:  $X^4 - 1 = (X^2 - 1)(X^2 + 1) = (X - 1)(X + 1)(X^2 + 1)$ . In

To understand the general situation, notice that the set of all cyclotomic roots is a cyclic group  $Z_n$  generated by  $\theta = e^{\frac{2\pi i}{n}}$ . The elements of order  $n$  in this group are called the *primitive*  $n$ th roots of unity, and it turns out that they are the roots of an irreducible polynomial  $\Phi_n(X)$  with integer coefficients. For instance, the primitive fourth roots of unity are  $\pm i$ , which are the roots of  $X^2 + 1$ . For each divisor  $d$  of  $n$ , the primitive  $d$ th roots of unity are exactly the elements of the group of order  $d$ . They satisfy  $\Phi_d(X) = 0$ . It follows that  $X^n - 1 = \prod_{d|n} \Phi_d(X)$ . For instance, the primitive square root of unity is  $-1$ , the root of  $X + 1$ , the primitive first root of unity is  $1$ , the root of  $X - 1$ , and

$$X^4 - 1 = (X - 1)(X + 1)(X^2 + 1).$$

**Theorem 47** *The polynomial  $X^n - 1$  factors as*

$$\prod_{d|n} \Phi_d(X)$$

where the roots of  $\Phi_d$  are exactly the primitive  $d$ th roots of unity. The polynomials  $\Phi_d$ , called the cyclotomic polynomials, are all irreducible over  $\mathbb{Q}$  with integer coefficients.

*Proof:* Define  $\Phi_d(X) = \prod (X - \alpha_i)$  over all primitive  $d$ th roots of unity. This is a monic polynomial, but for a moment the coefficients are unclear. However,  $X^n - 1 = \prod_{d|n} \Phi_d(X)$  for all  $n$  because every  $n$ th root of unity has some order  $d$  in the group of units.

These equations serve to recursively define the  $\Phi_i$ . We have  $\Phi_1 = X - 1$  and  $X^2 - 1 = (X - 1)(X + 1) = \Phi_1(X)\Phi_2(X)$ , so  $\Phi_2(X) = X + 1$ . In general

$$X^n - 1 = \left( \prod_{d|n; d < n} \Phi_d(X) \right) \Phi_n(X)$$

where the product is over known monic polynomials with integer coefficients, so by simple division we discover that  $\Phi_n$  has integer coefficients.

Therefore we need only prove

**Lemma 11** *Each  $\Phi_n(X)$  is irreducible over  $\mathbb{Q}$ .*

*Proof:* The proof isn't particularly difficult, but it is tricky.

Let  $\omega$  be a primitive  $n$ th root of unity, with minimal polynomial  $F(X)$  over  $\mathbb{Q}$ . If we can prove that all primitive  $n$ th roots are roots of  $F(X)$ , then  $F(X)$  must be divisible by  $\Phi_n$  over  $\mathbb{C}$ . Since both polynomials have rational coefficients, it must be divisible by  $\Phi_n$  over  $\mathbb{Q}$ . Since  $F$  is minimal, it must be equal to  $\Phi_n$ .

Every  $n$ th root of unity equals  $\omega^k$  for some  $k$ . The new root is primitive if and only if  $k$  is relatively prime to  $n$ . Indeed if  $j$  divides both  $k$  and  $n$ , then  $(\omega^j)^{n/j} = 1$  and so  $\omega^j$  is not a primitive root. Conversely if  $k$  and  $n$  are relatively prime, we can find integers  $a$  and  $b$  with  $1 = ak + bn$  and then  $\omega = (\omega^k)^a$  and so  $\omega^k$  is primitive.

Therefore every primitive root can be obtained by starting with  $\omega$  and a prime  $p$  not dividing  $n$ , forming  $\omega^p$ , finding a prime  $q$  not dividing  $n$ , raising the new number to the  $q$ th power, etc. So it is enough to prove the following lemma:

**Lemma 12** *Let  $F(X)$  be the minimal polynomial for a primitive  $n$ th root of unity  $\omega$ . Let  $p$  be a prime which does not divide  $n$ . Then  $F(\omega^p) = 0$ ,  $\omega^p$  is a primitive  $n$ th root, and  $F(X)$  is the minimal polynomial for  $\omega^p$ .*

Note that the lemma is clear provided we can prove that  $F(\omega^p) = 0$ .

Since  $\omega$  is a root of  $X^n - 1$ ,  $F(X)$  divides  $X^n - 1$  and  $X^n - 1 = F(X)G(X)$ . We claim  $F$  and  $G$  are monic with integer coefficients. This follows from Gauss' lemma on primitive polynomials.

If  $\omega^p$  is not a root of  $F$ , then it must be a root of  $G$ . So  $\omega$  is a root of  $G(X^p)$ . Since  $F(X)$  is the minimal polynomial of  $\omega$ , it follows that  $G(X^p) = F(X)H(X)$ . Repeating the Gauss technique, we see that since  $G(X^p)$  has integer coefficients, so do  $F(X)$  and  $H(X)$ .

Reduce modulo  $p$  to discover that in  $Z_p[X]$  we have  $X^n - 1 = F(X)G(X)$  and  $G(X^p) = G(X)^p = F[X]H[X]$ . Here we use the fact that  $G(X^p) = G(X)^p$  over  $Z_p$ .

But then any irreducible factor of  $F(X)$  is a factor of  $G(X)^p$  and thus a factor of  $G(X)$ , so  $X^n - 1$  has a repeated factor and thus over some extension field of  $Z_p$ , a multiple root. But this is impossible because  $X^n - 1 = (X - \alpha)^2 K(X)$  implies by formal differentiation that  $nX^{n-1} = 2(X - \alpha)K(X) + (X - \alpha)^2 K'(X)$  and so  $n\alpha^{n-1} = 0$ . This is impossible because  $\alpha^n - 1 = 0$  implies  $\alpha \neq 0$  and  $n \neq 0$  because  $p$  does not divide  $n$ . QED.

*Remark:* Once we know that  $\Phi_n(X)$  is irreducible over  $\mathbb{Q}$ , we get a Galois extension with a known Galois group. Indeed

**Theorem 48** *Let  $L$  be the splitting field of  $X^n - 1$  over  $\mathbb{Q}$ . Then  $L$  is a Galois extension, and its Galois group is the group of units in  $Z_n$ .*

*Proof:* Let  $\omega$  be a primitive  $n$ th root of unity and consider the root field of  $\Phi_n(X)$ . This field contains all powers of  $\omega$  and hence all roots of unity. In particular, the irreducible  $\Phi_n$  splits completely, so it is a splitting field over a field of characteristic zero, and hence a Galois extension.

An automorphism of this field is completely determined by the image of  $\omega$ . An image must be another root of  $\Phi_n(X)$  and so another primitive  $n$ th root of unity. Since the Galois group is transitive, all such roots occur.

Thus an automorphism has the form  $\omega \rightarrow \omega^k$  where  $k$  is only unique modulo  $n$ . The only  $k$  that can occur are those relatively prime to  $n$ . Said another way,  $k$  induces a unique element of  $Z_n$  and this element is a unit.

If  $j$  and  $k$  represent the automorphisms  $\omega \rightarrow \omega^j$  and  $\omega \rightarrow \omega^k$ , then their product is given by

$$\omega \rightarrow \omega^k \rightarrow (\omega^j)^k = \omega^{jk}$$

so the group product is given by multiplication in the ring  $Z_n$ .

*Remark:* The determination of the group structure of the ring of units in  $Z_n$  is a famous topic of elementary number theory. If  $n = p$  is prime, the group of units is  $Z_p^*$ , which is isomorphic to  $Z_{p-1}$ . Other interesting cases are  $Z_4^* = Z_2$  and  $Z_8^* = Z_2 \times Z_2$ . So the resulting group is not always cyclic.

The general result is given by the following theorem, which we will not prove:

**Theorem 49**

- If  $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots$ , then

$$Z_n^* = \left( Z_{p_1^{k_1}} \right)^* \times \left( Z_{p_2^{k_2}} \right)^* \times \left( Z_{p_3^{k_3}} \right)^* \times \dots$$

- If  $p$  is odd,

$$\left( Z_{p^k} \right)^* = Z_{p^k(p^k-1)}$$

- 

$$\left( Z_2 \right)^* = \{e\}$$

- 

$$\left( Z_4 \right)^* = Z_2$$

- For  $k \geq 3$ ,

$$\left( Z_{2^k} \right)^* = Z_2 \times Z_{2^{k-2}}$$

*Remark:* In particular, the Galois group of any cyclotomic extension is abelian. It follows that all subgroups are normal and abelian, and therefore any subgroup of the cyclotomic group produces a Galois extension  $Q \subset K$  with abelian Galois group.

