

Dirac's Belt Trick, Gyroscopes, and the iPad

Richard Koch

May 19, 2012

Dirac's Belt Trick

P. A. M. Dirac, 1902 - 1984

Nobel Prize (with Erwin Schrodinger) in 1933

Formulated Dirac equation, a relativistically correct quantum mechanical description of the electron, which predicted the existence of antiparticles.

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TeXShop is a Tex previewer for Mac OS X. For more information, click on

- [TeXShop Page](#)

Below are the slides from a talk given at OIMT on May 19, 2012.

- [OIMT Slides](#)

TeXShop for Macintosh; TeX by Donald Knuth for Everything

```
\documentclass[11pt]{amsart}
\usepackage[paper width = 6in, paperheight = 7in]{geometry}
\usepackage[parfill]{parskip}
\usepackage{graphicx}
```

```
\begin{document}
```

```
Using \TeX, we can typeset  $\sqrt{\{1 + x + x^2\}}$ 
\over  $\{e^{-2x + \sqrt{5}}\}$ 
and the matrix  $\left( \begin{array}{cc} 2 & 5 \\ \sqrt{10} & -7 \end{array} \right)$ .
```

According to calculus

```
$$\int_0^1 \{2x + 3x^2\} \ dx = 2 \ \hbox{and} \ \int_0^\infty e^{-x^2} \ dx = \frac{\sqrt{\pi}}{2}
= \frac{\sqrt{\pi}}{2}$$
```

```
The path of a particle in a gravitational field is given by
 $\gamma_i(t)$  where  $\frac{d^2 \gamma_i}{dt^2} + \sum_{jk} \Gamma_{jk}^i \frac{d\gamma_j}{dt} \frac{d\gamma_k}{dt} = 0$ 
```

```
\begin{figure}[htbp]
\centering
\includegraphics[width=2in]{MoveTest-math.jpg}
\end{figure}
\end{document}
```

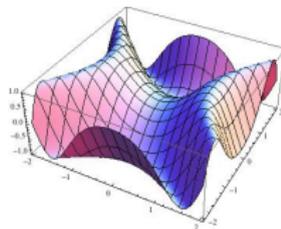
Using TeX, we can typeset $\sqrt{\frac{1+x+x^2}{e^{2x+\sqrt{5}}}}$ and the matrix $\left(\begin{array}{cc} 2 & 5 \\ \sqrt{10} & -7 \end{array} \right)$.

According to calculus

$$\int_0^1 2x + 3x^2 \, dx = 2 \quad \text{and} \quad \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}$$

The path of a particle in a gravitational field is given by $\gamma_i(t)$ where

$$\frac{d^2 \gamma_i}{dt^2} + \sum_{jk} \Gamma_{jk}^i \frac{d\gamma_j}{dt} \frac{d\gamma_k}{dt} = 0$$

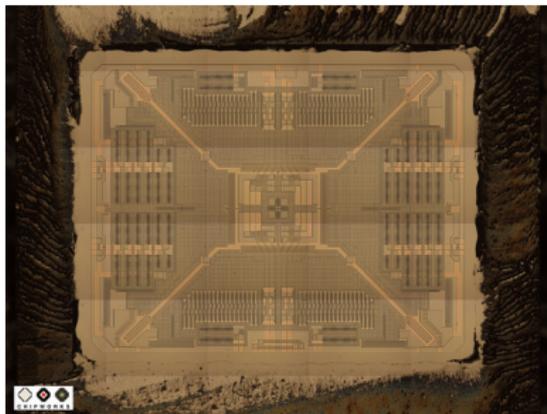


WWDC, Apple's Worldwide Developer's Conference



Gyroscopes in the iPhone and iPad

1. Announced at WWDC 2010
2. Now in iPhones and iPads
3. Actually a small chip



Secret Slide # 1

```
// Turn on gyroscope
motionManager = [[CMMotionManager alloc] init];
motionManager.deviceMotionUpdateInterval = 1.0 / 60.0;
[motionManager startDeviceMotionUpdates];

// Repeat as often as desired
newestDeviceMotion = motionManager.deviceMotion;
...

// Turn off gyroscope
[motionManager stopDeviceMotionUpdates];
[motionManager release];
```

Secret Slide # 2

NewestDeviceMotion contains three descriptions of the attitude of the device. Use whichever is most convenient.

