Of all conceptions Continuity is by far the most difficult for Philosophy to handle. You naturally cannot do much with a conception until you can define it. Now every man at all competent to express an opinion must admit as it seems to me that no definition of continuity up to quite recent times was nearly right, and I maintain that the only thoroughly satisfactory definition is that which I have been gradually working out, and of which I presented a first ébauche when I had the honor of reading a paper here in Cambridge in 1892,¹ and the final form of which I have given you sufficient hints in these lectures. But even supposing that my definition, which as yet has not received that sanction which can only come from the critical examination of the most powerful and exact intellects, is all wrong, still no man not in leading strings as to this matter can possibly think that there was anything like a satisfactory definition before the labors of Dr. Georg Cantor, which only began to attract the attention of the whole world about [1890].

But after a satisfactory definition of continuity has been obtained the philosophical difficulties connected with this conception only begin to [be] felt in all their strength. Those difficulties are of two kinds. First there is the logical difficulty, how we are to establish a method of reasoning about continuity in philosophy? and second there is the metaphysical difficulty, what are we to say about the being, and the existence, and the genesis of continuity?

As to the proper method of reasoning about continuity, the dictate of good sense would seem to be that philosophy should in this matter follow the lead of geometry, the business of which it is to study continua.
But alas! the history of geometry forces upon us some sad lessons about the minds of men. That which had already been called the Elements of geometry long before the day of Euclid is a collection of convenient propositions concerning the relations between the lengths of lines, the areas of surfaces, the volumes of solids, and the measures of angles. It concerns itself only incidentally with the intrinsic properties of space, primarily only with the ideal properties of perfectly rigid bodies, of which we avail ourselves to construct a convenient system of measuring space. The measurement of a thing was clearly shown by Klein, twenty-five years ago, to be always extrinsic to the nature of the thing itself. Elementary geometry is nothing but the introduction to geometrical metric, or the mathematical part of the physics of rigid bodies. The very early Greek geometers, I mean for example [Peirce left a blank to fill in later], who is said to have written the first Elements, I have no doubt, considered metric as the philosophical basis and foundation, not only of geometry, but of mathematics in general. For it is to be remarked that considerably the larger part of Euclid’s Elements is occupied with algebra not with geometry; and since he and all the Greeks had a much stronger impulse to get to the logical foundation of any object of study than we have, and since it is only the first book of Euclid in which the logic has been a matter of deep cogitation, it is plain that it was originally, at least, conceived that those geometrical truths in the first book of the Elements lay at the foundation even of algebra itself. But Euclid certainly, and in my opinion much earlier Greeks, had become acquainted with that branch of geometry which studies the conditions under which different rays of light indefinitely prolonged will intersect in common points or lie in common planes. There is no accepted name for this branch. It is sometimes called descriptive geometry; but that is in violent conflict with principles of nomenclature, since descriptive geometry is the accepted name of a branch of geometry invented by Monge and so named by him, — a branch closely allied to this other doctrine but not the same. Clifford called the branch of which we are speaking, Graphics (which conveys no implication); other writers call it synthetic geometry (though it may be treated analytically), geometry of position (which is the name of something else), modern geometry (where in fact it is ancient), intersectional geometry (though projection plays as great a role as section in it), projective geometry (though section is as important as projection), perspective geometry, etc. I would propose the name geometrical optic. Euclid, I say, and earlier Greeks were acquainted [with] this geometrical optic. Now to any person of discern-
ment in regard to intellectual qualities and who knows what the Greeks were, and especially what the Greek geometers were, and most particularly what Euclid was, it seems to me incredible that Euclid should have been acquainted with geometrical optic and not have perceived that it was more fundamental, — more intimately concerned with the intrinsic nature of space, — than metric is. And indeed a posteriori evidences that he actually did so are not wanting. Why, then, did Euclid not say a single word about this optic in his elements? Why did he altogether omit it even in cases where he must have seen that its propositions were indispensable conditions of the cogency of his demonstrations? Two possible explanations have occurred to me. It may be that he did not know how to prove the propositions of optic otherwise than by means of metric; and therefore, seeing that he could not make a thorough job, preferred rather ostentatiously and emphatically (quite in his style in other matters), omitting all mention of optical propositions. Or it may be that, being a university professor, he did not wish to repel students by teaching propositions that had an appearance of being useless. Remember that even the stupendous Descartes abandoned the study of geometry. And why? Because he said it was useless. And this he said a propos of conic sections! That he should have thought conic sections useless, is comparatively pardonable. But that he the Moses of modern thinkers should have thought that a philosopher ought not to study useless things is it not a stain of dishonor upon the human mind itself?

In modern times the Greek science of geometrical optic was utterly forgotten, all the books written about it were lost, and mathematicians became entirely ignorant that there was any such branch of geometry. There was a certain contemporary of Descartes, one Desargues, who rediscovered that optic and carried his researches into it very far indeed. He showed clearly and in detail the great utility of the doctrine in perspective drawing and in architecture, and the great economy that it would effect in the cutting of stones for building. On the theoretical side he pushed discovery to an advance of a good deal more than two centuries. He was a secular man. But he worked alone, with hardly the slightest recognition. (Without one word of encouragement, just as such a mind would do in America today.) Insignificant men treated him with vitriolic scorn. His works, though printed, were utterly lost and forgotten. The most voluminous historians of mathematics though compatriots did not know that such a man had ever lived, until one day Michel Chasles walking along the Quai des Grands Augustins probably after a meeting of the Institut, came across and bought for a franc a MS copy
of one of those printed books. He took it home and studied it. He learned from it the important theory of the Involution of Six Points; and from him the mathematical world learned it; and it has been a great factor in the development of modern geometry. There can be no possible doubt that this knowledge actually came from the book of Desargues, because the relation has always borne the strange name Involution which Desargues had bestowed upon it, — and the whole theory is in his book although it had been totally unmentioned in any known treatise, memoir, or programme previous to the lucky find of Chasles. When will mankind learn the lessons such facts teach? That had that doctrine not been lost to all those generations of geometers, philosophy would have been further advanced today, and that the nations would have attained a higher intellectual level, is undoubtedly true, — but that may be passed by as a bagatelle. But why will men not reflect that but for the stupidity with which Desargues was met, — many a man might have eaten a better dinner and have had a better bottle of wine with it? It needs not much computation of causes and effects to see that that must be so. Six collinear points are said to be in involution, provided that four points can be found such that every pair of them is
in one straight line with *one* of the six, but not with *all* the six. Thus AA’BB’CC’ are in involution because of the four points PQRS.