In 1853, Kronecker proved the converse of this theorem: Every Galois extension of  $Q$  with abelian Galois group is a subfield of  $Q(\zeta_n)$  for some primitive root of unity  $\zeta_m$ .

Kronecker's proof had gaps, by his own admission. The gaps were supposedly filled by Weber in 1886, but this proof also had an error. In 1896, Hilbert gave the first complete proof.

Every quadratic extension  $Q \subset K$  has Galois group  $Z_2$  and thus can be embedded in a cyclotomic extension. By comparing the factorization of prime ideals ( $p$ ) in the quadratic extension and in the cyclotomic extension, Gauss' law of quadratic reciprocity turns out to be a consequence of Kronecker's theorem.



Weber's theorem led to the development of *class field theory*, which is essentially the number theory of Galois extensions  $K \subset L$  with abelian Galois groups. This subject was developed in the first half of the twentieth century by Hilbert, Takagi, Artin, Hasse, and Chevalley. See the web document *History of Class Field Theory* by Keith Conrad. An important subfield of modern number theory flows from attempts to generalize class field theory to non-abelian extensions.

## 12.4 Galois Group of $X^p - a$ over $Q$

In our discussion of solving equations with radicals, we fixed a base field  $K$  containing all  $n$ th roots of unity. We then proved that the splitting field of  $X^n - a$  over  $K$  has a cyclic Galois group.

We now generalize to the case when these roots of unity are missing. We'll consider the special case where  $P(X) = X^p - a$ ,  $p$  is prime, and the base field is  $Q$ . Assume that  $a$  is not a  $p$ th power in  $Q$ . We'll determine the splitting field of  $P$  completely and compute its Galois group.

Since  $a$  is not a  $p$ th power,  $P(X)$  is irreducible over  $Q$  by an earlier result, so the splitting field contains all of its roots and is generated by these roots. The roots of  $P(X)$  are  $\omega^k \sqrt[p]{a}$  where  $\omega$  is a non-trivial  $p$ th root of unity. Dividing two roots, we find that the splitting field contains all roots of unity. So it is natural to construct the splitting field as an extension chain

$$Q \subset Q(\omega) \subset Q(\omega \sqrt[p]{a})$$

We know that  $Q \subset Q(\omega)$  is a Galois extension with multiplicative Galois group  $Z_p^*$ ; this group is cyclic and isomorphic to  $Z_{p-1}$ . We'll continue to write it multiplicatively; in that case the nonzero  $m \in Z_p^*$  corresponds to the automorphism determined by  $\omega^i \rightarrow \omega^{mi}$ .

We know that  $Q(\omega) \subset Q(\omega, \sqrt[p]{a})$  is also Galois with additive Galois group  $Z_p$ . The automorphism corresponding to  $n \in Z_p$  maps  $\omega^i \sqrt[p]{a} \rightarrow \omega^{i+n} \sqrt[p]{a}$ .

Since there are  $p$  roots, the Galois group is a subgroup of the permutation group  $S_p$ . This group can have more than  $p$  elements; indeed  $p \leq |G| \leq p!$ . Moreover,  $|G| = [Q(\omega, \sqrt[p]{a}) : Q]$ . It is natural to suppose that the Galois group is some sort of product of the Galois groups of extensions in the above chain, and thus has order  $p(p-1)$ . Indeed  $Q(\omega, \sqrt[p]{a})$  must contain higher powers of  $\sqrt[p]{a}$ . Writing  $\alpha = \sqrt[p]{a}$ , we rapidly discover that a basis is

$$\left\{ \omega_i \alpha^j \mid 0 \leq i \leq p-2 \text{ and } 0 \leq j < p \right\}$$

We require all of those powers of  $\alpha$  because a root field of  $X^p - a$  will have dimension  $p$  (recall that the root field adds only one root, not all roots). We only require powers of  $\omega$

up to the  $p-2$  power because  $\omega$  satisfies  $X^{p-1} + X^{p-2} + \dots + X + 1 = 0$  and thus  $\omega^{p-1}$  is a combination of lower powers of  $\omega$ .

We now extend the operations of  $m \in Z_p^*$  and  $n \in Z_p$  on roots of  $P$  in  $Q$  to the full field. Consider first  $n \in Z_p$ . We know this element fixes  $Q(\omega)$  and maps  $\alpha$  to  $\omega^n \alpha$ . It follows that it maps  $\alpha^j \rightarrow \omega^{nj} \alpha^j$ . So the full extension is

$$n \in Z_p : \omega^i \alpha^j \rightarrow \omega^{i+nj} \alpha^j$$

Next consider  $m \in Z_p^*$ . We know it maps  $\omega^i \rightarrow \omega^{mi}$ . Moreover, it maps  $\alpha \rightarrow \omega^n \alpha$  for some  $n$ , since it preserves roots of  $P(X)$ . We can multiply the automorphism by any automorphism fixing  $Q(\omega)$  and get another possible extension. One such automorphism maps  $\alpha \rightarrow \omega^{-n} \alpha$ . The composition of these maps fixes  $\alpha$ . Let us take this extension. So

$$m \rightarrow Z_p^* : \omega^i \alpha^j \rightarrow \omega^{mi} \alpha^j$$

Form the set  $Z_p^* \times Z_p$ . Typical elements of this product have the form  $\langle m, n \rangle$ . Define an action of this element on the basis elements by letting  $m$  act first and following with the action of  $n$ :

$$\omega^i \alpha^j \rightarrow \omega^{mi} \alpha^j \rightarrow \omega^{mi+nj} \alpha^j$$

In particular, on roots  $\langle m, n \rangle$  acts via

$$\langle m, n \rangle : \omega^i \alpha \rightarrow \omega^{mi} \alpha \rightarrow \omega^{mi+n} \alpha$$

Now notice that

$$\langle m_2, n_2 \rangle \langle m_1, n_1 \rangle : \omega^i \alpha \rightarrow \omega^{m_1 i + n_1} \alpha \rightarrow \omega^{m_2(m_1 i + n_1) + n_2} \alpha$$

We conclude that  $Z_p^* \times Z_p$  becomes a group with product

$$\langle m_2, n_2 \rangle \langle m_1, n_1 \rangle = \langle m_2 m_1, m_2 n_1 + n_2 \rangle$$

This is a semi-direct product induced by  $\varphi : Z_p^* \rightarrow \text{Aut}(Z_p)$  where  $\varphi(m)(n) = mn$ . Notice that  $\varphi(m)$  preserves addition and is one-to-one, hence onto.

*Note:* In general, let  $G_1$  and  $G_2$  be groups and let  $\varphi : G_1 \rightarrow \text{Aut}(G_2)$  be a group homomorphism. We define the semi-direct product of the two groups with respect to  $\varphi$  to be the set  $G_1 \times G_2$  with product

$$\langle \tilde{g}_1, \tilde{g}_2 \rangle \langle g_1, g_2 \rangle = \langle \tilde{g}_1 g_1, \varphi(\tilde{g}_1)(g_2) + \tilde{g}_2 \rangle$$

It is easy to see that this semidirect product is a group. Both  $G_1$  and  $G_2$  are subgroups of this group. The group  $G_2$  is a normal subgroup.

In our case  $Z_p^* \times Z_p$ , the  $Z_p$  is a normal subgroup of the Galois group whose fixed field is  $Q(\omega)$ . Therefore  $Q \subset Q(\omega)$  is Galois with Galois group  $(Z_p^* \times Z_p)/Z_p$ . Clearly this quotient is  $Z_p^*$ , as it ought to be.

## Chapter 13

# Constructing Regular Polygons

### 13.1 Constructing Regular Polygons

It is easy to inscribe a regular hexagon in a circle: take the compass used to draw the circle, and mark off six equal segments with the compass. Each straight segment joining two adjacent points has the same radius as the circle, so the interior angle is  $60^\circ$  and therefore six of these segments exactly fit around the circle. By joining every other point, a regular 3-gon can be inscribed.

Similarly, it is easy to inscribe a square. Draw the diameter, and the perpendicular to it at the center, and join the four intersection points of these lines with the circle. Bisecting each angle gives a regular 8-gon.

