The Octonions

Richard Koch

February 8, 2015

1 The Cayley-Dickson Construction

This material is taken from the paper *The Octonions* by John C. Baez, published in the Bulletin of the AMS in 2002 and also available on the web at `http://math.ucr.edu/home/baez/octonions/`

Our goal is describe an 8-dimensional algebra satisfying the conditions of Hurwitz's theorem. This algebra was discovered by a friend of Hamilton's, John T. Graves, on December 26th of 1843. It was independently discovered by Cayley. The algebra is known as the *octonions* or *Cayley numbers*.

We'll describe a later treatment of this algebra by Dickson.

An *admissible structure* on R^n is a bilinear product with unit on R^n and a conjugation operation $v \to \overline{v}$ on R^n, such that

- If u is the unit, $\overline{u} = u$
- We have $\overline{\overline{a}} = a$ for all a, and $\overline{ab} = \overline{b}\ \overline{a}$ for all a and b
- For any a, $a + \overline{a}$ is a multiple of the unit
- For any a, $a\overline{a}$ is $||a||^2$ times the unit.

Suppose we have a bilinear product on R^n. Suppose this product has a unit, and identify the real numbers with multiples of this unit. Suppose that we have a conjugation operation $v \to \overline{v}$ on R^n, satisfying $\overline{\overline{a}} = a$ and $\overline{vw} = \overline{w}\ \overline{v}$. Suppose that $a + \overline{a}$ is a real multiple of the unit, and $v\overline{v} = ||v||^2$ times the unit.

We identify the real numbers with the scalar multiples of the unit.

We'll describe a construction called the *Cayley-Dickson* construction, which produces a similar algebra on R^{2n}. By definition, an element of this new algebra is a pair (a, b) with

$a, b \in R^n$. Define

$$(a,b)(c,d) = (ac - d\bar{b}, \bar{a}d + cb)$$
$$\overline{(a,b)} = (\bar{a}, -b)$$

We now must show that this new algebra has all the properties required of the original algebra. If 1 is the unit of the original algebra, $(1,0)$ is the unit of the new algebra because

$$(1,0)(a,b) = (a, \bar{1}b) = (a,b) = (a,b)(1,0) = (a,b)$$

This product and conjugation have all the required properties. Indeed

$$\overline{(1,0)} = (\bar{1}, -0) = (1,0)$$

and

$$\overline{\overline{(a,b)}} = \overline{(\bar{a}, -b)} = (\bar{\bar{a}}, b) = (a,b)$$

Note also that

$$\overline{(a,b)(c,d)} = \overline{(ac - d\bar{b}, \bar{a}d + cb)} = (\overline{ac - d\bar{b}}, -\bar{a}d - cb) = (\bar{c}\ \bar{a} - b\ \bar{d}, -\bar{a}d - cb)$$

and

$$\overline{(c,d)}\ \overline{(a,b)} = (\bar{c}, -d)(\bar{a}, -b) = (\bar{c}\ \bar{a} - b\bar{d}, -cb - \bar{a}d)$$

We have $(a,b) + \overline{(a,b)} = (a,b) + (\bar{a}, -b) = (a + \bar{a}, 0)$, which is a multiple of $(1,0)$.

Finally, notice that

$$(a,b)\overline{(a,b)} = (a,b)(\bar{a}, -b) = (a\bar{a} + b\bar{b}, -\bar{a}b + \bar{a}b) = (a\bar{a} + b\bar{b}, 0) = ||a||^2 + ||b||^2 = ||(a,b)||^2$$

Examples

We can start the construction with the usual product and trivial conjugation on R. Then we get an algebra structure on R^2 satisfying

$$(a,b)(c,d) = (ac - d\bar{b}, \bar{a}d + cb) = (ac - bd, ad + bc)$$
$$\overline{(a,b)} = (\bar{a}, -b) = (a, -b)$$

Clearly this gives the complex numbers.