In 1859, Arthur Cayley showed that the whole of geometrical metric is but a special problem in geometrical optic. Namely, Cayley showed that there is a locus in space, — not a kind of a locus, but an individual place, — whose optical properties and relations to rigid bodies constitute those facts that are expressed by space-measurement.

That was in [1859.] It attracted the admiration and assent of the whole mathematical world, which has never since ceased to comment upon it, and develop the doctrine. Yet a few years ago I was talking with a man who had written two elementary geometries and who perhaps was, for aught I know still is, more influential than any other individual in determining how Geometry shall be taught in American schools at large, and this gentleman never heard of Projective Geometry neither the name nor the thing and his politeness never shone more than in his not treating what I said about Cayley with silent contempt.

But many years before Cayley made that discovery, a geometer in Göttingen, Listing by name, — a name which I will venture to say that Cayley, learned as he was in all departments of mathematics heard for the first time many years later, probably from Tait, who knew of him because he and Listing were both physicists, — this Listing had in 1847 four years before Riemann’s first paper discovered the existence of quite another branch of geometry, and had written two very long and rich memoirs about it. But the mathematical world paid no heed to them till half a century had passed. This branch, which he called Topology, but which I shall call Topic, to rhyme with metric and optic, bears substantially the same relation to optic that optic bears to metric. Namely, topic shows that the entire collection of all possible rays, or unlimited straight lines, in space, has no general geometrical characters whatever that distinguish it at all from countless other families of lines. Its only distinction lies in its physical relations. Light moves along rays; so do particles unacted on by any forces; and maximum-minimum measurements are along rays. But the whole doctrine of geometrical optic is merely a special case of a topical doctrine.

That which topic treats of is the modes of connection of the parts of continua. Geometrical topic is what the philosopher must study who seeks to learn anything about continuity from geometry.

I will give you a slight sketch of the doctrine. We have seen in a previous lecture what continuity consists in. There is an endless series of abnumeral multitudes, each related to the next following as M is
related to $2^N$, where we might put any other quantity in place of 2. The least of these abnumeral multitudes is $2^N$ [that is, 2 to the power $x_\infty$] where N is the multitude of all whole numbers. It is impossible that there should be a collection of distinct individuals of greater multitude than all these abnumeral multitudes. Yet every one of these multitudes is possible and the existence of a collection of any one of these multitudes will not in the least militate against the existence of a collection of any other of these multitudes. Why then, may we not suppose a collection of distinct individuals which is an aggregate of one collection of each [of] those multitudes? The answer is, that to suppose an aggregate of all is to suppose the process of aggregation completed, and that is supposing the series of abnumeral multitudes brought to an end, while it can be proved that there is no last nor limit to the series. Let me remind you that by the limit of an endless series of successive objects we mean an object which comes after all the objects of that series, but so that every other object which comes after all those objects comes after the limit also. When I say that the series of abnumeral multitudes has no limit, I mean that it has no limit among multitudes of distinct individuals. It will have a limit if there is properly speaking any meaning in saying that something that is not a multitude of distinct individuals is more than every multitude of distinct individuals. But, you will ask, can there be any sense in that? I answer, yes, there can, in this way. That which is possible is in so far general, and as general, it ceases to be individual. Hence, remembering that the word "potential" means indeterminate yet capable of determination in any special case, there may be a potential aggregate of all the possibilities that are consistent with certain general conditions; and this may be such that given any collection of distinct individuals whatsoever, out of that potential aggregate there may be actualized a more multititudinous collection than the given collection. Thus the potential aggregate is with the strictest exactitude greater in multitude than any possible multitude of individuals. But being a potential aggregate only, it does not contain any individuals at all. It only contains general conditions which permit the determination of individuals.

The logic of this may be illustrated by considering an analogous case. You know very well that $2^\infty$ is not a whole number. It is not any whole number whatever. In the whole collection of whole numbers you will not find $2^\infty$. That you know. Therefore, you know something about the entire collection of whole numbers. But what is the nature of your conception of this collection? It is general. It is potential. It is vague,
but yet with such a vagueness as permits of its accurate determination in regard to any particular object proposed for examination. Very well, that being granted, I proceed to the analogy with what we have been saying. Every whole number considered as a multitude is capable of being completely counted. Nor does its being aggregated with or added to any other whole number in the least degree interfere with the completion of the count. Yet the aggregate of all whole numbers cannot be completely counted. For the completion would suppose the last whole number was included, whereas there is no last whole number. But though the aggregate of all whole numbers cannot be completely counted, that does not prevent our having a distinct idea of the multitude of all whole numbers. We have a conception of the entire collection of whole numbers. It is a potential collection indeterminate yet determinable. And we see that the entire collection of whole numbers is more multitudinous than any whole number.

In like manner the potential aggregate of all the abnumerous multitudes is more multitudinous than any multitude. This potential aggregate cannot be a multitude of distinct individuals any more than the aggregate of all the whole numbers can be completely counted. But it is a distinct general conception for all that, — a conception of a potentiality.

A potential collection more multitudinous than any collection of distinct individuals can be[,] cannot be entirely vague. For the potentiality supposes that the individuals are determinable in every multitude. That is, they are determinable as distinct. But there cannot be a distinctive quality for each individual; for these qualities would form a collection too multitudinous for them to remain distinct. It must therefore be by means of relations that the individuals are distinguishable from one another.