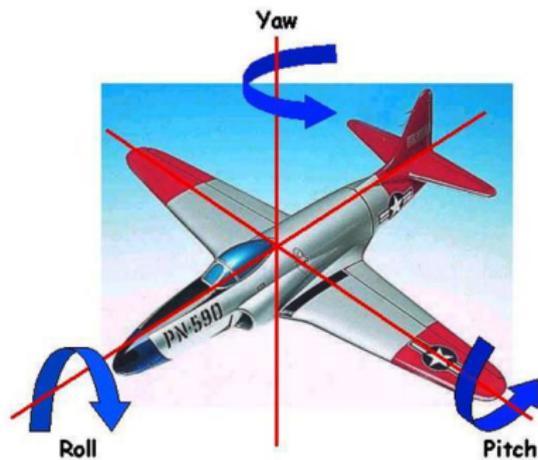
- ▶ Euler angles: roll, pitch, and yaw
- ▶ Rotation matrix
- ▶ Quaternion

Secret Slide # 3

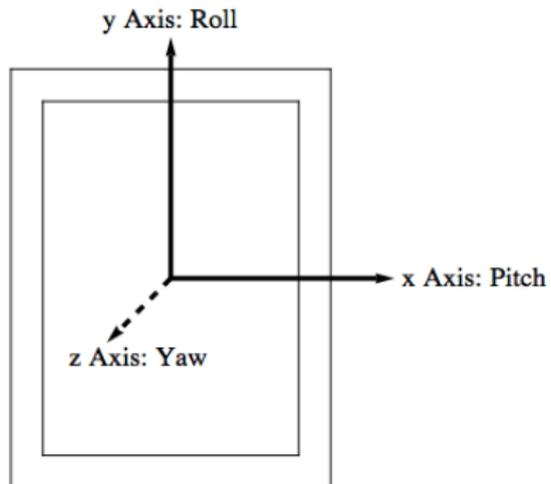
```
// Example code using roll, pitch, yaw
double r = newestDeviceMotion.attitude.roll;
double p = newestDeviceMotion.attitude.pitch;
double y = newestDeviceMotion.attitude.yaw;
```

```
// Example code using quaternions
double q0 = newestDeviceMotion.attitude.quaternion.w;
double q1 = newestDeviceMotion.attitude.quaternion.x;
double q2 = newestDeviceMotion.attitude.quaternion.y;
double q3 = newestDeviceMotion.attitude.quaternion.z;
```

Roll, Pitch, and Yaw

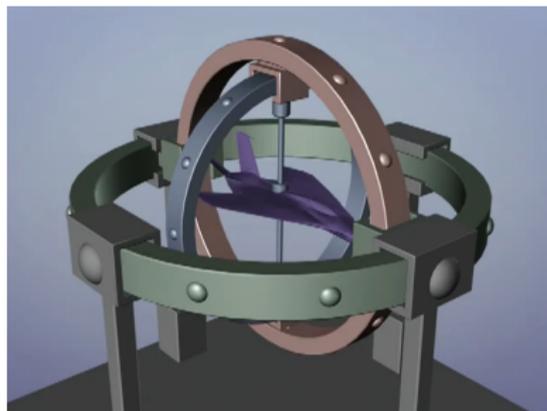
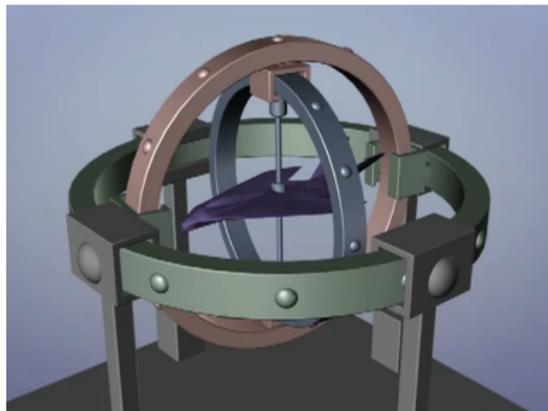