Next apply the Cayley-Dickson construction to the complex numbers. We claim that we get the quaternions. To see this, notice that $q = a + bi + cj - dk = (a + bi) + j(c + di) = A + jB$ where A and B are complex. Also notice that for complex A, $jA = \overline{A}j$. So

$$(A + jB)(C + jD) = AC + jBC + AjD + jBjD = AC + jBC + j\overline{A}D + j^2\overline{B}D$$

Thus

$$(A + jB)(C + jD) = (AC - \overline{B}D) + j(\overline{A}D + BC)$$

and since complex numbers commute, this agrees with the formula

$$(a, b)(c, d) = (ac - d\overline{b}, \overline{a}d + cb)$$

Moreover

$$\overline{(A + jB)} = \overline{A} + \overline{B}(-j) = \overline{A} - jB$$

agrees with the general formula

$$\overline{(a, b)} = (\overline{a}, -b)$$

Applying the construction once more gives an algebra structure on R^8. This structure is not associative, so great care is required when working with it. However, we will show that $||o_1 o_2||^2 = ||o_1||^2||o_2||^2$. It follows that it satisfies the conditions of Hurwitz's theorem, that non-zero elements have multiplicative inverses, and that the algebra has no zero divisors.

All the remaining Cayley-Dickson algebras have zero divisors.

2 The Octonions

By definition, the *octonions* or *Cayley numbers* are the result of applying the Cayley-Dickson construction to the quaternions.

We want to prove that the octonions satisfy the Hurwitz condition. To see this, let a and b be octonions. We want to prove that $||ab||^2 = ||a||^2||b||^2$. A naive proof would proceed as follows:

$$||ab||^2 = (ab)\overline{(ab)} = (ab)(\overline{b}\ \overline{a}) = a(b\overline{b})\overline{a}$$

Unfortunately, this last step uses associativity, which isn't always true in the octonions. But ignoring this, we could note that $b\overline{b} = ||b||^2$ is real and thus commutes with all octonions, so this is $||b||^2 a\overline{a} = ||b||^2||a||^2$.

Consequently, we try to prove this from first principles. Consider two octonions (a, b) and (c, d). We form

$$(a, b)(c, d) = (ac - d\overline{b}, \overline{a}d + cb)$$

Then

$$(a, b)(c, d)\overline{(a, b)(c, d)} = (ac - d\overline{b}, \overline{a}d + cb)(\overline{c}\ \overline{a} - b\overline{d}, -\overline{a}d - cb)$$

The second component of this product is

$$(\overline{c}\ \overline{a} - b\overline{d})(-\overline{a}d - cb) + (\overline{c}\ \overline{a} - b\overline{d})(\overline{a}d + cb) = 0$$

The first component of the product is

$$(ac - d\overline{b})(\overline{c}\ \overline{a} - b\overline{d}) - (-\overline{a}d - cb)(\overline{d}a + \overline{b}\overline{c})$$

This product has eight terms. Four are

$$||a||^2||c||^2 + ||b||^2||d||^2 + ||a||^2||d||^2 + ||b|^2|||d||^2 = (||a||^2 + ||b||^2)(||c||^2 + ||d||^2)$$

This is just

$$||(a,b)||^2||(c,d)||^2$$

exactly the result we desire The final four terms are

$$-acb\overline{d} - d\overline{b}\ \overline{c}\ \overline{a} + \overline{a}d\overline{b}\overline{c} + cb\overline{d}a$$

This can be rewritten

$$-2Re(acb\overline{d}) + 2Re(cb\overline{d}a)$$

and consequently equals a purely real quaternion. On the other hand, it is the real part of the difference

$$(cb\overline{d})a - a(cb\overline{d})$$

But the real part of a product of two quaternions $(r,v)(s,w)$ is $rs - v \cdot w$ and this does not depend on the order of the terms. So our real part is zero. QED.