Suppose, in the first place, that there is but one such distinguishing relation, ∼. Then since one individual is to be distinguished from another simply by this that one is ∼ of the other, it is plain that nothing is ∼ to itself. Let us first try making this ∼ a simple dyadic relation. If, then, of three individuals A, B, C, A is ∼ to B and B is ∼ to C, it must be that A is ∼ to C or else that C is ∼ to A. We do not see, at first, that there it matters which. Only there must be a general rule about it, because the whole idea of the system is the potential determination of individuals by means of entirely general characters. Suppose, first, that if A is ∼ to B and B is ∼ to C then in every case C is ∼ to A, and consequently A is not ∼ to C. Taken then any fourth individual, D, Either [see table on next page.]
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<th>A is $r$ to D</th>
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<td>If A is $r$ to D, since C is $r$ to A</td>
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<td>D is $r$ to C</td>
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<td>D is $r$ to B</td>
<td>Either B is $r$ to D</td>
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That rule, then, when you come to look into it will not work. The other rule that if $A$ is $r$ to $B$ and $B$ is $r$ to $C$ then $A$ is $r$ to $C$ leads to no contradiction, but it does lead to this, that there are two possible exceptional individuals[,] one that is $r$ to everything else and another to which everything else is $r$. This is like a limited line, where every point is $r$ that is, is to the right of every other or else that other is to the right of it. The generality of the case is destroyed by those two points of discontinuity, — the extremities. Thus, we see that no perfect continuum can be defined by a dyadic relation. But if we take instead a triadic relation, and say $A$ is $r$ to $B$ for $C$, say to fix our ideas that proceeding from $A$ in a particular way, say to the right, you reach $B$ before $C$, it is quite evident, that a continuum will result like a self-returning line

![Diagram of a self-returning line](image)

with no discontinuity whatever. All lines are simple rings and are topically precisely alike except that a line may have topical singularities. A topical singularity of a place is a place within that place from which the modes of departure are fewer or more than from the main collection of such places within the place. The topical singularities of lines are singular points. From an ordinary point on a line a particle can move two ways. Singular points are points from which a particle can move either no way, or in one way, or else in three ways or more. That is they are either, first, isolated points from which a particle cannot move in the line at all, or secondly, extremities, from which a particle can move but one way, or thirdly, f urcations,

![Diagram of a fork](image)

from which a particle can move in three or more ways. Those are the only topical distinctions there are among lines. Surfaces, or two dimensional continua, can also have singularities. These are either singular points or singular lines. The singular lines are either isolated lines, which
may have singular points at which they are not isolated, or they are bounding edges, or they are lines of which the surface splits into different sheets. These singular lines may themselves have singular points, which are subject to interesting laws. A student would find the singular lines of surfaces a good subject for a thesis. Isolated singular points of surfaces are either entirely detached from the surface or they are points at which different sheets or parts of the same sheet are tacked together. But aside from their singularities surfaces are of different kinds. In the first place, they are either perissid or artiad. A perissid surface is one which, although unbounded, does not enclose any space, that is, does not necessarily cut space into two regions, or what comes to the same thing, it has only one side. Such is the plane surface of geometrical optics, and in fact, such is every surface of odd order. The perissid surfaces are mathematically the simpler; but the artiad surfaces are the more familiar. A half twisted ribbon pasted together so that one side becomes continuous with the other side is an example of a bounded perissid surface. If you pass along a plane in geometrical optic, you finally come back to

the same point, only you are on the other side of the plane. An artiad surface, on the other hand, is for example the bounding surface between air and the stone of any finite stone, however, curiously it may be cut. Moreover, a surface may have a fornix or any number of fornice. A fornix is a part of the surface like a railway-tunnel which at once bridges over the interval between two parts of the surface, and so connects them, and at the same time, tunnels under that bridge so that a particle may move on the surface from one side of the bridge to the other without touching the bridge. A flat-iron handle, or any handle with two attachments has a surface which is a fornix of the whole surface of which it forms a part. Both perissid and artiad surfaces can equally have any number of fornice, without disturbing their artiad or perissid character.

If I were to attempt to tell you much about the different shapes which unbounded three dimensional spaces could take, I fear I might seem to talk gibberish to you, so different is your state of mental training and mine. Yet I must endeavor to make some things plain, or at least not leave them quite dark. Suppose that you were acquainted with no surface except the surface of the earth, and I were to endeavor to make
the shape of the surface of a double ring clear to you. I should say, you can imagine in the first place a disk with an outer boundary.

Then you can imagine that this has a hole or holes cut through it.

Then you can imagine a second disk just like this and imagine the two to be pasted together at all their edges, so that there are no longer any edges. Thus I should give you some glimmer of an idea of a double ring. Now I am going in a similar way to describe an unbounded three-dimensional space, having a different shape from the space we know. Begin if you please by imagining a closed cave bounded on all sides. In order not to complicate the subject with optical ideas which are not necessary, I will suppose that this cave is pitch dark. I will also suppose that you can swim about in the air regardless of gravity. I will suppose that you have learned this cave thoroughly; that you know it is pretty cool, but warmer in some places, you know just where, than others, and that the different parts have different odors by which they are known. I will suppose that these odors are those of neroli, portugal, limette, lemon, bergamot, and lemongrass,—all of them generically alike. I will further suppose that you feel floating in this cave two great balloons entirely separated from the walls and from each other, yet perfectly stationary. With the feeling of each of them and with its precise locality I suppose you to be familiarly acquainted. I will further suppose that you formerly inhabited a cave exactly like this one, except it was rather warm, that the distribution of temperature was entirely different, and that [the] odors in different localities in it with which you are equally familiar, were those of frankincense, benzoin, camphor, sandal-wood, cinnamon, and coffee, thus contrasting strongly with those of the other cave. I will further suppose the texture-feeling of the walls and of the
two balloons to be widely different in the two caves. Now, let us suppose that you, being as familiar with both caves as with your pocket, learn that works are in progress to open them into one another. At length, you are informed that the wall of one of the balloons has been reduced to a mere film which you can feel with your hand but through which you can pass. You being all this time in the cool cave swim up to that balloon and try it. You pass through it readily; only in doing so you feel a strange twist, such as you never have felt, and you find by feeling with your hand that you are just passing out through one of the corresponding balloons of the warm cave. You recognize the warmth of that cave[,] its perfume, and the texture of the walls. After you have passed backward and forward often enough to become familiar with the fact that the passage may be made through every part of the surface of the balloon, you are told that the other balloon is now in the same state. You try it and find it to be so, passing round and round in every way. Finally, you are told that the outer walls have been removed. You swim to where they were. You feel the queer twist and you find yourself in the other cave. You ascertain by trial that it is so with every part of the walls, the floor, and the roof. They do not exist any longer. There is no outer boundary at all.