iPad Conventions

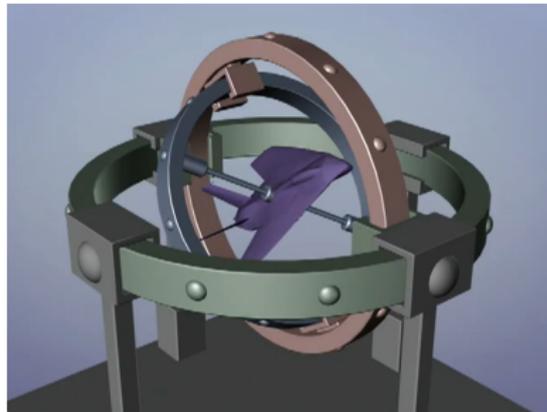
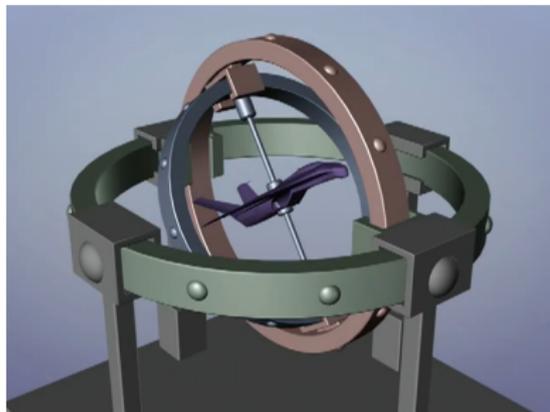
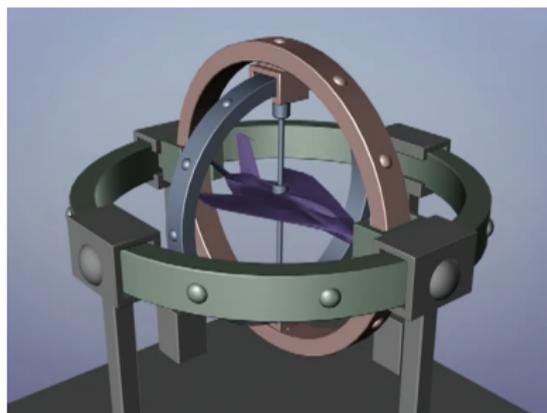
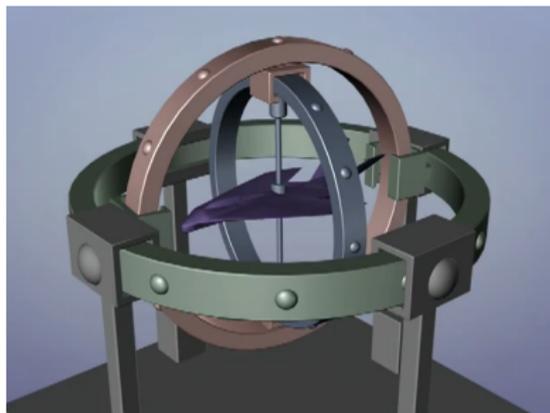


Changing Multiple Parameters

1. First yaw, then pitch, then roll
2. Move the pitch and roll axes when you yaw
3. Move the roll axis when you pitch
4. Don't move axes when you roll



Illustrations by Andrew Silke



Tweaking

Longitude and latitude are coordinates on the earth. Most of us don't know our current longitude and latitude, but we know how to tweak the values: “go one block West and two blocks North.”

Computer animators tweak roll, pitch, and yaw the same way: “yaw one more degree, pitch up two degrees, and roll backward half a degree.”

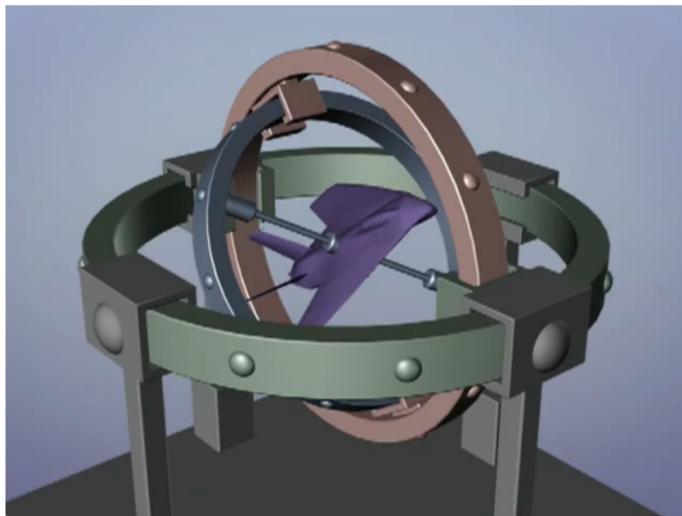
These tweaks use the conventions of the previous slide. A yaw tweak rotates about the vertical axis, *not* the axis perpendicular to the wings. A pitch tweak pitches the nose straight up or down, *not* perpendicular to the wings. But a roll tweak does rotate about the fuselage.