It is not so easy to inscribe a pentagon. Here's one way to proceed if you have completely forgotten what you learned in high school. We need to construct  $\cos(\frac{2\pi}{5}) = \cos \theta$ . Then  $(\cos \theta + i \sin \theta)^5 = 1$ . Taking imaginary parts,  $5i \cos^4(\theta) \sin(\theta) - 10i \cos^2(\theta) \sin^3(\theta) + i \sin^5(\theta) = 0$ . Factoring out  $i \sin(\theta)$ , we obtain

$$\left(5 \cos^4(\theta) - 10 \cos^2(\theta)(1 - \cos^2(\theta)) + (1 - \cos^2(\theta))^2\right) = 0$$

or

$$16 \cos^4(\theta) - 12 \cos^2(\theta) + 1 = 0$$

So  $\cos \theta$  satisfies  $16X^4 - 12X^2 + 1$ . Consequently  $X^2 = \frac{12 \pm \sqrt{144 - 64}}{32} = \frac{12 \pm \sqrt{80}}{32} = \frac{3 \pm \sqrt{5}}{8}$ . Since  $\frac{3 + \sqrt{5}}{8} = \frac{1}{4} \left(\frac{1 + \sqrt{5}}{2}\right)^2$ , we have  $\cos \theta = \frac{1}{2} \left(\frac{1 + \sqrt{5}}{2}\right)$ . This number can be constructed by our methods, so a regular pentagon can be inscribed in a circle. Consult the internet for many beautiful constructions. Incidentally,  $\frac{1 + \sqrt{5}}{2}$  is the Golden Mean.

**Theorem 50**

- If an  $n$  sided polygon can be inscribed in a circle using straightedge and compass, then a  $2^k n$  sided polygon can also be inscribed
- If an  $m$  sided polygon and an  $n$  sided polygon can be inscribed and  $m$  and  $n$  are relatively prime, then an  $mn$  sided polygon can be inscribed.

*Proof:* Bisect the angles of a polygon to double the number of sides. If  $m$  and  $n$  are relatively prime, integers  $a$  and  $b$  exist with  $am + bn = 1$ . These integers may be positive or negative or both. Then

$$a \left( \frac{2\pi}{n} \right) + b \left( \frac{2\pi}{m} \right) = (am + bn) \left( \frac{2\pi}{mn} \right) = \frac{2\pi}{mn}$$

Hence we can obtain a side of the  $mn$ -sided polygon by taking a multiple of the angle of one of the polygons and reversing by a different multiple of the angle of the other. QED.

*Remark:* It follows that up to 20, we can construct polygons with 3, 4, 5, 6, 8, 10, 12, 15, 16, and 20 sides, leaving 7, 9, 11, 13, 14, 17, 18, and 19 in doubt.

When he was very young, Gauss succeeded in constructing a 17 sided polygon. This was the first progress in the theory since the Greeks, and it caused Gauss to decide to become a mathematician. The theory of this polygon and its generalizations is covered in the last chapter of Gauss's book on number theory, written when he was 24 years old.

**Theorem 51** *A polygon with  $n$  sides can be constructed using straightedge and compass if and only if*

$$n = 2^k p_1 p_2 \dots p_k$$

where the  $p_k$  are distinct odd primes of the form  $2^{2^n} + 1$ .

*Remark:* This theorem rules out  $n = 9, 18$  since the factorization of each contains a repeated off prime.

Primes of the form  $2^{2^n} + 1$  are called *Fermat primes* because Fermat conjectured that all such numbers are prime. The first few such numbers are 3, 5, 17, 257, and 65537 and each is prime. The next number is 4294967297 which was too large to factor by hand in Fermat's day. Euler later proved that it is not prime by showing that it does not satisfy Fermat's little theorem for 3. No other prime has been found in the sequence of Fermat numbers, but it is unknown whether others exist. If one assumes that the  $2^{2^n} + 1$  are random and takes into account the rough distribution of the primes, the likely number of primes in the sequence is only 4 or 5, but this is just a heuristic argument.

*Proof:* Starting with  $(0, 0)$  and  $(1, 0)$ , we want to construct  $(\cos \frac{2\pi}{n}, \sin \frac{2\pi}{n})$ . It suffices to construct the cosine, for then the sin can be found by taking one square root. Starting from

$Q$ , suppose a sequence of quadratic extensions can be found with the final one containing  $\cos \frac{2\pi}{n}$ . By taking one more quadratic extension (this one complex), we can get a tower whose last element contains  $\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} = \omega$  where  $\omega$  is a primitive  $n$ th root of unity. The dimension of the final element of the tower over  $Q$  is a power of 2, so  $[Q(\omega) : Q]$  must equal a power of 2.

In section 12.3 on cyclotomic fields, we discovered that  $X^n - 1 = \prod_{d|n} \Phi_d(X)$  where each  $\Phi_d$  is irreducible over  $Q$ . Moreover  $[Q(\omega) : Q] = \text{degree}(\Phi_n(X))$ . The numbers  $\varphi(d) = \text{degree}(\Phi_d(X))$  define what is called the *Euler Phi Function*. Recall that  $\varphi(d)$  is the number of generators of  $Z_d$ , or alternately the number of  $a \in Z_d$  with  $a$  relatively prime to  $d$ .

**Lemma 13**

- If  $p$  is prime,  $\varphi(p^k) = p^k - p^{k-1} = p^k \left(1 - \frac{1}{p}\right)$
- if  $m$  and  $n$  are relatively prime,  $\varphi(mn) = \varphi(m)\varphi(n)$

*Proof of theorem from lemma:* Factor  $n$  into primes:  $n = 2^k p_1^{k_1} \dots p_m^{k_m}$ . Then  $\varphi(n)$  is a product of terms and each must be a power of 2 if an  $n$ -sided can be constructed. We have  $\varphi(2^k) = 2^k - 2^{k-1} = 2^k(2 - 1)$ , which is a power of 2. We have  $\varphi(p^k) = p^k - p^{k-1} = p^{k-1}(p - 1)$ . If  $k > 1$ , the first term is odd and the polygon is ruled out. If  $k = 1$ , the  $p - 1$  must be a power of 2, and thus  $p = 2^k + 1$ . But such a number cannot be prime unless  $k$  is itself a power of 2, for if  $b$  is odd prime then for

$$2^{2^k b} + 1 = (2^{2^k})^b + 1 = \left((2^{2^k} + 1)\right) \left( (2^{2^k})^{b-1} - (2^{2^k})^{b-2} + \dots - (2^{2^k}) + 1 \right)$$

Consequently the condition of the theorem is necessary.

To prove that this condition is also sufficient, it suffices to study the case when  $n = p = 2^{2^n} + 1$  by the previous theorem. Consider  $Q \subset Q(\omega)$ . This is a Galois extension, with Galois group  $Z_p^* = Z_{p-1}$ . Since  $p = 2^{2^n} + 1$ ,  $p - 1$  is a power of 2. The subgroups of  $Z_{p-1}$  correspond to divisors of this power of two, and therefore form a single chain  $e \subset Z_2 \subset Z_4 \subset \dots \subset Z_{p-1}$ . Therefore the subfields of  $Q(\omega)$  form a single chain  $Q \subset K_1 \subset K_2 \subset \dots \subset K_{n-1} \subset Q(\omega)$  and there are no other subfields to mess up this picture.

But we can easily find a candidate for  $K_{n-1}$ , namely  $Q(\cos \frac{2\pi}{p})$ . Indeed, the extension is generated by  $X = i \sin \frac{2\pi}{p}$  and  $X^2 = -\sin^2 \frac{2\pi}{p} = \cos^2 \frac{2\pi}{p} - 1$ . So  $K_{n-1}$  is a field of real numbers, and therefore every other extension in the chain was obtained by adding a real square root. So  $\cos \frac{2\pi}{p}$  can be constructed by straightedge and compass. QED.

*Proof of lemma:* Recall that  $\varphi(p^n)$  is the number of  $a \in Z_{p^n}$  relatively prime to  $p^n$ . There are  $p^n$  elements altogether; those not relatively prime to  $p^n$  are multiples of  $p$  and so those

of the form  $pb$  for  $1 \leq b < p^{n-1}$ . Since there are  $p^{n-1}$  of the latter, there are  $p^n - p^{n-1}$  relatively prime to  $p^n$ .