Now all this is quite contrary to the geometry of our actual space. Yet it is not altogether inconceivable even sensuously. A man would accustom himself to it. On the mathematical side, the conception presents no particular difficulty. In fact mathematically our own shaped space is by no means the easiest to comprehend. That will give you an idea of what is meant by a space shaped differently from our space. The shape may be further complicated by supposing the two balloons to have the shape of anchor-rings and to be interlinked with one another.

After what I have said, you cannot have much difficulty in imagining that in passing through one of the balloons you have a choice of twisting yourself in either of two opposite ways, one way carrying you into the second cave and the other way into a third cave. That balloon surface is then a singular surface.

I will not attempt to carry you further into geometrical topic. You can readily understand that nothing but a rigidly exact logic of relations can be your guide in such a field. I will only mention that the real complications of the subject only begin to appear when continua of higher dimensionality than 3 are considered. For then first we begin to have systems of relations between the different dimensions.

A continuum may have any discrete multitude of dimensions what-
soever. If the multitude of dimensions surpasses all discrete multitudes there cease to be any distinct dimensions. I have not as yet obtained a logically distinct conception of such a continuum. Provisionally, I identify it with the uralt vague generality of the most abstract potentiality.

Listing, that somewhat obscure Göttingen professor of physics, whose name must forever be illustrious as that of the father of geometrical topic, — the only intrinsic science of Space, [—] invented a highly artificial method of thinking about continua which the great Riemann independently fell into in considering the connectivity of surfaces. But Riemann never studied it sufficiently to master it thoroughly. This method is not all that could be desired for continua of more than 3 dimensions, which Listing never studied. Even for our space, the method fails to throw much light on the theory of Knots; but it is highly useful in all cases, and is almost all that could be desired for tridimensional space. This method consists in the employment of a series of numbers which I proceed to define. By a figure, modern geometers do not mean what Euclid meant at all. We simply mean any place or places considered together. An indivisible place is a point. A movable thing which at any one instant occupied a point is a particle. The place which a particle can occupy during a lapse of time, one point at one instant and another at another instant, is a line. A movable thing which at any one instant occupies a line is a filament. The place which a filament can occupy in a lapse of time is a surface. A movable thing which at any one instant occupies a surface is a film. The place which a film can occupy in a lapse of time is a space, or as I would call it a triton. A movable body which at any one instant occupied a triton is a solid, or as I would call it a trion. Thus for higher dimensional places we have the tetrapon, the pentapon, etc. And for the movable things in them the tetron, penton, etc.

Now then for Listing’s numbers.

I call the first of them the Chorisis. He calls it simply the number of separate pieces. I give it a name to rhyme with his Cyclosis and Periphraxis; and with the analogous name I give it an analogous definition. Namely, the chorisis of a D-dimensional figure, is the number of simplest possible D-dimensional places that must be removed from it, in order to leave room for no particle at all. That is but a roundabout way of expressing the number of separate pieces. At first, I wished to define it as the number of simple D-dimensional places that must be removed in order to leave no room for a pair of particles which cannot move within the figure so as to coalesce. This would be one less than the number of pieces. But I subsequently surrendered to Listing’s own view.

The second Listing number he calls the Cyclosis. The cyclosis of
a D-dimensional figure is the number of simplest possible places of \( D - 1 \) dimensions which have to be cut away from the figure, to preclude the existence in the remaining place of a filament without topical singularities which cannot gradually move within that place so as to collapse to a particle. For example, this line

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has a cyclosis equal to 1. For you must cut it through to preclude a ring shaped filament which is incapable of collapsing by any gradual motion within the figure. On the other hand the surface of the blackboard has its cyclosis equal to zero, because any ringshaped filament in it has room gradually to shrink to a particle. It is true that when the ringshaped filament shrinks to a particle there is a breach of continuity at the last instant; but when we define cyclosis we except that final breach of continuity. A similar remark applies to all the other numbers. An annular surface bounded by two rings has a cyclosis equal to 1. For it has to be cut through on one line to prevent the existence of a ringshaped filament which cannot gradually shrink within the annular surface to a particle. The surface of an anchor-ring has a cyclosis equal to 2. For to preclude the existence of the noncollapsible filament it is necessary first to cut round the bar of the ring and after that to slit the bar along all its length. The space which the solid iron of the anchor-ring fills has a cyclosis equal to 1. For simply sawing it through in any plane is sufficient to preclude a noncollapsible filament. [A] spiral line having one end and winding in toward the centre at such a rate as to be infinitely long, has a cyclosis zero. For although it be infinitely long in measure, measure does not concern topical geometry. The line has an end at the centre, and its infinite windings will not prevent any filament in it from shrinking in it to a particle. The plane of imaginary quantity which the theory of functions studies has a cyclosis equal to zero. For take a straight line extending through the zero point and the point at infinity

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though the modulus of its point of greatest modulus is infinity, yet
measure does not concern topical geometry and it may continuously
contract to a circle and finally to the origin. But the plane of perspective
geometry is of an entirely different shape. Its cyclosis is 1. For consider
a ray or unlimited straight line. That ray cuts the ray at infinity. In the
plane of imaginary quantity there is no line at infinity, but only a point.
Here, there is a ray at infinity. Here it is