Singular Points

- ▶ A *singular point* is a position in which tweaking fails
- ▶ The south pole is a singular point for longitude and latitude; if you are there and I say “go North,” you can go anywhere. If I say “go East” , you don’t move.
- ▶ There are similar singular positions for the roll, pitch, yaw coordinates. You are in *Gimbal Lock* if you are in such a position.

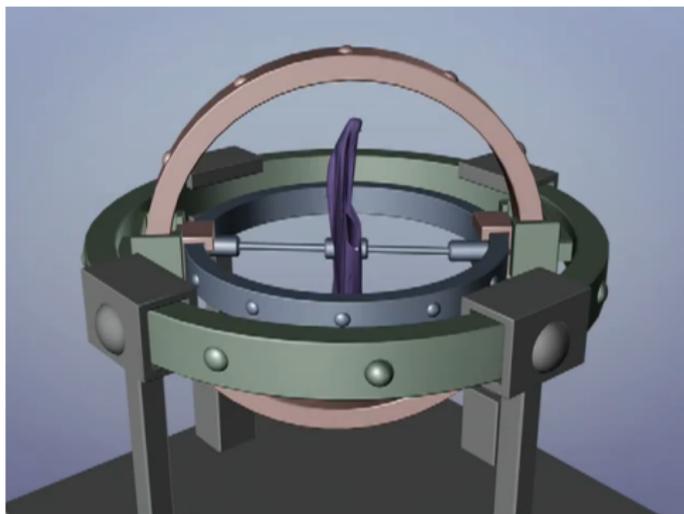
Gimbal Lock Preliminary

In a general rotation, the red pitch circle is perpendicular to the green yaw circle, and perpendicular to the blue roll circle. But the inner and outer yaw and roll circles are not necessarily perpendicular.



Gimbal Lock

Gimbal lock occurs when the yaw and roll circles are in the same plane. At that case, tweaking with extra yaw or extra roll do the same thing (they roll the plane about its fuselage). Tweaking with extra pitch pulls the nose in the red direction. No tweak can push the nose perpendicular to the red.



Gimbal Lock and Quaternions

- ▶ Quaternions solve the gimbal lock problem by providing four coordinates q_0, q_1, q_2, q_3 . They satisfy

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

- ▶ Most of the time, any three numbers can be used for tweaking, with the fourth automatically determined by the above equation.
- ▶ When one of the q_i is ± 1 , it cannot be used for tweaking, but the other three numbers work fine.
- ▶ During the first moon landing, Michael Collins circled the moon while the other two astronauts landed. Collins ran into gimbal lock problems during the orbits, and radioed Houston “All I want for Christmas is a fourth gimbal.”

Complex Numbers

Quaternions are sort of like complex numbers. Recall the complex rules:

- ▶ A point in the plane like $(2, 3)$ can be written $2 + 3i$. This i is a way to keep track of the second coordinate.
- ▶ The square of the distance to the origin is given by the Pythagorean theorem: $2^2 + 3^2 = 13$.
- ▶ If $c = a + bi$, it is sometimes useful to define $\bar{c} = a - bi$
- ▶ Multiply these numbers using the rule $i^2 = -1$. For instance

$$(2+3i)(4+5i) = 8+10i+12i+15i^2 = 8+22i-15 = -7+22i$$

Everything about Quaternions

- ▶ A *quaternion* is formed by four numbers q_0, q_1, q_2, q_3 . We always write

$$q = q_0 + q_1i + q_2j + q_3k$$

For instance, $q = 2 + 3j + 5k$ is the quaternion determined by the four numbers 2, 0, 3, 5.

- ▶ The square of the distance to the origin is given by $q_0^2 + q_1^2 + q_2^2 + q_3^2$.
- ▶ If $q = q_0 + q_1i + q_2j + q_3k$, we define $\bar{q} = q_0 - q_1i - q_2j - q_3k$
- ▶ Multiply these numbers using Hamilton's multiplication rules

$$i^2 = -1 \quad j^2 = -1 \quad k^2 = -1$$

$$ij = k = -ji \quad jk = i = -kj \quad ki = j = -ik$$

About Hamilton

William Rowan Hamilton discovered the quaternions in Dublin during a walk in 1843. He immediately carved the equations on the Boughton Bridge. They vanished, but the bridge remains.