Suppose  $m$  and  $n$  are relatively prime. Then  $Z_{mn}$  is isomorphic to  $Z_m \times Z_n$ .

An element of  $Z_s$  is relatively prime to  $s$  if and only if it is a unit in the multiplicative group  $Z_s^*$ . Indeed if  $a$  and  $s$  are relatively prime, we can find  $A$  and  $B$  with  $Aa + Bs = 1$ . Then in  $Z_s$  we have  $Aa = 1$ , so  $a$  is a unit. Conversely if  $Aa = 1$  in  $Z_s$ , then  $Aa - 1 = Bs$ , so  $a$  and  $s$  are relatively prime.

The number of units in  $Z_{mn}$  is  $\varphi(mn)$ . Clearly  $(a, b) \in Z_m \times Z_n$  is a multiplicative unit if and only if  $a$  is a unit in  $Z_m^*$  and  $b$  is a unit in  $Z_n^*$  and the number of such units is  $\varphi(m)\varphi(n)$ .

*Remark:* We can now see more about trisection of angles:

**Theorem 52** *There is a straightedge and compass construction to divide every angle into  $n$  equal pieces if and only if  $n$  is a power of 2.*

*Proof:* If there were such a construction and we applied it to the angle  $2\pi$ , we'd obtain the vertices of an inscribed  $n$ -gon. Or analogously, we could apply the construction to a right angle and get a  $4n$ -gon, from which an  $n$ -gon is easily obtained. So  $n = 2^k p_1 \dots p_k$  where the  $p_i$  are distinct odd primes. By connecting vertices with similar gaps, we could construct a  $p$ -gon. But then by dividing each interior angle into  $p$  pieces we'd have a  $p^p$  gonon, and odd primes would be repeated, which is impossible. So no  $p_i$  can occur. QED.

*Remark:* In high school, we learn that a small number of angles have precise trigonometric values. For example,  $\cos 0^\circ = 1$ ,  $\cos 30^\circ = \frac{\sqrt{3}}{2}$ ,  $\cos 45^\circ = \frac{1}{\sqrt{2}}$ ,  $\cos 60^\circ = \frac{1}{2}$ . An earlier result in these notes gave  $\cos 72^\circ = \frac{1}{2} \frac{1+\sqrt{5}}{2}$ .

Gauss' results show that a small number of other values can be expressed using only square roots. In his number theory book, Gauss worked out the exact value of  $\cos \frac{2\pi}{17}$ . The value is

$$\cos \frac{2\pi}{17} = \frac{1}{16} \left[ -1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}} \right]$$

## Chapter 14

# Normal and Separable Extensions

### 14.1 Normal Extensions, Separable Extensions, and All That

In traditional treatments of Galois theory, an extension  $K \subset L$  is called a Galois extension if  $L$  is *normal* and *separable* over  $K$ .

**Definition 11** *A finite extension  $K \subset L$  is said to be normal if whenever  $Q(X)$  is an irreducible polynomial over  $K$  with a root in  $L$ , then  $Q(X)$  splits completely over  $L$ .*

**Theorem 53** *Let  $K \subset L$  be a finite extension.*

- *If  $L$  is the splitting field of some polynomial  $P(X)$ , then  $L$  is normal.*
- *Conversely, if  $L$  is normal, then  $L$  is the splitting field of some polynomial  $P(X)$  over  $K$ .*

*Proof:* Suppose  $L$  is a splitting field of  $P(X)$ , and suppose  $Q(X)$  is irreducible over  $K$  and has a root in  $L$ . Let  $K \subset M$  be the splitting field of  $Q(X)P(X)$ . Then  $M$  contains all roots of  $P(X)$ , and these generate  $L$ , so  $K \subset L \subset M$ . Suppose  $a$  is a root of  $Q(X)$  in  $L$ . By assumption  $Q(X)$  splits completely in  $M$ ; let  $b \in M$  be a second root.

By corollary 3 in section 4.3, there is an automorphism  $\sigma$  of  $M$  over  $K$  taking  $a$  to  $b$ . Since  $\sigma$  is an automorphism over  $K$ , it must map roots of  $P(X)$  to other roots. But the roots of  $P(X)$  generate  $L$ , so  $\sigma$  maps  $L$  to  $L$ . Hence  $b \in L$ . This holds for any root of  $Q(X)$ , so  $Q(X)$  splits completely in  $L$ .



The other direction is essentially trivial. If  $L = K$ , we are done. Otherwise find  $a_1 \in L$  and let  $P_1(X)$  be its minimal polynomial over  $K$ . By assumption all roots of  $Q$  are in  $L$ . We have  $K \subset K(a_1) \subset L$ . If  $K(a_1)$  is not  $L$ , find  $a_2 \in L$  not in  $K(a_1)$ . Let  $P_2(X)$  be its minimal polynomial. This polynomial splits completely in  $L$ . Then  $K \subset K(a_1) \subset K(a_1, a_2) \subset L$ . Continue. The sequence of fields is strictly increasing, so eventually we get  $L$ , and then  $P(X) = P_1(X)P_2(X) \dots$  is the splitting polynomial.

**Definition 12** Let  $P(X)$  be a polynomial over a field  $K$ . We say  $P(X)$  is separable if all of its roots are distinct in the splitting field of  $P(X)$ .

*Remark:* So  $P(X)$  is separable if and only if the roots are “separated” in  $L$ , i.e., there are no multiple roots.

**Definition 13** Let  $K \subset L$  be a finite extension. We say the extension is separable if whenever  $\theta \in L$  and  $P(X)$  is its minimal polynomial, then  $P(X)$  is separable.

**Theorem 54** A finite extension  $K \subset L$  is a Galois extension if and only if it is normal and separable.

*Proof:* Suppose  $K \subset L$  is normal and separable. Since the extension is normal, there is a polynomial  $P(X)$  such that  $L$  is the splitting field of  $P$  over  $K$ . Factor  $P$  into irreducible polynomials  $P_1(X) \dots P_k(X)$  over  $K$ . If there are redundant factors, we can eliminate repetitions, so assume the factors are distinct.

Since the extension is separable, no irreducible factor has repeated roots. Suppose both  $P_i$  and  $P_j$  share a root. Then both are minimal polynomials of this root, and by uniqueness of the minimal polynomial they are equal. So  $P(X)$  is a splitting polynomial with no repeated roots, and thus  $K \subset L$  is Galois.

Conversely, suppose  $K \subset L$  is Galois. Then there is a polynomial  $P(X)$  making  $L$  a splitting field of  $P$ , so the extension is normal.

The most difficult step comes last; we want to prove that  $L$  is separable. Let  $\alpha \in L$  and let  $Q(X)$  be its minimal polynomial. We need to prove that all roots of  $Q$  are distinct. Since  $K \subset L$  is normal, all roots of  $Q$  are in  $L$ .

Extend  $\alpha$  to a set of generators of  $L$ :  $\alpha = \alpha_1, \alpha_2, \dots, \alpha_n$ . No assumption is being made about where these generators come from. We have a chain

$$K \subset K(\alpha_1) \subset K(\alpha_1, \alpha_2) \subset \dots \subset K(\alpha_1, \dots, \alpha_n) = L$$

For each  $i$ , let  $Q_i(X)$  be the minimal polynomial of  $\alpha_i$  over  $K(\alpha_1, \dots, \alpha_{i-1})$ . In particular,  $Q(X) = Q_1(X)$ . Normality of  $K \subset L$  again implies that all roots of  $Q_i(X)$  are in  $L$ .

Let  $n_i$  be the number of distinct roots of  $Q_i$  in  $L$  and let  $d_i$  be the degree of  $Q_i$ . Clearly  $n_i \leq d_i$ .

We have  $[L : K] = [K(\alpha_1, \dots, \alpha_n) : K(\alpha_1, \dots, \alpha_{n-1})] \dots [K(\alpha_1) : K] = d_n d_{n-1} \dots d_1$ . Since  $K \subset L$  is Galois, this is also the number of automorphisms of  $L$ . On the other hand, an automorphism  $\sigma$  must map  $\alpha_1$  to another root of  $Q_1$ . There are  $n_1$  choices for this root, and thus  $n_1$  homomorphisms  $\varphi_i : K(\alpha_1) \rightarrow L$ . We then extend these maps to  $K(\alpha_1, \alpha_2)$  by choosing possible images of  $\alpha_2$ . There are  $n_2$  such choices, and so  $n_1 n_2$  possible homomorphisms from  $K(\alpha_1, \alpha_2)$  into  $L$ . Continue. We conclude that the total number of automorphisms of  $L$  is at most  $n_1 n_2 \dots n_n$ . This can only equal  $d_1 d_2 \dots d_n$  is each  $n_i = d_i$ , and in particular  $n_1 = d_1$ . So  $Q(X)$  cannot have repeated roots. QED.

