MATH 635
PROOF OF EXCISION
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This is an expanded version of the proof of excision from class, using relatively few formulas. This is not meant to be entirely self-contained, but rather to be read in addition to Hatcher's more explicit argument. This has not been proofread, and likely has typos.

Our goal was to prove:

Proposition 1. *Let $\mathcal{U}$ be a collection of subsets of X whose interiors cover X, and let $C_*^{\mathcal{U}}(X)$ be the subcomplex of $C_*(X)$ generated by the singular simplices $\sigma\colon \Delta^n \to X$ so that $\sigma(\Delta^n)$ is contained in some $U_i \in \mathcal{U}$. Then the inclusion $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ is a chain homotopy equivalence.*

Recall that we defined the barycentric subdivision of a simplex as in Hatcher. We then defined a *barycentric subdivision operator* B_n on $C_n(X)$ by

$$B_n(\sigma) = \sum_{\delta^n} \pm\sigma|_{\delta^n}$$

where the sum is over the n-simplices of the barycentric subdivision of Δ^n. Implicitly, we are identifying each δ with the standard n-simplex. It's important we do this in a somewhat consistent way. A simplex δ corresponds to a sequence of faces of Δ^n, $f_0 \subsetneq f_1 \subsetneq \cdots \subsetneq f_n = \Delta^n$: the vertices of δ are the barycenters of these faces. Identify δ^n with the standard simplex by the unique linear homeomorphism which sends (the vertex corresponding to) f_0 to v_0, (the vertex corresponding to) f_1 to v_1, and so on.

Lemma 2. *There are choice of signs in the definition of B_n so that the B_n form a chain map, and B_0 is the identity.*

Proof. We prove this by induction on n. Suppose we have chosen signs so that for $i < n$, $\partial \circ B_i = B_{i-1} \circ \partial$. Let Y denote the barycentric subdivision of Δ^n. We distinguish two kinds of $(n-1)$-simplices in Y: *boundary facets*, which lie on ∂Y, and *internal facets*, which do not. More precisely, a facet of Y corresponds to a sequence $f_0 \subsetneq f_1 \subsetneq \cdots \subsetneq f_{n-1}$ of faces of Δ^n. A facet is internal if $f_{n-1} = \Delta^n$, the top-dimensional face, and is a boundary face if $\dim f_{n-1} = n-1$. There is a bijection between boundary facets of Y and n-simplices in Y: a boundary facet $f_0 \subsetneq f_1 \subsetneq \cdots \subsetneq f_{n-1}$ corresponds to $f_0 \subsetneq f_1 \subsetneq \cdots \subsetneq f_{n-1} \subsetneq \Delta^n$.

By construction, $B_{n-1}(\partial(\sigma))$ is the sum of the restrictions of σ to the boundary facets of Y, with some signs. So, there is a unique way to choose signs in the definition of B_n so that

$$\partial B_n(\sigma) = B_{n-1}(\partial\sigma) + \alpha$$

where α is a linear combination of (restrictions of σ to) internal facets. We will show that $\alpha = 0$, completing the proof. (Actually, there's a unique way to choose signs as long as σ is, say, injective; in general (e.g., if $X = \{pt\}$), there might be some unexpected cancellation.

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Here and in the rest of the proof, we'll assume we're making the universal choice, i.e., the one that doesn't depend on some extra cancellation from something specific about σ.) We have

$$0 = \partial(\partial(B_n(\sigma))) = \partial(B_{n-1}(\partial(\sigma))) + \partial(\alpha) = B_{n-1}(\partial(\partial(\sigma))) + \partial(\alpha) = \partial(\alpha),$$

so α is a cycle. However, each internal facet has a unique boundary facet on the boundary of Δ^n: given an internal facet $f_0 \subsetneq f_1 \subsetneq \cdots \subsetneq f_{n-1} = \Delta^n$, one of its boundaries is $f_0 \subsetneq f_1 \subsetneq \cdots \subsetneq f_{n-2}$. Moreover, each internal facet corresponds to a different codimension-2 face on the boundary of Δ^n. So, the fact that $\partial(\alpha) = 0$ implies that, in fact, $\alpha = 0$. (Really, the same caveat as just above applies.) $\square$

The construction of B does not really depend on X: it really happened inside the simplex Δ^n. This is made precise by two observations:

- Write $[\Delta^n]$ to mean the identity map of Δ^n, viewed as an element of $C_n(\Delta^n)$. Then for any singular simplex $\sigma\colon \Delta^n \to X$, $\sigma = \sigma_\#([\Delta^n])$.
- The operation B is natural in the sense that given a map $f\colon X \to Y$, $f_\# \circ B = B \circ f_\#$.

Both properties are immediate from our construction. Combined, they mean that $B(\sigma) = \sigma_\# B([\Delta^n])$, so B is determined by what it does to Δ^n.

One reason B helps is that repeating it enough times does subdivide a simplex. That is:

Lemma 3. *Given a singular n-simplex $\sigma\colon \Delta^n \to X$, there is an integer k so that $\overbrace{B \circ \cdots \circ B}^{k}(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$.*

Proof sketch. First, if we take repeated barycentric subdivisions of the n-simplex then the diameters of the sub-simplices go to zero uniformly. Now, consider the cover $\sigma^{-1}\mathcal{U}$ of Δ^n. By the Lebesgue number lemma, there is an ϵ so that any ball of radius ϵ is contained in some element of this cover. Choose k so that in the k^{th} barycentric subdivision of Δ^n, every simplex has diameter less than ϵ. Then $\overbrace{B \circ \cdots \circ B}^{k}(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$. $\square$

The other key ingredient we needed was a chain homotopy S between B and the identity map:

Lemma 4. *There is a chain homotopy S between B and the identity map. Moreover, we can choose S so that:*

(1) The chain homotopy S is natural in the sense that given $f\colon X \to Y$, $f_\# \circ S = S \circ f_\#$.
(2) Given a singular n-simplex $\sigma\colon \Delta^n \to X$, the image of $S(\sigma)$ agrees with the image of σ.