The reals, complex numbers, and quaternions satisfy most standard rules of arithmetic from grade school. It isn't possible to do this in any dimensions but 1, 2, and 4.



Quaternions and Rotations in Three Dimensions

Fact 1: Every rotation of three space can be described by a quaternion of length 1. Thus the iPad can give its attitude by providing $q = q_0 + q_1i + q_2j + q_3k$ where

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

Fact 2: The set of attitudes of the iPad looks like a sphere, except that it is a sphere in *four* dimensions rather than a sphere in *three* dimensions.

Fact 3: If $v = (x, y, z)$ is a point in three dimensions, write this point as a quaternion with zero constant term: $v = xi + yj + zk$. Then under rotation by q , v maps to a new point with no constant term and three space coordinates,

$$qv\bar{q}$$

Example

If you understand this example, then you completely understand rotations using quaternions. Suppose a rotation is given by $q = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$. Notice that $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$, so this is a legal rotation quaternion.

Let's rotate the point $(1, 0, 0) = i$. The formula is

$$\begin{aligned}qv\bar{q} &= \left(\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j\right) i \left(-\frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j\right) = -\frac{1}{2}(i+j)i(i+j) \\ &= -\frac{1}{2}(i+j)(-1+k) = -\frac{1}{2}(-i-j+ik+jk) = j = (0, 1, 0)\end{aligned}$$

More calculation gives $i \rightarrow j$ and $j \rightarrow i$ and $k \rightarrow -k$. A little thought shows that this is exactly what happens if we rotate the iPad by 180 degrees around the axis $(1, 1, 0)$ in the xy -plane. Notice that the quaternion q points along this axis.

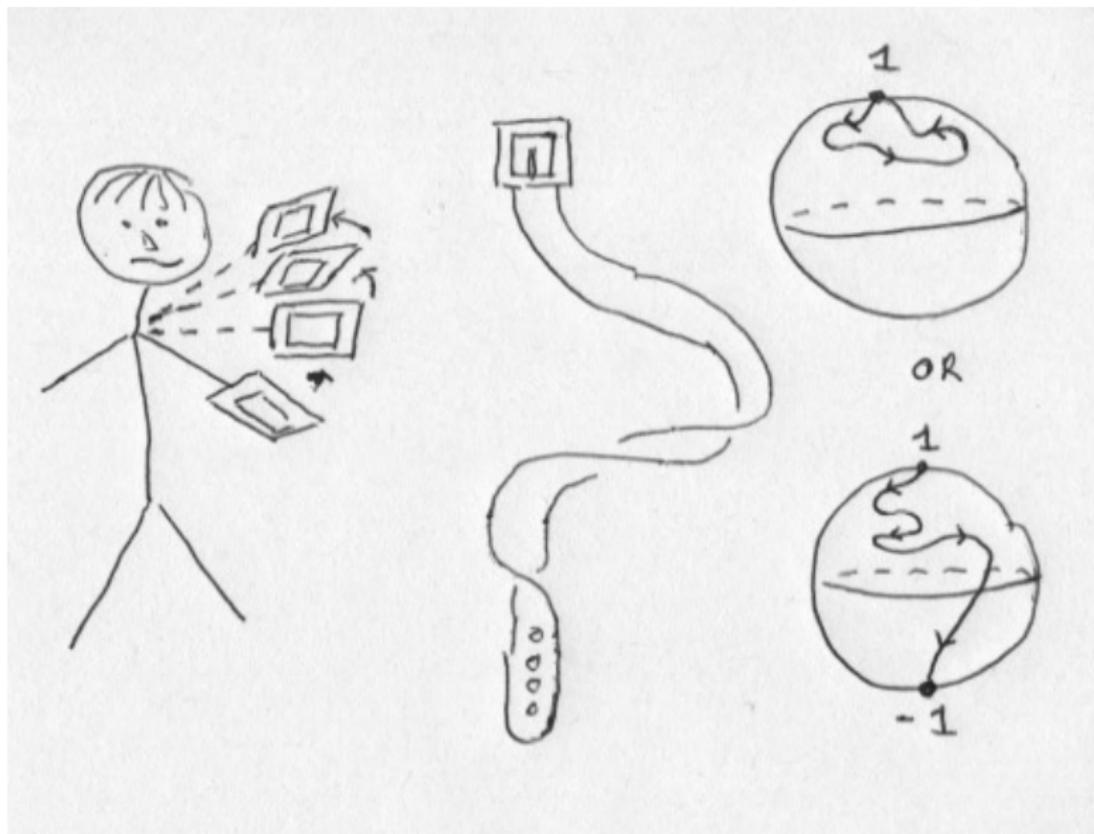
A Final Complication

Let us rotate the iPad and display quaternions as we rotate. At the start, $q = 1$ because we are at the initial position and there is no rotation. Perform a 360 degree yaw. At the end, we are right back where we started, but surprisingly, the iPad now gives $q = -1$.