Here is the movable filament cutting it

That filament cuts that ray once and once only. But the ray at infinity
returns into itself as anybody would see who stood upon a boundless
desert plane and viewed his horizon round and round. That horizon
would be that ray. Now, disturb and move it as you will, the filament
will always cut that fixed ray in an odd number of points. Thus it can
ever cease to cut it. For if it did it would cut it in zero points and zero
is an even number. Thus we see that any ray, or any curve of odd order,
is the place of an unshrinkable filament. But every such line cuts every
other. Hence if the surface were separated along a single line of that
kind, no unshrinkable filament could any longer exist in it. Those of
you who are acquainted with non-Euclidean geometry might ask me
how it would be in hyperbolic space. I reply it is the same thing precisely.
Perspective geometry no more concerns itself with measure than does
Topical, or Intrinsic, geometry. It is true that according to the mode of
measurement used in hyperbolic geometry, all the parts of the plane at
finite distance are enclosed within one circle. The parts of the plane
outside the circle have then no existence, that is to say we are utterly cut
off from volitional reactions with them. But they are none the less real
on that account. You must recognize them, or fill your mind with sense-
less exceptions to all the laws of optics. Two rays continue to intersect
in one point, although that point may be outside our universe. Only it
now becomes a matter of indifference to us what the shape of the plane
may be, since whether it have a cyclosis zero or a cyclosis 1, the part of
the plane in our universe is a simple disk.
Listing's third number he calls the Periphraxis. The periphraxis of a figure of D dimensions is the number of simple places of D – 2 dimensions which must be taken away to prevent a non-singular film from gradually collapsing to a filament within the figure. Thus a cave with 2 balloons in it has a periphraxis equal to zero. For it is necessary to build out two barriers to preclude a sac from containing one balloon or the other, so that sac cannot collapse. The periphraxis of the space of perspective geometry is 1. But that of the geometry of quaternions is zero.

Listing's fourth number he calls the Immensity. It might be called the "fourth Listing." It is the number of simple places of D – 3 dimensions which must be taken away to preclude a non-singular noncollapsible solid, or trion. As Listing remarks, for all figures in our space this number is equal to zero excepting only for the entirety of space itself, for which it is 1.

For anybody who wishes to study this subject I will say that one of Listing's two papers is in the Göttinger Studien page 771, and my impression is that it is in the excellent library of Columbia University. The other memoir which relates to Listing's Census Theorem, which is mainly if not wholly an artificial theorem, — true indeed but yet a mere formality, or affair of book-keeping (although this memoir is most important and it is in it that the author develops his numbers), is in the Göttinger Abhandlungen. I think vol. VII. Both memoirs are full of interest and excessively easy to read. Listing himself makes the Cyclosis and Periphraxis of space equal to zero, showing how little he knew of mathematics.

I have occupied far too large a part of my hour with this matter and must now leave entirely untouched two methods of my own invention for treating problems upon which Listing's numbers fail to throw much light.

Every attempt to understand anything, — every research, — supposes, or at least hopes, that the very objects of study themselves are subject to a logic more or less identical with that which we employ.

That the logic of the universe is more rudimentary than our subjective logic is a hypothesis which may be worth examination in some stage of culture, but it is too violently at war with all the lessons which this age has learned for any man nowadays to embrace it with that ardor with which a man must embrace the theory which he is to devote his best powers to developing and bringing to the test of experience. Whatever else may be said for or against that hypothesis, that which we of
these times ought to try is rather the hypothesis that the logic of the universe is one to which our own aspires rather than attains.

Now continuity is shown by the logic of relatives to be nothing but a higher type of that which we know as generality. It is relational generality.

How then can a continuum have been derived? Has it for example been put together? Have the separated points become welded, or what?

Looking upon the course of logic as a whole we see that it proceeds from the question to the answer, — from the vague to the definite. And so likewise all the evolution we know of proceeds from the vague to the definite. The indeterminate future becomes the irrevocable past. In Spencer's phrase the undifferentiated differentiates itself. The homogeneous puts on heterogeneity. However it may be in special cases, then, we must suppose that as a rule the continuum has been derived from a more general continuum, a continuum of higher generality.

From this point of view we must suppose that the existing universe with all its arbitrary secondness is an offshoot from, or an arbitrary determination of, a world of ideas, a Platonic world; not that our superior logic has enabled us to reach up to a world of forms to which the real universe with its feeble logic was inadequate.

If this be correct, we cannot suppose the process of derivation, a process which extends from before time and from before logic, we cannot suppose that it began elsewhere than in the utter vagueness of completely undetermined and dimensionless potentiality.

The evolutionary process is, therefore, not a mere evolution of the existing universe, but rather a process by which the very Platonic forms themselves have become or are becoming developed.

We shall naturally suppose, of course, that existence is a stage of evolution. This existence is presumably but a special existence. We need not suppose that every form needs for its evolution to emerge into this world, but only that it needs to enter into some theatre of reactions, of which this is one.

The evolution of forms begins, or at any rate, has for an early stage of it, a vague potentiality; and that either is or is followed by a continuum of forms having a multitude of dimensions too great for the individual dimensions to be distinct. It must be by a contraction of the vagueness of that potentiality of everything in general but of nothing in particular that the world of forms comes about.

We can hardly but suppose that those sense-qualities that we now experience, colors, odors, sounds, feelings of every description, loves,
grievances, surprise, are but the relics of an ancient ruined continuum of qualities, like a few columns standing here and there in testimony that here some old-world forum with its basilica and temples had once made a magnificent ensemble. And just as that forum, before it was actually built, had had a vague under-existence in the mind of him who planned its construction, so too the cosmos of sense qualities which I would have you to suppose in some early stage of being was [as] real as your personal life is this minute, had in an antecedent stage of development a vaguer being, before the relations of its dimensions became definite and contracted.

The sense-quality is a feeling. Even if you say it is a slumbering feeling, that does not make it less intense; perhaps the reverse. For it is the absence of reaction, — of feeling another, — that constitutes slumber, not the absence of the immediate feeling that is all that it is in its immediacy. Imagine a magenta color. Now imagine that all the rest of your consciousness, memory, thought, everything except this feeling of magenta is utterly wiped out, and with that is erased all possibility of comparing the magenta with anything else or of estimating it as more or less bright. That is what you must think the pure sense quality to be. Such a definite potentiality can emerge from the indefinite potentiality only by its own vital Firstness, and spontaneity. Here is this magenta color. What originally made such a quality of feeling possible? Evidently nothing but itself. It is a First.

