MATH 636 SPRING 2024
HOMEWORK 4
DUE APRIL 29, 2024

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Required problems:

(1) Hatcher 4.1.3 (p. 358).
(2) Hatcher 4.1.4 (p. 358).
(3) Hatcher 4.1.11 (pp. 358–359).
(4) (a) Suppose that M is a closed, connected m-manifold. Suppose further that M is *triangulated*, i.e., is an m-dimensional simplicial complex. Suppose $p \in M$ is on a codimension-1 face (or facet) $\partial_i\sigma$ of some simplex σ. Show that p is on a codimension-1 face of exactly two simplices σ, σ'.
  (b) With notation as in the previous part, let $\alpha = \sum \sigma_i \in C_m(M; \mathbb{F}_2)$ be the sum of the m-simplices in M, viewed, via their characteristic maps, as maps $\sigma_i\colon \Delta^m \to M$. Show that α is a cycle.
  (c) Show that for any point p in the interior of some m-simplex σ_i, the image of α in $H_m(M, M \setminus \{p\}; \mathbb{F}_2)$ is a generator. Deduce that α is the generator of $H_m(M; \mathbb{F}_2) \cong \mathbb{F}_2$ so α is a (in fact, the) mod-2 fundamental class for M.
  (d) Now, suppose that $M^m \subset N^n$ is a closed submanifold of a manifold N, and that N is triangulated in such a way that $M \subset N$ is a subcomplex. Show that $i_*[M] \in H_m(N; \mathbb{F}_2)$, the homology class represented by M, is the sum of the m-simplices in N which are contained in M. (Hint: this is easy.)

Optional problems:

Some good qual-level problems:

- Hatcher 4.1.5, 4.1.8.
- In class, we showed that if $f\colon Y \to Z$ is a weak homotopy equivalence then for any CW complex X, $f_*\colon [X, Y] \to [X, Z]$ is injective. We also sketched a proof that f_* is surjective. Fill in the details of that proof.
- With notation as in the previous problem, fill in the proof that the map $f_*\colon [(X, x_0), (Y, y_0)] \to [(X, x_0), (Z, z_0)]$ of based homotopy classes of maps is bijective.

Some more problems to think about but not turn in:

- Hatcher 4.1.10.
- Extend Problem 4 to the case that instead of the manifold M being triangulated, M is an n-dimensional CW complex and $\alpha \in C_n^{\text{cell}}(M; \mathbb{F}_2)$ is the sum of the n-cells in M. (You'll have to follow α through the isomorphism between cellular and singular homology.)
- Extend Problem 4 to $\mathbb{Z}$-coefficients.

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