Proof. Write S_n^X for the map $S\colon C_n(X) \to C_{n+1}(X)$, so Point (1) reads more precisely

$$f_\# \circ S_n^X = S_n^Y \circ f_\#.$$

To satisfy Point (1), we must define $S_n^X(\sigma) = \sigma_\# S_n^{\Delta^n}([\Delta^n])$. Moreover, such an S_n will automatically satisfy Point (2).

We will construct the S_n inductively. Since $B_0 = \mathbb{I}$, we can take $S_0 = 0$. Now, suppose we have constructed S_i for $i < n$, so that Point (1) is satisfied and

$$\partial S_i + S_{i-1} \circ \partial = B_i - \mathbb{I}$$

for $i < n$. We first construct $S_n^{\Delta}([\Delta^n])$, so that

$$\partial(S_n^{\Delta^n}([\Delta^n])) = B_n([\Delta^n]) - [\Delta^n] - S_{n-1}^{\Delta^n}(\partial[\Delta^n]).$$

To do so, observe that

$$\begin{aligned}\partial\big(B_n([\Delta^n]) - [\Delta^n] - S_{n-1}^{\Delta^n}(\partial[\Delta^n])\big) &= \partial B_n([\Delta^n]) - \partial[\Delta^n] - \partial S_{n-1}^{\Delta^n}(\partial[\Delta^n]) \\ &= B_{n-1}(\partial[\Delta^n]) - \partial[\Delta^n] + S_{n-2}^{\Delta^n}(\partial^2[\Delta^n]) - B_{n-1}(\partial[\Delta^n]) + \partial[\Delta^n] \\ &= 0.\end{aligned}$$

But since $H_n(\Delta^n) = 0$, this implies that $B_n([\Delta^n]) - [\Delta^n] - S_{n-1}^{\Delta^n}(\partial[\Delta^n])$ is a boundary. So, we can define $S_n^{\Delta}([\Delta^n])$ to be an $n+1$-chain whose boundary is $B_n([\Delta^n]) - [\Delta^n] - S_{n-1}^{\Delta^n}(\partial[\Delta^n])$.

Now, define S_n^X using Point (1), i.e., by $S_n^X(\sigma) = \sigma_\# S_n^{\Delta^n}([\Delta^n])$. We claim that the result is a chain homotopy between B_n and $\mathbb{I}$, for any X. Indeed,

$$\begin{aligned}\partial S_n^X(\sigma) + S_{n-1}^X(\partial\sigma) &= \partial\sigma_\# S_n^{\Delta^n}([\Delta^n]) + \sigma_\# S_{n-1}^{[\Delta^n]}(\partial[\Delta^n]) \\ &= \sigma_\#\partial S_n^{\Delta^n}([\Delta^n]) + \sigma_\# S_{n-1}^{[\Delta^n]}(\partial[\Delta^n]) \\ &= \sigma_\#(B_n([\Delta^n]) - [\Delta^n]) = B_n(\sigma) - \sigma.\end{aligned}$$

(The first equality takes a little thought.) Finally, it is immediate from the definitions that Point (1) holds for any map $f\colon X \to Y$. $\square$

It's now easy to see that the inclusion $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ induces an isomorphism on homology. To see the map on homology is surjective, suppose α is a cycle in $C_n(X)$, representing some homology class $[\alpha]$. Then we can write α as a finite linear combination of singular n-simplices in X. So, by Lemma 3, there is a k so that $B^k(\alpha) \in C_n^{\mathcal{U}}(X)$. But by Lemma 4,

$$B(\alpha) - \alpha = \partial(S(\alpha)) + S(\partial(\alpha)) = \partial(S(\alpha)),$$

so $[B(\alpha)] = [\alpha]$ and, inductively, $[B^k(\alpha)] = [\alpha]$.

Similarly, to see that the map on homology is injective, suppose $\alpha \in C_n^{\mathcal{U}}(X)$ is the boundary of some $\beta \in C_{n+1}(X)$. Then

$$\alpha = \partial(\beta) = \partial(B(\beta) - S(\partial(\beta)) - \partial(S(\beta))) = \partial(B(\beta)) - \partial(S(\alpha)).$$

Since $S(\alpha) \in C_*^{\mathcal{U}}(X)$, α and $\alpha + \partial(S(\alpha)) = \partial(B(\beta))$ represent the same element of $H_n^{\mathcal{U}}(X)$, and this is the boundary of $B(\beta)$. Repeating this k times, α and $\partial(B^k(\beta))$ represent the same element of $H_n^{\mathcal{U}}(X)$. But for large enough k, $B^k(\beta) \in C_{n+1}^{\mathcal{U}}(X)$, so $[\partial(B^k(\beta))] = 0$ in $H_n^{\mathcal{U}}(X)$. So, α represents the trivial class, as desired.

Finally, to get that the map is a homotopy equivalence, we invoked:

Lemma 5. *If a chain map $f\colon C_* \to D_*$ of bounded-below chain complexes of free modules induces an isomorphism on homology then it is a chain homotopy equivalence.*

I promised a proof soon, and still do.

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