Yet we must not assume that the qualities arose separate and came into relation afterward. It was just the reverse. The general indefinite potentiality became limited and heterogeneous. Those who express the idea to themselves by saying that the Divine Creator determined so and so, may be incautiously clothing the idea in a garb that is open to criticism, but it is, after all, substantially the only philosophical answer to the problem. Namely, they represent the ideas as springing into a preliminary stage of being by their own inherent firstness. But so springing up, they do not spring up isolated; for if they did, nothing could unite them. They spring up in reaction upon one another, and thus into a kind of existence. This reaction and this existence these persons call the mind of God. I really think there is no objection to this except that it is wrapped up in figures of speech, instead of having the explicitness that we desire in science. For all you know of “minds” is from the actions of animals with brains or ganglia like yourselves or at furthest like a cockroach. To apply such a word to God is precisely like the old pictures which show him like an aged man leaning over to look out from above
a cloud. Considering the vague intention of it, as conceived by the non-
theological artist, it cannot be called false, but rather ludicrously figu-

rative.

In short, if we are going to regard the universe as a result of evolu-
tion at all, we must think that, not merely the existing universe, that
locus in the cosmos to which our reactions are limited, but the whole
Platonic world which in itself is equally real, is evolutionary in its origin,
too. And among the things so resulting are time and logic. The very first
and most fundamental element that we have to assume is a Freedom, or
Chance, or Spontaneity, by virtue of which the general vague nothing-
in-particular-ness that preceded the chaos took a thousand definite quali-
ties. The second element we have to assume is that there could be acci-
dental reactions between those qualities. The qualities themselves are
more eternal possibilities. But these reactions we must think of as events.
Not that Time was. But still, they had all the here-and-nowness of events.
I really do not see how the metaphysician can explain either of those
elements as results, further than this, that it may be said that the acciden-
tal reaction was, at first, one of the special determinations that came
about by pure spontaneity or chance.

Let me here say one word about Tychism, or the doctrine that
absolute chance is a factor of the universe. There is one class of objectors
to it who are so impressed with what they have read in popular books
about the triumphs of science, that they really imagine that science has
proved that the universe is regulated by law down to every detail. Such
men are theologians, perhaps, or perhaps they have been brought up
in surroundings where everything was so minutely regulated that they
have come to believe that every tendency that exists at all in Nature
must be carried to its furthest limit. Or, there is I know not what other
explanation of their state of mind; but I do know one thing; they cannot
be real students of physical science, — they cannot be chemists, for exam-
ple. They are wrong in their logic. But there is another class of objectors
for whom I have more respect. They are shocked at the atheism of
Lucretius and his great master. They do not perceive that that which
offends them is not the Firstness in the swerving atoms, because they
themselves are just as much advocates of Firstness as the ancient Atom-
ists were. But what they cannot accept is the attribution of this firstness
to things perfectly dead and material. Now I am quite with them there.
I think too that whatever is First is ipso facto sentient. If I make atoms
swerve, — as I do, — I make them swerve but very very little, because
I conceive they are not absolutely dead. And by that I do not mean exactly that I hold them to be physically such as the materialists hold them to be[,] only with a small dose of sentiency superadded. For that, I grant, would be feeble enough. But what I mean is, that all that there is is First, Feelings; Second, Efforts; Third, Habits; — all of which are more familiar to us on their psychical side than on their physical side; and that dead matter would be merely the final result of the complete induration of habit reducing the free play of feeling and the brute irrationality of effort to complete death. Now I would suppose that that result of evolution is not quite complete even in our beakers and crucibles. Thus, when I speak of chance, I only employ a mathematical term to express with accuracy the characteristics of freedom or spontaneity.

Permit me further to say that I object to having my metaphysical system as a whole called Tychism. For although tychism does enter into it, it only enters as subsidiary to that which is really, as I regard it, the characteristic of my doctrine, namely, that I chiefly insist upon continuity, or Thirdness, and in order to secure to thirdness its really commanding function, I [find it indispensable] that it is a third, and that Firstness, or chance, and Secondness, or Brute reaction, are other elements without the independence of which Thirdness would not have anything upon which to operate. Accordingly, I like to call my theory Synechism, because it rests on the study of continuity. I would not object to Tritism. And if anybody can prove that it is trite, that would delight me the chiefest degree.

All that I have been saying about the beginnings of creation seems mildly confused enough. Now let me give you such slight indication as brevity permits of the clue to which I trust to guide us through the maze.

Let the clean blackboard be a sort of Diagram of the original vague potentiality, or at any rate of some early stage of its determination. This is something more than a figure of speech; for after all continuity is generality. This blackboard is a continuum of two dimensions, while that which it stands for is a continuum of some indefinite multitude of dimensions. This blackboard is a continuum of possible points; while that is a continuum of possible dimensions of quality, or is a continuum of possible dimensions of a continuum of possible dimensions of quality or something of that sort. There are no points on this blackboard. There are no dimensions in that continuum. I draw a chalk line on the board. This discontinuity is one of those brute acts by which alone the original
vagueness could have made a step toward definiteness. There is a certain
element of continuity in this line. Where did this continuity come from?
It is nothing but the original continuity of the black board which makes
everything upon it continuous. What I have really drawn there is an
oval line. For this white chalk-mark is not a line, it is a plane figure in
Euclid sense, — a surface, and the only line [that] is there is the line
which forms the limit between the black surface and the white surface.
Thus discontinuity can only be produced upon that blackboard by the
reaction between two continuous surfaces into which it is separated, the
white surface and the black surface. The whiteness is a Firstness, — a
springing up of something new. But the boundary between the black
and white is neither black, nor white, nor neither, nor both. It is the
pairedness of the two. It is for the white the active Secondness of the
black; for the black the active Secondness of the white.