*Remark:* Now the theorem implying that separability is a weak requirement, often automatically true.

### Theorem 55

- In characteristic zero, every irreducible polynomial is separable and every extension is separable.
- In characteristic  $p$ , an irreducible polynomial  $P(X)$  is separable unless there is a  $Q(X)$  with  $P(X) = Q(X^p)$ .
- If every element in  $K$  is a  $p$ th power, then every irreducible polynomial is separable and every extension is separable.
- If  $K$  is a finite field, every irreducible polynomial is separable and every extension is separable.

*Proof:* Let  $\alpha$  be a repeated root of an irreducible  $P(X)$ . Since  $P(X)$  is irreducible, it is the minimal polynomial for  $\alpha$ . But if  $\alpha$  is a repeated root, then it is also a root of the formal derivative  $P'(X)$ , which cannot vanish in characteristic zero. Contradiction.

In characteristic  $p$ , the same argument works unless  $P'(X)$  is identically zero. This can only happen if the only terms in  $P$  are  $(X^p)^k$ .

In a field of characteristic  $p$ , the map  $\sigma : K \rightarrow K$  defined by  $k \rightarrow k^p$  is a one-to-one homomorphism. Indeed,  $(a + b)^p = a^p + b^p$  because all remaining binomial coefficients are divisible by  $p$ ; clearly  $(ab)^p = a^p b^p$ . If  $a^p = 0$ , then  $a = 0$ .

Suppose each element of  $K$  is a  $p$ th power. Then if an irreducible polynomial  $P(X)$  is not separable, it equals

$$Q(X^p) = a_0 + a_1 X^p + \dots + a_k (X^p)^k = b_0^p + b_1^p X^p + \dots + b_k^p (X^p)^k = (b_0 + b_1 X + \dots + b_k X^k)^p$$

but this is not irreducible.

If  $K = \mathbb{Z}_p$ , and  $a \in K$ , then  $a^p = a$  by Fermat's Little Theorem. If  $L$  is a finite field of characteristic  $p$ , then  $\mathbb{Z}_p \subset K$  is finite dimensional and  $\sigma(k) = k^p$  is a one-to-one map from  $K \rightarrow K$ , hence onto. QED.

**Corollary 6** *If  $K$  has characteristic zero or  $K$  is a finite field, then every splitting field  $K \subset L$  of an arbitrary polynomial  $P(X)$  is a Galois extension.*

*Proof:* Obvious.

**Definition 14** *Let  $K \subset L$  be a finite extension. A primitive element is a  $\theta \in L$  such that  $L = K(\theta)$ .*

*Remark* Several of our results guarantee the existence of a primitive element. They are all covered by the following:

**Theorem 56** *If  $K \subset L$  is finite and separable, it has a primitive element.*

*Proof:* The theorem is obvious if  $K$  and  $L$  are finite fields because  $L^*$  is cyclic. So we can assume that  $K$  is infinite.

Write  $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$  where the  $\alpha_i$  have no special properties. Notice that  $K(\alpha_1, \dots, \alpha_i)$  is separable for each  $i$ . We prove that each  $K(\alpha_1, \dots, \alpha_i)$  has a primitive element inductively. Clearly, it suffices to study the case  $K(\alpha, \beta)$ . In this case, we look for a primitive element of the form  $\alpha + c\beta$ .

Let  $P_1(X)$  be the minimal polynomial of  $\alpha$  over  $K$  and let  $P_2(X)$  be the minimal polynomial of  $\beta$  over  $K$  and let  $P(X)$  be the product of these polynomials. Then  $P(X)$  is a polynomial over  $K(\alpha, \beta)$ . Let  $S$  be its splitting field, so

$$K \subset K(\alpha) \subset K(\alpha, \beta) \subset S$$

The key observation of the proof is that we can find exactly  $[K(\alpha, \beta) : K]$  one-to-one homomorphisms  $\sigma_i : K(\alpha, \beta) \rightarrow S$ . Suppose for a moment that this has been done.

Next we find  $c \in K$  such that  $\sigma_i(\alpha + c\beta) \neq \sigma_j(\alpha + c\beta)$  for all  $i \neq j$ . This is easy. If the two expressions are equal, then

$$\sigma_i(\alpha) - \sigma_j(\alpha) = (-c)(\sigma_i(\beta) - \sigma_j(\beta))$$

This equation cannot hold if  $\sigma_i(\beta) - \sigma_j(\beta)$  because it would imply that  $\sigma_i$  and  $\sigma_j$  agree on both  $\alpha$  and  $\beta$  and thus everywhere, but  $\sigma_i \neq \sigma_j$ . Otherwise the equation rules out exactly one  $c$  for each  $i$  and  $j$ , and so only finitely many  $c$ 's. Since  $K$  is infinite, we can find an appropriate  $c$ .

Consider

$$K \subset K(\alpha + c\beta) \subset K(\alpha, \beta) \subset S$$

Let  $R(X)$  be the minimal polynomial of  $\alpha + c\beta$  over  $K$  and let it have degree  $d$ . Since  $R$  has a root in the separable  $K(\alpha, \beta)$ , its roots in  $S$  are distinct. Any homomorphism  $K(\alpha + c\beta) \rightarrow S$  must map  $\alpha + c\beta$  to a root of  $R$ , so there are at most  $d$  such homomorphisms. But there are  $[K(\alpha, \beta) : K]$  homomorphisms, so  $[K(\alpha, \beta) : K] \leq d = [K(\alpha + c\beta) : K]$  and we conclude that  $K(\alpha + c\beta) = K(\alpha, \beta)$ .

Now we prove the existence of the  $\sigma_i$ .

Since  $K \subset L$  is separable and  $P_1(X)$  is irreducible over  $K$ , all of its roots are distinct in  $L$ . Call them  $\alpha = \alpha_1, \dots, \alpha_{d_1}$ , where  $d_1$  is the degree of  $P_1(X)$ . We can find exactly  $n$  one-to-one homomorphisms  $\varphi_i : K(\alpha) \rightarrow S$ , since such a map is determined by the image of  $\alpha$ , which can be any root of  $P_1(X)$ .

Our goal is now to extend the  $\varphi_i$  to one-to-one homomorphisms  $K(\alpha, \beta) \rightarrow S$ , and count the number of ways this can be done. The polynomial  $P_2(X)$  is irreducible over  $K$ , so all of its roots are distinct in  $S$ . It might be reducible in  $K(\alpha_i)$ . Factor  $P_2(X)$  over  $K(\alpha)$  and let  $Q_2(X)$  be the irreducible factor with root  $\beta$ . Since  $P_2$  has coefficients in  $K$ ,  $P = \varphi_i(P)$  factors in  $K(\alpha_i)$ , and  $\varphi_i(Q_2(X))$  is a factor over  $K(\alpha_i)$ . Call the roots of this polynomial  $\beta_{ij}$  in  $S$ . For a fixed  $i$ , these roots are distinct and their number  $d_2$  is the degree of  $\varphi_i(Q_2(X))$ , which equals the degree  $d_2$  of  $Q_2(X)$ . We can extend  $\varphi$  to a map  $K(\alpha, \beta) \rightarrow S$  sending  $\alpha$  to  $\alpha_i$  and sending  $\beta$  to some  $\beta_{ij}$ . The number of such maps is  $d_1 d_2 = [K(\alpha : K)][K(\alpha, \beta) : K(\alpha)]$ . QED.

**Corollary 7** *Every finite  $K \subset L$  has a primitive element if the characteristic is zero, or if the fields are finite.*

*Remark:* Let  $P(X)$  be a polynomial, and let  $K \subset L$  be its splitting field. The Galois group of  $P$  is then a subgroup of the permutation group on the roots of  $P$ . We have found cases when this group is the full  $S_5$ . It is thus highly transitive, since any 5 elements can be taken to any other 5 elements.

This is a particular “permutation representation” of  $G$ , but it is not the only one. By the previous theorem, we can write the splitting field as  $K \subset K(\gamma)$ . Let  $M(X)$  be the minimal polynomial of  $\gamma$ . Then each automorphism is completely determined by the value on  $\gamma$ , so the Galois group is a simply transitive subgroup on  $5! = 120$  elements.

