1: Let $\gamma(t) = (t, 0)$. Then $\gamma'(t) = (1, 0)$ and $||\gamma'(t)||^2 = g_{11} = \frac{4}{(1-r^2)^2}$. So the length is

$$
\int_0^r ||\gamma'(t)|| \, dt = \int_0^r \frac{2}{1-r^2} \, dt = \int_0^r \left\{ \frac{1}{1-t} + \frac{1}{1+t} \right\} \, dt = [-\ln(1-t) + \ln(1+t)]^r_0 = \ln \frac{1+r}{1-r}
$$

2: If $d = \ln \frac{1+r}{1-r}$, then $e^d = \frac{1+r}{1-r}$, so $e^d - re^d = 1 + r$ and $e^d - 1 = r(e^d + 1)$. Therefore

$$
r = \frac{e^d - 1}{e^d + 1}.
$$

3: Now we must change the meaning of $t$; ignore the previous parameterization of the curve. At time $t$ we will be at a spot $(r, 0)$. This $r$ depends on $t$ in some way. When we compute the distance traveled, we’ll get $d = \ln \frac{1+r}{1-r}$ and we want this to actually be $t$. So $t = \ln \frac{1+r}{1-r}$ and therefore $r = \frac{e^t - 1}{e^t + 1}$. So the parameterized curve which travels at unit speed is

$$
\gamma(t) = \left( \frac{e^t - 1}{e^t + 1}, 0 \right)
$$

Notice that $\gamma(0) = (0, 0)$. The first component is always smaller than 1 because $e^t - 1 < e^t + 1$. As $t$ approaches infinity, $e^t$ becomes enormously large, and the numerator and denominator are large numbers within two of each other, so their quotient approaches 1. Hence $\lim_{t \to \infty} \gamma(t) = (1, 0)$.

4: We have $g_{11} = \frac{4}{(1-r^2)^2}, g_{12} = 0, g_{22} = \frac{4r^2}{(1-r^2)^2}$. Then

$$
\Gamma^k_{ij} = \frac{1}{2}(g^{-1})^k_{kk} \left( \frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} \right)
$$

Using our standard analysis, the only nonzero terms are $\Gamma^1_{11}, \Gamma^1_{22}, \Gamma^2_{12}$.

Then

$$
\Gamma^1_{11} = \frac{1}{2} \frac{(1-r^2)^2}{4} \frac{\partial g_{11}}{\partial r} = \frac{1}{2} \frac{(1-r^2)^2}{4} \frac{-2}{(1-r^2)^3} = \frac{-2r}{1-r^2}
$$

$$
\Gamma^1_{22} = \frac{1}{2} \frac{(1-r^2)^2}{4} \left( -\frac{\partial g_{22}}{\partial r} \right) = \frac{1}{2} \frac{(1-r^2)^2}{4} \frac{8r}{(1-r^2)^3} = \frac{r(1+r^2)}{1-r^2}
$$

$$
\Gamma^2_{12} = \frac{1}{2} \frac{(1-r^2)^2}{4r^2} \frac{\partial g_{22}}{\partial r} = \frac{1}{2} \frac{(1-r^2)^2}{4r^2} \frac{8r}{(1-r^2)^3} = \frac{1+r^2}{r(1-r^2)}
$$

1
5:
\[
\frac{d^2 r}{dt^2} + \frac{2r}{1-r^2} \left( \frac{dr}{dt} \right)^2 - \frac{r(1+r^2)}{1-r^2} \left( \frac{d\theta}{dt} \right)^2 = 0
\]
\[
\frac{d^2 \theta}{dt^2} + 2 \frac{1+r^2}{r(1-r^2)} \frac{dr}{dt} \frac{d\theta}{dt} = 0
\]

6: If $\theta$ is constant, then $\frac{d\theta}{dt} = \frac{d^2 \theta}{dt^2} = 0$ and the second equation is trivially true.

7: The remaining equation now reads
\[
\frac{d^2 r}{dt^2} + \frac{2r}{1-r^2} \left( \frac{dr}{dt} \right)^2 = 0
\]

By symmetry, we can suppose that the radial line is the $x$-axis. So our curve has the form $\gamma(t) = (r(t), 0)$. Notice that $\gamma'(t) = (r'(t), 0)$ and $||\gamma'||^2 = g_{11}(r')^2 = \frac{4}{(1-r^2)^2}(r')^2$. Therefore
\[
||\gamma'(t)|| = \frac{2r'(t)}{1-r^2}
\]

This length will be constant just in case its derivative with respect to $t$ is zero. The derivative is
\[
\frac{(1-r^2)2r'' - 2r'(-2r)r'}{(1-r^2)^2}
\]

and the condition that this is zero is
\[
(1-r^2)2r'' + 4r(r')^2 = 0
\]

which simplifies to
\[
\frac{d^2 r}{dt^2} + \frac{2r}{1-r^2} \left( \frac{dr}{dt} \right)^2 = 0
\]

This is exactly the remaining equation.