Now the clue that I mentioned consists in making our thought dia-
grammatic and mathematical, by treating generality from the point of
view of geometrical continuity, and by experimenting upon the di-
agram.

We see the original generality like the ovum of the universe seg-
mentated by this mark. However, the mark is a mere accident, and as
such may be erased. It will not interfere with another mark drawn in
quite another way. There need be no consistency between the two. But
no further progress beyond this can be made, until a mark will stay for
a little while; that is, until some beginning of a habit has been established
by virtue of which the accident acquires some incipient staying quality,
some tendency toward consistency.

This habit is a generalizing tendency, and as such a generalization,
and as such a general, and as such a continuum or continuity. It must
have its origin in the original continuity which is inherent in potentiality.
Continuity, as generality, is inherent in potentiality, which is essentially
general.

The whiteness or blackness, the Firstness, is essentially indifferent
as to continuity. It lends itself readily to generalization but is not in itself
general. The limit between the whiteness and blackness is essentially
discontinuous, or antigeneral. It is insistently this here. The original pot-
tentiality is essentially continuous, or general.

Once the line will stay a little after it is marked, another line may
be drawn beside it. Very soon our eye persuade us there is a new line,
the envelope of those others.
This rather prettily illustrates the logical process which we may suppose takes place in things, in which the generalizing tendency builds up new habits from chance occurrences. The new curve, although it is new in its distinctive character, yet derives its continuity from the continuity of the black board itself. The original potentiality is the Aristotelian matter or indeterminacy from which the universe is formed. The straight lines as they multiply themselves under the habit of being tangent to the envelope, gradually tend to lose their individuality. They become in a measure more and more obliterated and sink into mere adjuncts to the new cosmos in which they are individuals.

Many such reacting systems may spring up in the original continuum; and each of these may itself act as first line from which a larger system may be built in which it in turn will merge its individuality.

At the same time all this, be it remembered, is not of the order of the existing universe, but is merely a Platonic world, of which we are, therefore, to conceive that there are many, both coordinated and subordinated to one another; until finally out of one of these Platonic worlds is differentiated the particular actual universe of existence in which we happen to be.

There is, therefore, every reason in logic why this here universe should be replete with accidental characters, for each of which in its particularity there is no other reason than that it is one of the ways in which the original vague potentiality has happened to get differentiated.

But, for all that, it will be found that if we suppose the laws of nature to have been formed under the influence of a universal tendency of things to take habits, there are certain characters that those laws will necessarily possess.

As for attempting to set forth the series of deductions I have made upon this subject, that would be out of the question. All that I have any thought of doing is to illustrate, by a specimen or two, chosen among
those which need the least explanation, some of the methods by which such reasoning may be conducted.

Various continuua to which the inquirer's attention will be directed in the course of this investigation must be assumed to be devoid of all topical singularities. For any such singularity is a locus of discontinuity; and from the nature of the continuum there may be no room to suppose any such secondness. But now, a continuum which is without singularities must, in the first place, return into itself. Here is a remarkable consequence.

Take, for example, Time. It makes no difference what singularities you may see reason to impose upon this continuum. You may, for example, say that all evolution began at this instant, which you may call the infinite past, and comes to a close at that other instant, which you may call the infinite future. But all this is quite extrinsic to time itself. Let it be, if you please, that evolutionary time, our section of time, is contained between those limits. Nevertheless, it cannot be denied that time itself, unless it be discontinuous, as we have every reason to suppose it is not, stretches on beyond those limits, infinite though they be, returns into itself, and begins again. Your metaphysics must be shaped to accord with that.

Again, the lowest Listing number, the number of separate pieces, cannot be zero; for such a hypothesis would annul the whole continuum. Nor can the highest Listing number be zero, unless the continuum has singularities. But the intermediate Listing numbers may be zero or almost any numbers. If metaphysics is really to be made a definite science, and not child's play, the first inquiry concerning any general must be, first, what its dimensionality is, and secondly, what those intermediate Listing numbers are; and whatever your answer is, it will generally be found to lead you into those difficult but definite questions, out of which we are accustomed in inductive science to think that the true theory is pretty sure to grow. It is one of the great merits of the method of thought that the logic of relatives inculcates that it leads to such definite questions.

For example, take the continuum of all possible sense qualities after this has been so far restricted that the dimensions are distinct. This is a continuum in which firstness is the prevailing character. It is also highly primitive; and therefore we ought to suppose, till the contrary is proved, that the intermediate Listing numbers are all unity. For zero is distinctly a dualistic idea. It is mathematically A - A, i.e. the result of the inverse process of subtraction. Now an inverse process is a Second process. It is
true that there is another sort of zero which is a limit. Such is the vague zero of indeterminacy. But a limit involves Secondness prominently, and besides that, Thirdness. In fact, the generality of indeterminacy marks its Thirdness. Accordingly, zero being an idea of Secondness, we find, as we should expect, that any continuum whose intermediate Listing numbers are zero is equivalent to a pair of continua whose Listing numbers are 1. For instance, a perspective plane has a cyclosis equal to 1, while a ball has a cyclosis equal to 0. Now a ball is, topically speaking, of the same shape as two planes after the singularity of the pair has been removed. I will show you that this is true. Let the one plane be that of the blackboard, and let the other be oblique to it. Let this mark represent their ray of intersection. This ray is a singular line upon the two planes considered as one surface. In order to remove this singularity, we must split it down, so as to leave the right hand side of the blackboard plane, joined along the right hand parts of the split line to that part of the oblique plane that is in front, while the left hand part of the black board plane is joined along the left hand parts of the split line to the part of the oblique plane behind the black board. Thus this ray becomes two rays. But two rays intersect. So that a singular point still remains. We must, then, cut through that singular point, making two points of it; and leaving the right side of the black board plane joined to the forward part of the oblique plane and the left side joined to the other part. We now move apart those two hyperbolic branches that the two rays have made until they have made nearly a completed circuit of the plane. They no longer cut the ray at infinity, and we have an eggshaped solid which is topically just like a ball. Thus I have shown how secondness enters into the zero cyclosis.