The Galois group is an abstract group; it makes sense to think of it as a permutation group only when we have a particular generating polynomial for the extension.

## Chapter 15

# Galois Theory and Reduction Modulo $p$

### 15.1 Preview

We often want to prove that a particular  $P(X)$  over  $Q$  is irreducible and then find its Galois group. In these notes, we introduced a small number of tools to do these tasks. For instance, irreducibility was mainly proved using Eisenstein's theorem.

In this chapter, we discuss new methods for both calculations. The methods involve reducing modulo a prime  $p$ . We factor  $P(X)$  over  $Z_p$  and compute the Galois groups of the resulting irreducible factors. Brute force methods and computer programs are available for these tasks. By varying the prime  $p$ , we gain additional information. Then this information can be assembled to provide information about the original problems over the rationals.

Consider the problem of factoring over  $Q$ . When a polynomial has rational coefficients, it does not make sense to reduce modulo  $p$ . But luckily Gauss associated to each polynomial with rational coefficients an associated primitive polynomial with integer coefficients, and proved that factoring the primitive polynomial over the integers is equivalent to factoring the original polynomial over the rational numbers. We can certainly reduce the primitive polynomial modulo a prime  $p$ . Suppose the resulting polynomial is irreducible. Then so is the primitive polynomial over  $Z$ , and also the original polynomial over  $Q$ .

But even if the reduced polynomial factors, we gain important information. For instance, suppose a primitive polynomial factors as a cubic times a quartic over  $Z_5$ , and factors as a quadratic times an irreducible polynomial of degree 6 over  $Z_7$ . Factorization over  $Z$  must

be consistent with both of these results, and a little thought shows that our polynomial must be irreducible.

At the moment, it is unclear how we could start with a splitting field  $Q \subset L$  and reduce modulo  $p$ . The essential idea is to introduce a notion of “integers” in both fields, so the field extension induces a ring inclusion  $I(Q) = Z \subset I(L)$  and then reduce modulo  $p$ . Details we will provide in a later section. The ultimate theorem we obtain is due to Dedekind:

**Theorem 57 (Dedekind)** *Let  $P(X)$  be a monic irreducible polynomial with integer coefficients. Suppose the reduction of  $P$  modulo a prime  $p$  factors as a product of irreducible polynomials over  $Z_p$  of degrees  $d_1, d_2, \dots, d_k$  over  $Z_p$ . Suppose this reduced polynomial has distinct roots in its splitting field over  $Z_p$ . If we regard the Galois group of the original  $P$  as a permutation group on its roots, then this group has an element  $\tau$  that can be written as a product of  $k$  distinct cycles of degrees  $d_1, d_2, \dots, d_k$ .*

## 15.2 $X^5 - X - 1$

We postpone the proof of Dedekind’s theorem to the end of this chapter, and begin with a series of applications.

The most famous example of a polynomial which cannot be solved by radicals is

$$P(X) = X^5 - X - 1$$

This example was discovered by Artin. It has only one real root, so the techniques of section 9.6 do not suffice to find its Galois group. We will prove it irreducible with Galois group  $S_5$ .

We begin by proving this polynomial irreducible. The symbolic algebra program Mathematica has a built-in function to Factor polynomials modulo a prime. Let’s factor over  $Z_5$ . Here is the command to run the function, and the resulting output:

```
Factor[X^5 - X - 1, Modulus -> 5]
X^5 + 4X + 4
```

It follows that  $X^5 - X - 1$  is irreducible over  $Z_5$ , and thus irreducible over  $Q$ .

Just this once, imagine factoring over  $Z_5$  by brute force. If  $a \in Z_5$ ,  $a^5 = a$ . Therefore, for any  $a$  we have  $a^5 - a - 1 = -1$ . It follows that the reduced polynomial modulo 5 has no linear factors. If it factors at all, it must have one quadratic and one cubic factor.

We can ignore quadratic candidates which factor. So it suffices to divide by  $X^2 + aX + b$  when  $\sqrt{a^2 - 4b} \notin Z_5$ , and thus when  $a^2 + b = 2, 3$ . If  $a = 0$ , this gives  $X^2 + 2$  and  $X^2 + 3$ . If  $a = 1$  it gives  $X^2 + X + 1$  and  $X^2 + X + 2$ . If  $a = 2$  it gives  $X^2 + 2X + 3$  or  $X^2 + 2X + 4$ .

If  $a = 3$  it gives  $X^2 + 3X + 3$  and  $X^2 + 3X + 4$  and if  $a = 4$  it gives  $X^2 + 4X + 1$  and  $X^2 + 4X + 2$ . So it suffices to test the following list, and the reader can show that none of these is a factor of the reduced  $X^5 - X - 1$ .

$$X^2 + 2, X^2 + 3, X^2 + X + 1, X^2 + X + 2, X^2 + 2X + 3, X^2 + 2X + 4$$

$$X^2 + 3X + 3, X^2 + 3X + 4, X^2 + 4X + 1, X^2 + 4X + 2$$

We now compute the Galois group of  $X^5 - X - 1$ . The argument in section 9.6 showing that the group contains a 5-cycle is still valid here. But the argument that it has a 2-cycle fails. We will use Dedekind's theorem to find a 2-cycle. Then section 9.6 shows that the Galois group is the full  $S_5$ .

Let us ask Mathematica to factor  $X^5 - X - 1$  over  $Z_7$ . Here is the session:

```
Factor[X^5 - X - 1, Modulus -> 7]
{X^2 + 6 X + 3} {X^3 + X^2 + 5X + 2}
```

We can apply Dedekind's theorem, since  $X^5 - X - 1$  has non-zero formal derivative in  $Z_7$  and thus cannot have multiple roots in an extension field. Dedekind's theorem allows us to find an element  $\tau$  in the Galois group with disjoint 2-cycle and 3-cycle. An example would be  $\tau = (12)(345)$ . Notice that  $\tau^3$  then has a 2-cycle but no 3-cycle. Hence the Galois group has a 2-cycle and a 5-cycle, and so must be the full  $S_5$ .

*Remark:* Curiously, the similar polynomial  $X^5 + X + 1$  is solvable by radicals. Indeed if  $\omega$  is a primitive third root of unity, then  $\omega^5 = \omega^2$ , so substituting in  $P$  gives  $\omega^2 + \omega + 1 = 0$ . It follows that  $X^2 + X + 1$  divides  $P$ , and indeed

$$X^5 + X + 1 = (X^2 + X + 1)(X^3 - X^2 + 1)$$

*Remark:* If we replace  $X$  by  $-X$  in  $X^5 + X - 1$ , we obtain  $-(X^5 + X + 1)$  and it immediately follows that  $X^5 + X - 1$  factors and is solvable by radicals. If we replace  $X$  by  $-X$  in  $X^5 - X + 1$ , we obtain  $-(X^5 - X - 1)$  and it immediately follows that  $X^5 - X + 1$  has Galois group  $S_5$  and is not solvable by radicals.

It turns out that all quintics can be reduced to  $X^5 + aX + b$  by transformations which preserve the Galois group. If  $a$  or  $b$  is zero, the quintic can be solved by radicals. If  $|a| = |b| = 1$ , we obtain one pair  $X^5 + X + 1$  and  $X^5 + X - 1$  of solvable quintics, and one pair  $X^5 - X - 1$  and  $X^5 - X + 1$  of unsolvable quintics.

### 15.3 A Tricky Point Concerning Straightedge and Compass Constructions

If  $Q \subset L$  is an extension generated by constructible numbers, then the degree  $[L : Q]$  is a power of 2. The converse of this theorem is false, and the purpose of this section is to explain.

Suppose first that  $Q \subset L$  is a Galois extension of degree a power of 2. Then the Galois group of the extension has order a power of two, and thus is solvable by the Sylow theorems. So we can find a composition series with composition quotients  $Z_2$ . The fundamental theorem of Galois theory then gives a series of quadratic extensions  $Q \subset K_1 \subset K_2 \subset \dots \subset L$ . So every element of  $L$  can be constructed by straightedge and compass.

However, suppose  $r$  is algebraic with minimal polynomial a power of 2. We can form the root field  $Q(r)$ , but this root extension need not be Galois, and we cannot employ Galois theory to obtain a tower of quadratic extensions. We now show by example that in this situation,  $r$  may not be constructible by straightedge and compass.

