MATH 636 SPRING 2024
HOMEWORK 3
DUE APRIL 22, 2024

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Required problems:

(1) Hatcher 3.3.16 (p. 259).
(2) Hatcher 3.3.24 (p. 259).
(3) Hatcher 3.3.30 (p. 260).
(4) Hatcher 3.3.33 (p. 260).

Optional problems:

Some good qual-level problems:

- Hatcher 3.3.20, 3.3.21 (minus the parenthetical question), 3.3.25, 3.3.26.

Some more problems to think about but not turn in:

- Hatcher 3.3.22, 3.3.23, 3.3.27–29.
- Let X be a topological space. A *locally-finite singular n-chain* in X is a (possibly infinite) formal linear combination $\sum k_\sigma \sigma \in \prod_{\sigma\colon \Delta^n\to X}\mathbb{Z}$ so that for each compact set $C\subset X$,
$$\{\sigma \mid k_\sigma \neq 0 \text{ and } \sigma^{-1}(C)\neq\emptyset\}$$
is finite. Let $C_n^{BM}(X)$ denote the abelian group of locally-finite singular n-chains in X.
  (1) Show that the "obvious" boundary operator $\partial\colon C_n^{BM}(X)\to C_{n-1}^{BM}(X)$,
$$\partial\sum_\sigma k_\sigma\sigma = \sum_\sigma\sum_{i=0}^n (-1)^i k_\sigma \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$
is well-defined and makes $C_n^{BM}(X)$ into a chain complex. (*Hint.* This would *not* be true without the "locally-finite" condition, so your proof had better use it!)
  (2) The homology of (C_*^{BM},∂) is called the *Borel-Moore homology of M*, and denoted $H_*^{BM}(X)$. Observe that if X is compact then $H_*^{BM}(X)\cong H_*(X)$.
Compute $H_*^{BM}(\mathbb{R})$.
  (3) Show that if M is a connected, orientable n-manifold, not necessarily compact, then $H_n^{BM}(M)\cong\mathbb{Z}$. In particular, any oriented n-manifold M has a fundamental class $[M]\in H_n^{BM}(M)$.
  (4) Show that there is a well-defined cap product
$$H_{i+j}^{BM}(X)\otimes H^i(X)\to H_j^{BM}(X).$$
  (5) Imitate Hatcher's proof of Poincaré duality to show that for any oriented n-manifold M,
$$[M]\cap\cdot\colon H^i(X)\to H_{n-i}^{BM}(X)$$
is an isomorphism. In particular, this gives an alternative proof of Poincaré duality for compact manifolds.

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