MATH 636 SPRING 2024
HOMEWORK 1
DUE APRIL 8, 2024

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Required problems:

(1) Use the Künneth theorem to compute the homology groups of the following spaces. Then use the universal coefficient theorem to compute their cohomology groups:
  (a) $S^1 \times M_g$, where M_g is the closed, orientable surface of genus g.
  (b) $K \times K$, where K is the Klein bottle.
  (c) $M(\mathbb{Z}/p\mathbb{Z}, m) \times M(\mathbb{Z}/q\mathbb{Z}, n)$ (a product of two Moore spaces), where p and q are (not necessarily distinct) primes.
  (d) $M(\mathbb{Z}/4\mathbb{Z}, 2) \times M(\mathbb{Z}/6\mathbb{Z}, 3)$.

(2) Hatcher 3.2.15 (p. 230). (You can skip the "and the spaces in the preceding three exercises" part.)

(3) Hatcher 3.B.3 (p. 280).

Optional problems:

Some good qual-level problems:

(4) Some good qual-level problems: Hatcher 3.2.16, 3.2.18, 3.B.1.

Some more problems to think about but not turn in:

(5) Let $F, G\colon C_*(X) \otimes C_*(Y) \to C_*(X) \otimes C_*(Y)$ be chain maps, for each pair of spaces X, Y, with the following properties:
  (a) F and G are natural, in the sense that given $f\colon X \to X'$, $g\colon Y \to Y'$,
$$F(f_*(\alpha) \otimes g_*(\beta)) = (f_* \otimes g_*)(F(\alpha \otimes \beta))$$
$$G(f_*(\alpha) \otimes g_*(\beta)) = (f_* \otimes g_*)(G(\alpha \otimes \beta)).$$
  (b) On $C_0(X) \otimes C_0(Y)$, F and G are the identity map.

Prove that F is chain homotopic to G. (This was one of the steps in our brief sketch of the Eilenberg-Zilber theorem.)

(6) By the Eilenberg-Zilber theorem, $C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$; let $E\colon C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ be a one of the chain homotopy equivalences constructed in the proof of the Eilenberg-Zilber theorem. There is an induced chain homotopy equivalence
$$E^T\colon C^*(X) \otimes C^*(Y) \to C^*(X \times Y).$$
Now, define a map $\cup\colon C^i(X) \otimes C^j(X) \to C^{i+j}(X)$ to be the composition
$$C^i(X) \otimes C^j(X) \xrightarrow{E^T} C^{i+j}(X \times X) \xrightarrow{\Delta^*} C^{i+j}(X),$$
where $\Delta\colon X \to X \times X$ is the diagonal map.
  (a) Show that $\cup$ induces a map $\cup\colon H^i(X) \otimes H^j(X) \to H^{i+j}(X)$.

(b) Show that the map $\cup$ on homology is independent of the choice of chain homotopy equivalence E.

(c) Show that the this cup product on cohomology is natural, unital, and associative. (Associativity probably takes a little work.)

(d) Show that this definition of the cup product agrees with the one given in Hatcher.

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