I have been pretending that there is a one-to-one correspondence between rotations of space and quaternions of length one. But the truth is that each rotation is given by *two* quaternions, which differ in sign. We should have suspected that because $qv\bar{q} = (-q)v(-q)$ since the minus signs cancel.

In particular, if you start with the iPad in its initial position and rotate it anyway you like until you come back to the initial position in the end, the starting q will be 1 and the ending q will be either 1 or -1 .

The Key Picture



Dirac's Belt

We can encode a motion of the iPad from a starting time to a later stopping time by using a belt. Think of time as starting at my foot on the leather end, and ending in my hand as I hold the buckle in the air. Imagine that the belt is decorated with little iPad icons, one after another from beginning to end. Twist the belt so its little iPad icon is oriented at an intermediate spot exactly as the actual iPad is oriented at at the corresponding time during its motion. If the actual iPad starts and ends at its initial position, then the start of the belt under my foot and the buckle in my hand at the end will be parallel, as in the belt trick.

Each intermediate position of the iPad corresponds to an attitude of the belt at the appropriate height. This attitude is described by a quaternion in the four dimensional sphere. As the iPad moves from its starting to its ending position, the belt moves from the foot at the bottom to the buckle at the top, and the quaternion traces a path in the sphere.

More on the Picture

Incidentally, our picture of the sphere is wrong because the quaternions form *a sphere in four dimensional space* rather than a sphere in three dimensional space. But the picture shows the correct idea, so I'm not going to worry about it.

If the attitude of the iPat starts and ends at the same initial position, the ends of the belt start and end at the same attitude, i.e., are parallel, and the path on the sphere starts at 1 and ends at either 1 or -1 .

Suppose the rope starts and ends at 1 on the sphere. Then we can gradually push the rope north until it is all piled up at the pole. This corresponds to gradually twisting the belt until finally all the little pieces are oriented in the same way. In other words, it corresponds to untwisting the belt without turning its ends.

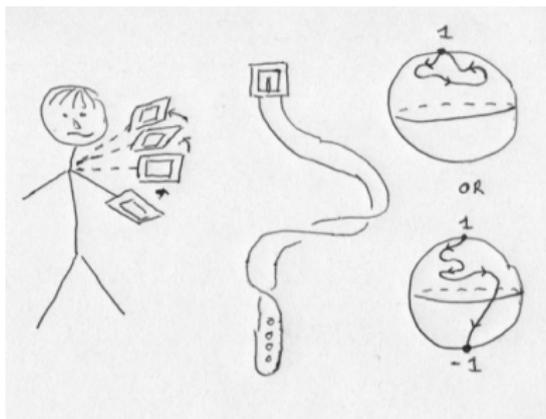
Still More on the Picture

Conversely, suppose we can untwist the belt without turning its ends. This corresponds to gradually pushing the rope without moving its ends. In the end, the belt has no twists, and so the rope is all piled at the same spot. So it must be piled at 1. Therefore the original rope before it was pushed around must have started and ended at 1.

We have proved that the belt can be untwisted if and only if the corresponding quaternionic path begins and ends at 1. Direct experiments with the iPad show that one 360 rotation around an axis changes the sign of the initial quaternion, but two twists leave it alone. Therefore you cannot untwist a belt with one twist, but you can untwist a belt with two twists.

The Moral of the Story

Look one final time at the picture below. This picture illustrates a deep fact about rotations in three dimensional space: “there is a difference between an odd number of twists and an even number of twists.” This fact is virtually invisible if you rotate an actual object like an iPad. It becomes visible, but mysterious, in the belt trick. It becomes obvious and understandable in the sphere picture of quaternions. The goal of mathematics is to discover subtle features of the world, and finds way of thinking about them that make the features obvious.