It is the same with the other intermediate Listing numbers; and we must assume that all the Listing numbers of the continuum of sense-qualities are equal to 1. This is confirmed by carrying the evolution of the continuum and its definiteness a step further. Namely, we will now suppose that each quality has acquired a settled identity in all its different degrees, so that the continuum is ready for the application of measure-
ment. This measurement is a network figure imposed upon the blank continuum. It is true that it is in large measure arbitrary. It is our creation. Nevertheless, we shall adapt our creation as far as possible to the real properties of the continuum itself. Besides, there are certain modes of measurement which are impossible without breach of continuity in certain shapes of continua. For example, anybody can see that the same system of coördinates which could be applied to defining positions of points on a sphere, say latitude and longitude, would have to be modified in order to apply it to the definition of positions on an anchor-ring. On the sphere, longitude returns into itself after every 360°, and there are two points, the poles, whose longitudes are indeterminate; while latitude extends through 180° and then stops. But on the ring there will be one series of lines which will go round the bar of the ring without ever cutting one another and another series going round the hole of the ring without cutting one another. This is a much simpler system of measurement than any that is possible on the sphere. Now in the network figure of coördinates which conforms best to the properties of the continuum of pure quality, there is a line for each quality, along which line that quality only varies in intensity. All these lines come together at the absolute zero of quality. For in the zero of intensity quality is indistinguishable in its inmost nature. But those lines meet nowhere else. In the infinite degree qualities may dazzle our senses; but in themselves they are different. Hence, the continuum of quality is such that unlimited lines may cut one another an odd number of times, namely, once only. Now this would be impossible were the intermediate Listing numbers even, say zero. Our hypothesis that they are odd is therefore confirmed. I must add that the measurement of quality is evidently hyperbolic which weakens considerably the force of the last argument.

As another example consider the continuum of Space. In my lecture on the subject I pointed out to you how though it is a continuum, and therefore a Thirdness, the whole nature and function of space refers to Secondness. It is the theatre of the reactions of particles, and reaction is Secondness in its purity. For this and other reasons, which I omit for the sake of brevity, we must as our first reproduction assume that the intermediate Listing numbers for space are all zero. When we come to consider the principles of hydrodynamics we find that view confirmed. I cannot enter into details; but the motions of a frictionless incompressible fluid is as though it were composed of interpenetrating parts shot out in straight lines from sources and disappearing into sinks. But that implies that all the straight lines radiating from a single point
will meet again in another single point which supposes the Cyclosis and Periphraxis of Space to be zero. There will be some difficulties connected with this view, but I do not think them serious; and at any rate this will serve as another illustration of the manner in which reasoning about continuity can be applied to give real vitality to metaphysical reasoning, and to cure it of its deathly impotency.

I should have been glad if I could have set forth all this in greater detail; but that would have required mathematics. I should have liked to interest you in a number of my scientifically important and philosophically significant results which I have been obliged to leave altogether unmentioned. I wish I could also have expounded some theories of other thinkers which, although I cannot accept them, seem to me to be well worthy of the most careful consideration. But to treat a theory like this, the whole life of which lies in minute diagrammatic reasoning, in eight lectures was inevitably to make it seem excessively abstruse and, at the same time, to do no more than exhibit a fragment here and there selected as being comparatively easy of presentation. The subject of mathematical metaphysics, or Cosmology, is not so very difficult, provided it be properly expanded and displayed. It deeply concerns both physicist and psychist. The physicist ought to direct his attention to it, in order that he may be led to contemplate the intellectual side of his own science. Especially, the chemist, whose attention is forced to theory, needs above all to study the theory of theorizing. Psychologists have not yet dropped their excellent habit of studying philosophy; but I venture to think that they are not fully alive to all the value for their science of certain higher mathematics and to the virtues of mathematical thinking. The failure of Herbart's whose attempt was made before either Mathematics or Psychology was ripe for it, does not argue that no success can be attained in that line. I have presented, — or no, I have not presented anything in these lectures, but I have talked about the most abstract parts of Cosmology; but this subject embraces many topics which have not that character, such as the question of the present state of the evidences of the Conservation of Energy and the question of the nature of the influences which hold together the constituent elements of chemical compounds. In short there is a great variety of different ways in which Cosmology is both curious and useful for widely different classes of minds. We all know the kind of man who is warranted never to be interested in it, the man who lays out a system of ideas in his youth and stands on his platform with stalwart constancy like Casablanca on the burning deck. But if a mind is not absolutely argon and helium, but is
capable of being drawn by any means within an alien sphere of attraction, no study is more calculated to bring about that event than this. It is decidedly a difficult subject on which to break ground for oneself. Economy of time, avoidance of a terrible waste, requires the student to take counsel of the experience here of a mathematician, there of a logician, again of a physicist or chemist, and continually of a psychologist. It is, by the way, precisely in psychology, where you are the strongest that I have to confess myself the weakest. For that reason, in these lectures I have touched as little as possible upon psychology, preferring to deal with topics of Cosmology where I should be more at home, although you were less so. Crabbed and confused as all these circumstances have caused these conferences to become, you have been kind enough to listen to them, and really I dare not acknowledge, as it is in my heart to do the whole warmth of my thanks, for fear you might think it out of measure. But should it happen to any of you to select for his life's explorations a region very little trodden, he will, as a matter of course, have the pleasure of making a good many discoveries of more fundamental importance than at all remain to be made in any ground that has long been highly cultivated. But on the other hand, he will find that he has condemned himself to an isolation like that of Alexander Selkirk. He must be prepared for almost a lifetime of work with scarce one greeting, and I can assure him that if, as his day is sinking, a rare good fortune should bring a dozen men of real intellect, some men of great promise others of great achievement, together to listen to so much of what he has learned as his long habit of silence shall have left him the power of expressing in the compass of eight lectures, he will know then an almost untasted joy and will comprehend then what gratitude I feel at this moment.