8.2a If $X$ is finite, then every set is open and $X$ has the discrete topology. Thus if $x \neq y$, the open sets $U = \{x\}$ and $V = \{y\}$ separate $x$ and $y$.

Conversely, suppose $X$ is Hausdorff and let $x \neq y$. Choose disjoint $U$ and $V$ with $x \in U$ and $y \in V$. Notice that $U = X - A$ and $V = X - B$ where $A$ and $B$ are finite. Then $U \cap V = X - (A \cup B) = \emptyset$ and so $X = A \cup B$ is also finite.

8.14a Since $f : X \to Y$ is continuous, $U \subseteq Y$ open implies $f^{-1}(U)$ open. Conversely, suppose $f^{-1}(U)$ open. Then $X - f^{-1}(U)$ is closed in the compact set $X$, so compact. By one of our theorems, $f(X - f^{-1}(U)) \subseteq Y$ is compact in the Hausdorff space $Y$, so closed. But $f(X - f^{-1}(U)) = Y - U$ since $f$ is onto. So $Y - U$ is closed and $U$ is open.

8.14b Suppose $Y$ is Hausdorff. Whenever $x \times y \notin D$ we have $x \neq y$ and we can find open $x \in U$ and $y \in V$ with $U \cap V = \emptyset$. It follows that $x \times y \in U \times V \subseteq Y \times Y$ and $(U \times V) \cap D = \emptyset$. The union of all such $U \times V$ is an open set which is exactly $Y \times Y - D$, so $D$ is closed.

Conversely, suppose $D$ is closed. Then $W = Y \times Y - D$ is open. If $x \neq y$, then $x \times y \in W$, so by definition of open sets in the product topology there is a rectangle $U \times V$ with $x \times y \in U \times V \subseteq Y \times Y - D$. Since $U \times V$ does not intersect $D$, $U \cap V = \emptyset$.

8.14c Consider the map $f \times f : X \times X \to Y \times Y$. Since $Y$ is Hausdorff, the diagonal $D \subseteq Y \times Y$ is closed, so $(f \times f)^{-1}(D)$ is closed. But this is exactly the set of all $x \times y \in X \times X$ such that $f(x) \times f(y) \in D$, that is $f(x) = f(y)$.

8.14e Suppose $X$ is compact, $Y$ is Hausdorff, and $f : X \to Y$ is continuous and onto. If $A \subseteq X$ is closed, then $A$ is compact, so $f(A) \subseteq Y$ is compact in a Hausdorff space and so closed.

Conversely, suppose $f$ takes closed sets to closed sets. Apply theorem 8.11: If $Y$ is the quotient space of the compact Hausdorff space $X$ determined by an onto map $f : X \to Y$, and if $f$ is a closed mapping, then $Y$ is Hausdorff.

Continuation of 8.14e Now we must prove that under the same hypotheses, $Y$ is Hausdorff if and only if $E = \{ x_1 \times x_2 \ | \ f(x_1) = f(x_2) \}$ is closed. Half of this was proved in 8.14c. We must still prove that if this set is closed, then $Y$ is Hausdorff. This will follow from the first part of the problem if we can prove that $E$ closed implies that $f$ is a closed mapping.

So suppose $A \subseteq X$ is closed. We must prove that $f(A)$ is closed; since $Y$ has the quotient topology, we must prove that $f^{-1}(f(A))$ is closed, and thus that

$$\{ x \in X \ | \ \exists a \in A \text{ with } f(x) = f(a) \}$$
is closed. But \( A \) closed in \( X \) implies that \( X \times A \) is closed in \( X \times X \). We are assuming \( E \) is closed, so \( (X \times A) \cap E \) is closed. This set is \( \{x \times a \mid f(x) = f(a)\} \). Since the natural projection \( \pi : X \times X \to X \) is continuous and since \( X \) is compact Hausdorff, \( \pi \) maps closed sets to closed sets, so \( \pi(A \cap E) \) is closed. But this set is exactly \( \{x \in X \mid \exists a \in A \text{ with } f(x) = f(a)\} \).

**8.14i** Notice that \( x \sim x \) since \( x - x = 0 \in Q \). If \( x \sim y \), then \( x - y \in Q \) and so \( -(x - y) = y - x \in Q \), so \( y \sim x \). Finally if \( x \sim y \) and \( y \sim z \), then \( x - y \in Q \) and \( y - z \in Q \), so \( (x - y) + (y - z) = x - z \in Q \), so \( x \sim z \).