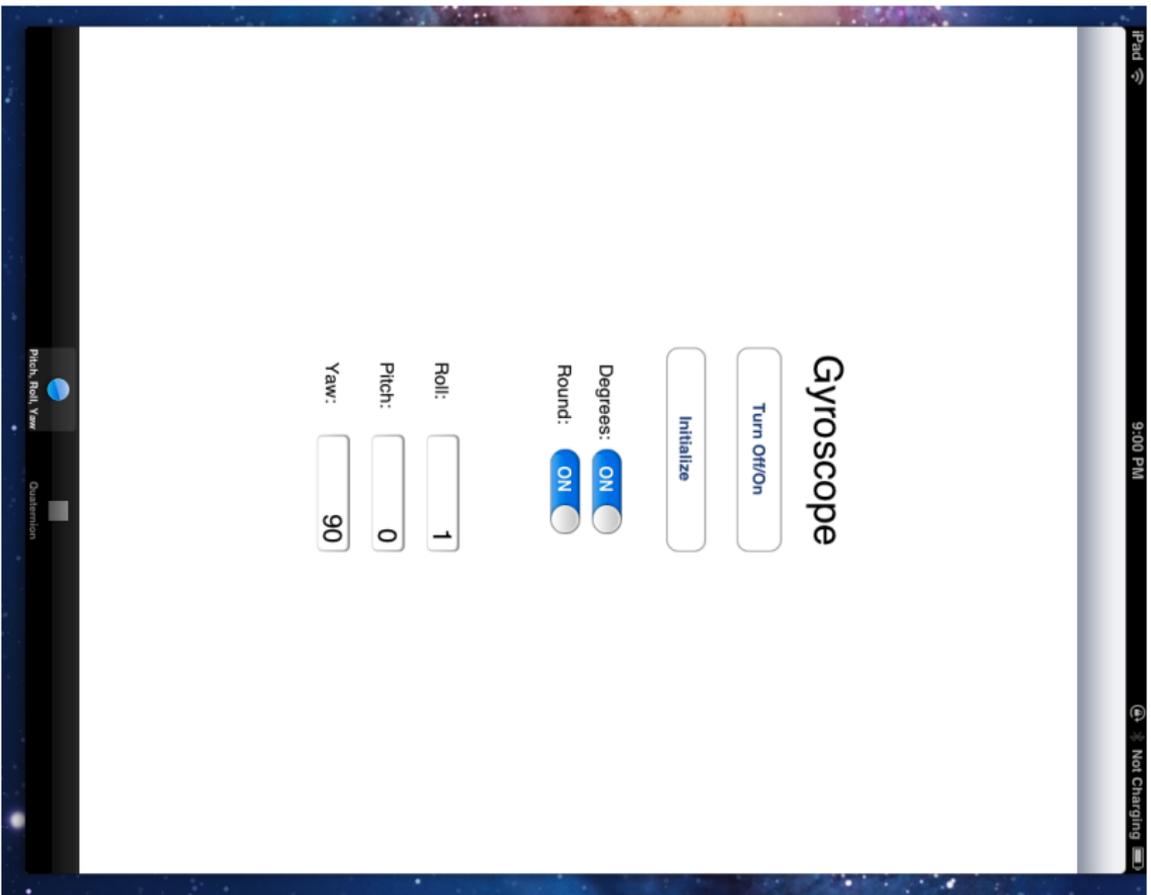
Quaternion from Rotation

Every rotation in three space rotates about an axis (a_1, a_2, a_3) by an angle θ . Let me explain how to obtain the quaternion from this description (I skipped this in the talk).

The axis forms the ijk -part of the quaternion, but we must choose an appropriate length. Set $(q_1, q_2, q_3) = (\lambda a_1, \lambda a_2, \lambda a_3)$ for λ to be chosen later.

The constant term gives the amount of rotation: $q_0 = \cos\left(\frac{\theta}{2}\right)$. This formula implies that as we rotate between zero and 360 degrees, q_0 goes between 1 and -1 . To make the length of the quaternion equal 1, we then must choose $\lambda = \frac{1}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \sin\left(\frac{\theta}{2}\right)$.

iPad; Yaw by Ninety Degrees



iPad; Rotation by $\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j$

8:59 PM Not Charging

Gyroscope

Turn On/Off

Initialize

Round: OFF

Quaternion:

q i q:

q j q:

q k q:

Peek, Roll, View Question