Consider the polynomial  $X^4 + 8X + 12$  and let  $\theta$  be a root. We claim that the polynomial is irreducible, so  $[Q(\theta) : Q] = 4 = 2^2$ . We also claim that the Galois group of the splitting field  $L$  is  $A_4$ . Assuming this is correct, suppose we can find a field  $Q \subset K \subset Q(\theta) \subset L$ . By the fundamental theorem of Galois theory, the field  $K$  would correspond to a subgroup of  $A_4$  of order 6. But  $A_4$  is the group of the 12 rotational symmetries of a tetrahedron, and this group contains no subgroup of order 6. QED.

To prove  $X^4 + 8X + 12$  irreducible, we factor over  $Z_5$  and  $Z_{17}$ .

```
Factor[X^4 + 8 X + 12, Modulus -> 5]
{X + 1} {X^3 + 4 X^2 + X + 2}
```

```
Factor[X^4 + 8 X + 12, Modulus -> 17]
{X^2 + 4 X + 7} {X^2 + 13 X + 9}
```

Over  $Z$ ,  $X^4 + 8X + 12$  cannot have a linear factor, since there is no linear factor over  $Z_{17}$ . It cannot have a quadratic factor, since there is no quadratic factor and no product of two linear factors, over  $Z_5$ . So  $X^4 + 8X + 12$  is irreducible.

The discriminant of our polynomial is  $576^2$ , so the Galois group is a subgroup of  $A_4$ . We want to prove that the Galois group is the full  $A_4$ , and for that it is enough to prove that the order of the group is divisible by both 3 and 4, since then it has at least 12 elements and thus must be all of  $A_4$ .

If  $L$  is the splitting field of  $P$ , we have  $Q \subset Q(r) \subset L$ . Then  $|G| = [L : Q] = [L : Q(r)][Q(r) : Q]$ . Since  $[Q(r) : Q] = 4$ , the order of  $G$  is divisible by 4.



Apply Dedekind's theorem. The derivative of our polynomial over  $Z_5$  is not zero, so it has no multiple roots over an extension. By the above factorization, there is a  $\tau$  in the Galois group with a one cycle and a three cycle. This element has order 3, so the order of the Galois group is divisible by 3.

## 15.4 Polynomials with Galois Group $S_n$

**Theorem 58** *For each  $n > 0$ , there is a polynomial of degree  $n$  with integer coefficients whose Galois group is the full  $S_n$ .*

*Proof:* This result is trivial for  $n = 1$  and  $n = 2$ .

Suppose  $p$  is prime and  $n \geq 1$  is an integer. We first claim there is a monic polynomial over  $Z$  which is irreducible in  $Z_p[X]$ . Indeed, we proved that a field  $GF(p^n)$  of order  $p^n$  exists. So  $Z_p \subset GF(p^n)$  is a Galois extension. By an earlier result, there exists  $\theta \in GF(p^n)$  generating this extension. The minimal polynomial of  $\theta$  is monic in  $Z_p[X]$  and irreducible there. Lift it to a monic polynomial over  $Z$ .

Find  $P_1(X)$  a monic polynomial of degree  $n$  with integer coefficients such that  $P_1$  is irreducible over  $Z_2$ . Find a monic polynomial  $P_2$  of degree  $n$  which factors in  $Z_3[X]$  into a linear polynomial and an irreducible polynomial of degree  $n - 1$ . If  $n$  is odd, find a monic polynomial  $P_3$  of degree  $n$  which factors over  $Z_5$  as a product of an irreducible quadratic and an irreducible polynomial of degree  $n - 2$ . If  $n$  is even and  $n \neq 4$ , find a similar polynomial which factors over  $Z_5$  as an irreducible quadratic times a product of two irreducible polynomials of distinct odd degrees. If  $n = 4$ , let  $P_3(X)$  be a product of an irreducible quadratic and two linear terms with distinct roots. Let  $P(X) = -15P_1(X) + 10P_2(X) + 6P_3(X)$ . Notice that this polynomial is monic of degree  $n$ . Since the polynomial reduced modulo 2 is irreducible,  $P(X)$  is irreducible.

Notice that modulo 2,  $P(X) = P_1(X)$ . Also modulo 3,  $P(X) = P_2(X)$ . Finally modulo 5,  $P(X) = P_3(X)$ . Since  $P_1(X)$  is irreducible over  $Z_2$ , all of its roots are distinct by theorem 55 in section 14.1. Since  $P_2(X)$  factors over  $Z_3$  into a linear factor and an irreducible factor, the linear factor cannot be one of the roots of the irreducible factor and the roots of the irreducible factor are all distinct by theorem 55. So all roots are distinct. When  $n$  is odd,  $P_3(X)$  factors over  $Z_5$  as a product of an irreducible quadratic and an irreducible polynomial of degree  $n - 2$ . Each irreducible factor has distinct roots by theorem 55, and the two polynomials cannot have a common root, else they would be minimal polynomials for this root, but one has even degree and one has odd degree.

If  $n$  is even and not four,  $P_3(X)$  factors over  $Z_5$  as an irreducible quadratic and two irreducible polynomials of distinct odd degrees. The same reasoning shows that all roots are distinct. If  $n = 4$ ,  $P_3(X)$  still factors into terms with distinct roots.

Hence we can apply Dedekind's theorem. It first implies that the Galois group has an  $n$ -cycle. Second, it implies that the group contains an  $(n - 1)$ -cycle. When  $n$  is odd, it implies that the group has a 2-cycle and an  $(n - 2)$ -cycle. When  $n$  is even and not 4, it implies that the group has a 2-cycle and two odd cycles. When  $n = 4$  it implies that the group has a 2-cycle.

By taking powers of elements, we conclude that the group always contains a 2-cycle, an  $n$ -cycle, and an  $(n - 1)$ -cycle. For instance, if it contains a 2-cycle  $\sigma$  multiplies by a disjoint cycle  $\tau$  of odd order  $k$ , then  $(\sigma\tau)^k = \sigma$ .

We now prove that such a group must be all of  $S_n$ . Numbering the elements appropriately, we can assume that the  $n - 1$ -cycle is  $(1\ 2\ \dots\ n - 1)$ . If the 2-cycle is  $\sigma = (i\ j)$  with  $1 \leq i < j < n$ , then find  $\tau \in G$  mapping  $n$  to  $j$ ; this is possible since  $G$  is transitive on roots. Then  $\tau^{-1}\sigma\tau$  maps  $n$  to  $\tau^{-1}(i)$  and  $\tau^{-1}(i)$  to  $n$  and leaves everything else fixed. So this product is  $(\tau^{-1}(i)\ n)$ . In short, we may assume that  $G$  contains  $(1\ 2\ 3\ \dots\ n - 1)$  and  $(k\ n)$  for some  $k < n$ .

We now show that such a group must be all of  $S_n$ . Let  $\tau = (1\ 2\ \dots\ n - 1)$  and let  $\sigma = (k\ n)$ . If  $j < n$ , we can find a power  $\tau^s$  mapping  $j$  to  $k$ . Then  $\tau^{-s}\sigma\tau^s$  maps  $j$  to  $n$  and maps  $n$  to  $j$  and leaves everything else fixed. So  $G$  contains  $(j\ n)$  for all  $j < n$ .

Now notice that  $(in)(jn)(in) = (ij)$ . Hence all transpositions are in  $G$ , so  $G = S_n$ .

## 15.5 Every Finite $G$ is a Galois Group

**Theorem 59** *If  $G$  is a finite group, there is a Galois extension  $Q \subset K \subset L$  such that the Galois group of  $K \subset L$  is  $G$*

*Proof:* Assume  $G$  has  $n$  elements. Define an action of  $G$  on itself by

$$g : g_1 \rightarrow gg_1$$

This is a group homomorphism from  $G$  to  $S_n$ , the permutation group on  $n$  symbols. It is one-to-one, because if  $gg_1 = g_1$  for all  $g_1$ , then  $g = e$ . So we can identify  $G$  with its image in  $S_n$ . Notice that this subgroup has order  $n$ , while  $S_n$  has order  $n!$ .

Find a polynomial  $P(X)$  whose splitting field  $Q \subset L$  has Galois group  $S_n$ . By the fundamental theory of Galois theory, the subgroup  $G$  corresponds to a subfield  $Q \subset M \subset L$ . Indeed  $M \subset L$  is a Galois extension with Galois group  $G$ . QED.