Let \( \pi : R \to R/\sim \) and give \( R/\sim \) the quotient topology. Call the resulting space \( Y \). Notice that \( Y \) is uncountable, since each equivalence class contains only countably many elements. In particular, \( Y \) has two distinct points. I will prove that \( Y \) has the concrete topology: the only open sets are \( \emptyset \) and \( Y \). If so, it will follow that \( Y \) is not Hausdorff because there exist distinct elements \( y_1 \neq y_2 \) and yet the only possible open neighborhoods of \( y_1 \) and \( y_2 \) are \( Y \) and \( Y \) and these are not disjoint.

Let \( U \subseteq Y \) be nonempty and open. We will prove \( U = Y \). Notice first that \( \pi^{-1}(U) \) is nonempty and open; call this set \( V \). The set \( V \) contains a nontrivial interval \( (a, b) \). I claim that every real number is equivalent to an element in \( (a, b) \). Indeed, if \( r \in R \) then the interval \( (a - r, b - r) \) contains a rational \( q \), so \( a - r < q < b - r \) and then \( a < r + q < b \) and \( r + q \sim r \). Any \( y \in Y \) is represented by some real number, which we can assume is in \( (a, b) \) and thus \( y \in \pi(\pi^{-1}(U)) = U \). So \( U = Y \).

**8.14j** To understand this problem, consider first the case when \( X \) is a large closed rectangle about the origin and \( U \) is a smaller open disk inside this rectangle.
The space $X/(X-U)$ is formed by gluing all points in $X-U$ together. This means that all of the points which are darker gray become a single point. In particular, all the points on the boundary of $U$ become a single point, so the light gray disk becomes a sphere. Notice that $U^\infty$ is also a sphere, since $U$ is homeomorphic to $\mathbb{R}^n$ and thus $U^\infty$ is homeomorphic to $(\mathbb{R}^n)^\infty$, which is a sphere.

Now we give the general argument. As a set, $U^\infty = U \cup \{\infty\}$. Next we analyze $X/(X-U)$ as a set. Notice that $X = U \cup (X-U)$. Each point in $U$ represents a unique point in $X/(X-U)$ and all of the points in $X-U$ represent the same point, $p$. Therefore, as a set $X/(X-U)$ is $U \cup \{p\}$ where $p$ is the point represented by all elements of $X-U$. Map $U^\infty$ to $X/(X-U)$ by sending points in $U$ to themselves and sending $\infty$ to $p$. This map is clearly one-to-one and onto. We must show that this map induces a one-to-one correspondence between open sets $V \subseteq U^\infty$ and open sets $W \subseteq X/(X-U)$.

Incidentally, the previous argument assumes that $X-U$ is nonempty. Otherwise there would be no $p$ and we would be in trouble.

There are two types of open sets in $U^\infty$. First there are open sets $V \subseteq U$. Second there are open sets of the form $V = (U-A) \cup \{\infty\}$ where $A \subseteq U$ is compact and closed.

There are also two types of open sets in $X/(X-U)$, namely open sets which do not contain the special point $p$ and open sets which contain this point. Each point of a subset $W$ which does not contain $p$ is represented by a unique point in $U$, so we can identify such subsets of $X/(X-U)$ with subsets $W \subseteq U$, and such a set is open in $X/(X-U)$ exactly when its inverse image in $X$ is open, i.e., exactly when $W \subseteq U$ is open.

The open sets of $X/(X-U)$ which contain $p$ have the form $W = V \cup \{p\}$ where $V \subseteq U$. This set is open exactly when its inverse image in $X$ is open. The inverse image is $V \cup (X-U) = X - (U \cap V^c)$ and is open just in case $U \cap V^c$ is closed in $X$. Since $X$ is compact Hausdorff, this happens just in case $U \cap V^c$ is compact. Call this set $A$ and notice that $V = U-A$.

To summarize, the open sets in $X/(X-U)$ have the form $W \subseteq U$ where $W$ is open, or $W = (U-A) \cup \{p\}$ where $A \subseteq U$ is closed and compact.

It is immediately clear that our map sets up a one-to-one correspondence between open sets in $U^\infty$ and open sets in $X/(X-U)$.

**Continuation of 8.14j** Consider the special case $U = X-\{p\}$. Then $U^\infty = (X-\{p\})^\infty$ is homeomorphic to $X/(X-(X-\{p\})) = X/\{p\}$. This last space is $X$ with the point $p$ glued to itself, i.e., just $X$. 

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