Assignment 7; Due Friday, November 11

Read section nine on connected spaces, and the beginning of section ten on pancake problems. Then do the following problems:

- 9.8 abd
- 9.8 ef (graduate students only)
- 9.8 h
- 9.8 i (graduate students only)
- 10.7 a

- Let \( p \in X \). Prove that the union of all connected subsets of \( X \) containing \( p \) is itself connected. This union is thus the largest connected subset of \( X \) containing \( p \). It is called the connected component of \( p \).

- If \( p, q \in X \), prove that the connected component of \( p \) and the connected component of \( q \) are either equal or else disjoint. Conclude that \( X \) can be written uniquely as a disjoint union of connected components.

- What are the connected components of \( \{ (x, y) \in \mathbb{R}^2 \mid y \neq 0 \} \)? What are the connected components of \( \mathbb{Q} \)?

- (graduate students only) Show that each connected component is closed. Show by example that connected components may or may not be open.

- (graduate students only) Let \( O(n) \) be the set of all linear transformations \( A : \mathbb{R}^n \to \mathbb{R}^n \) which preserve distance, so \( ||Av|| = ||v|| \). This set is a group, the orthogonal group. Denote the standard dot product on \( \mathbb{R}^n \) by \( <v, w> \). Prove that \( ||v + w||^2 - ||v||^2 - ||w||^2 = 2<v, w> \). Conclude that a linear transformation \( A \) is in \( O(n) \) if and only if \( <Av, Aw> = <v, w> \) for all vectors \( v \) and \( w \). Using this result, prove that a matrix \( A \) represents an element of \( O(n) \) if and only if \( A^tA = I \).

Give \( O(n) \) a topology by noticing that each \( n \times n \) matrix has \( n^2 \) components, so \( O(n) \subseteq \mathbb{R}^{n^2} \). Prove that \( O(n) \) is compact.

Find the connected components of \( O(n) \). I’ll give two hints:

Suppose \( A \subseteq X \) and suppose that whenever \( p \) and \( q \) are points in \( A \), there is a continuous map \( \gamma : [0, 1] \to A \) such that \( \gamma(0) = p \) and \( \gamma(1) = q \). Then \( A \) is connected. This result is easy to prove.
You may use without proof the following result from linear algebra: If $A \in O(n)$, there exist matrices $B$ and $C$ in $O(n)$ such that $A = BC B^{-1}$ where $C$ has $1 \times 1$ and $2 \times 2$ blocks on the diagonal and is otherwise zero, and the $1 \times 1$ blocks are $(\pm 1)$ and the $2 \times 2$ blocks are
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]