*Remark:* It is unknown whether every  $G$  is the Galois group of a Galois extension of  $Q$ . Hilbert proved that all  $S_n$  and all  $A_n$  are realizable. Shafarevich proved that all solvable groups are realizable. It is known that all sporadic groups except possibly the Mathieu group  $M_{23}$  are realizable.

## 15.6 Proof of Dedekind's Theorem

The proof we give is due to John Tate, from two web expositions of the proof.

*Step 1:* Let  $K \subset L$  be the splitting field of  $P(X)$  and denote the roots of  $P$  by  $\{r_1, \dots, r_n\}$ . Define

$$D = \sum_{0 \leq e_1, \dots, e_n < n} Zr_1^{e_1} r_2^{e_2} \dots r_n^{e_n}$$

Since the degree of  $P$  is  $n$ , powers of  $r_i$  of size  $n$  or larger can be written in terms of lower powers. Consequently  $D$  is a subring of  $L$ . It contains  $Z$  since  $r_1^0 r_2^0 \dots r_n^0 = 1$ .

Notice that the  $r_1^{e_1} \dots r_n^{e_n}$  need not be linearly independent. But we easily deduce the general structure of  $D$ . It is a finitely generated abelian group. Moreover, it has no torsion elements since it is contained in a field of characteristic zero. So by the classification of finitely generated abelian groups, it equals  $Z \times Z \times \dots \times Z$ . Said another way, it has a basis  $d_1, \dots, d_N$  of linearly independent elements and equals  $Zd_1 \oplus Zd_2 \oplus \dots \oplus Zd_N$ .

Fix a prime  $p$ . Clearly  $pD$  is a proper ideal of  $D$ , since it contains linear combinations of the  $d_i$  whose coefficients are multiples of  $p$ . Let  $\mathfrak{m}$  be a maximal ideal containing  $pD$ . Then  $E = (D/\mathfrak{m})$  is a finite field. This field has characteristic  $p$ , since  $p$  times any element of  $E$  is zero. So  $(Z/pZ) \subset (D/\mathfrak{m})$ .

*Step 2:* Let  $\varphi : D \rightarrow D/\mathfrak{m}$  be the natural map. We have  $Z \subset D$ ;  $\varphi$  sends integers to their values modulo  $p$ , and thus maps  $P(X)$  to  $P_p(X)$ . But  $P$  splits and all of its roots belong to  $D$ , so  $\varphi(P) = P_p = \prod (X - \varphi(r_i))$ . We are assuming that  $P_p$  has distinct roots, so the  $\varphi(r_i)$  are distinct.

*Step 3:* Let  $R$  be the set of roots of  $P$  in  $L$ , and let  $R_p$  be the set of roots of  $P_p$  in  $E$ . It follows from the above that  $\varphi$  maps  $R$  to  $R_p$  in a one-to-one and onto manner.

*Step 3:* Let  $G$  be the Galois group of  $Q \subset L$ . If  $\sigma \in G$ ,  $\sigma$  permutes the  $r_i$ . So it maps  $D \rightarrow D$ , inducing an automorphism of  $D$ . It follows that  $\varphi \circ \sigma$  is a ring homomorphism from  $D$  to  $E$ . This map again maps  $R$  to  $R_p$  bijectively. Moreover, if  $\sigma \neq \tau$ , then  $\varphi \circ \sigma$  and  $\varphi \circ \tau$  are unequal because induce different maps from  $R$  to  $R_p$ . But we know that an element of the Galois group is completely determined by the resulting permutations of the roots.

*Step 4:* Let  $G = \{\sigma_1, \dots, \sigma_N\}$ . It follows that  $\{\varphi \circ \sigma_1, \varphi \circ \sigma_2, \dots, \varphi \circ \sigma_N\}$  are distinct homomorphisms from  $D$  to  $E$ . Notice that one of these is the original  $\varphi$  since  $e \in G$ .

*Step 5:* We claim that the  $\varphi \circ \sigma_i$  are the only homomorphisms from  $D$  to  $E$  sending the identity to the identity. This is the most difficult step in the proof, so we will postpone its

proof until the bitter end.

*Step 6:* The Galois group of  $E$  is generated by the Frobenius automorphism  $\pi(e) = e^p$ . If  $\varphi : D \rightarrow E$  is a homomorphism, then so is  $\pi \circ \varphi$ . Hence by the previous result there is an automorphism  $\tau \in G$  such that  $\pi \circ \varphi = \varphi \circ \tau$ . But  $\tau$  and  $\pi$  induce permutations of  $R$  and  $R_p$  and  $\varphi$  sets up a bijection between these sets of roots. The equation  $\pi \circ \varphi = \varphi \circ \tau$  then implies that  $\tau$  and  $\pi$  have the same cycle structure as permutations.

*Step 7:* Notice that  $P_p$  is not irreducible; it factors as  $P_1 P_2 \dots \prod_k$ . Each of these has different roots, so the roots  $R_p$  separate into  $k$  pieces, each left invariant by  $\pi$ . On each of these pieces,  $\pi$  generates a transitive cycle. So the cycle structure of  $\pi$  corresponds to the factorization of  $P_p$ . This completes the argument.

*Step 8:* To finish, we need only prove step 5.

We have  $D = Zd_1 \oplus Zd_2 \oplus \dots \oplus Zd_N$ . The  $d_i$  are linearly independent over  $Z$ , but then they are linearly independent over  $Q$  because a dependent relation  $q_1 d_1 + \dots + q_N d_N = 0$  could be written with common denominator  $\frac{a_1}{b} d_1 + \dots + \frac{a_N}{b} d_N = 0$ , implying  $a_1 d_1 + \dots + a_N d_N = 0$ .

Consider  $Qd_1 + \dots + Qd_N$ . This is a ring containing  $Q$  and finite dimensional over  $Q$ , so it is a field. This field contains  $D$ , hence all powers of the roots of  $P$ . So it must be  $L$ . In particular,  $[L : Q] = N$ , so the Galois group  $G$  of  $Q \subset L$  has  $N$  elements.

We want to prove that the  $N$  homomorphisms already introduced are all there are. Suppose not and suppose there is another,  $\psi_{N+1}$ .

*Step 9:*

**Lemma 14** *The maps  $\psi_1, \dots, \psi_{N+1} : D \rightarrow E$  are linearly independent over  $E$ .*

*Proof:* We prove this by induction on the number of maps. It suffices to study the induction step. Suppose  $\sum_{i=1}^{N+1} e_i \psi_i = 0$ .

Since  $\psi_1 \neq \psi_{N+1}$ , they are not equal on some non-zero element  $k_0$ . Then  $\sum e_i \psi_i(k_0) = 0$  and so

$$\begin{aligned} \sum e_i \psi_i(k) \psi_i(k_0) &= 0 \\ \sum e_i \psi_i(k) \psi_{N+1}(k_0) &= 0 \end{aligned}$$

Subtracting

$$\sum e_i \psi_i(k) \left( \psi_i(k_0) - \psi_{N+1}(k_0) \right) = 0$$

This is a dependence relation on the first  $N$  terms, so by induction all coefficients are zero. In particular,  $e_1 \left( \psi_1(k_0) - \psi_{N+1}(k_0) \right) = 0$ . By the choice of  $k_0$  we conclude that  $e_1 = 0$ .

Repeat the argument, selecting  $k_0$  such that  $\psi_2(k_0) \neq \psi_{N+1}(k_0)$ . We conclude that  $e_2 = 0$ . Continue. Eventually  $e_i = 0$  for  $i \leq N$ . Consequently  $e_{N+1}\psi_{N+1} = 0$ . Since the  $\psi$  map the identity to itself,  $e_{N+1} = 0$ , proving the lemma.

*Step 10:* We have  $N + 1$  linearly independent maps from  $D$  to  $E$ , where  $D$  has a basis  $d_1, \dots, d_N$  over  $Z$ , and these elements are actually linearly independent elements of  $L$  over  $Q$ . Consider the equations

$$\sum_{i=1}^{N+1} x_i \psi_i(d_j) = 0$$

These are  $N$  linear equations in unknowns  $x_1, \dots, x_{N+1}$  belonging to  $E$ . Consequently, they have a non-zero solution. This solution is a dependence relation among the  $\psi_i$ , which holds for each  $d_i \in D$  and consequently is identically true in  $D$ . This contradiction proves step 5, and consequently the entire theorem. QED.