# MATH 636 SPRING 2024
# HOMEWORK 6
# DUE MAY 13, 2024

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**Required problems**:

(1) Hatcher 4.2.9 (p. 389). (For the second part, I found it easier to ignore Hatcher's hint.)
(2) Hatcher 4.2.12 (p. 389).
(3) Hatcher 4.2.16 (p. 389).
(4) Hatcher 4.2.18 (p. 389).

**Optional problems**:
Some good qual-level problems:

- Hatcher 4.2.2, 4.2.8, 4.2.13, 4.2.15 (p. 389).

Some more problems to think about but not turn in:

- Homology with local coefficients might be helpful context for understanding the orientation cover $M_R$ of a manifold $M$. Read Hatcher's Section 3.H and try problems 3.H.1 and 3.H.2.
- This problem assumes you know the basics of Čech cohomology of sheaves. A *good cover* of a space $X$ is a cover by open sets so that every finite intersection of sets in the cover is either empty or contractible. Given a bundle of groups $p\colon E \to X$, define a sheaf $\mathscr{F}$ with $\mathcal{F}(U)$ the group of sections of $E$ over $U$ (i.e., continuous maps $s\colon U \to E$ with $p \circ s = \mathbb{I}_U$). The goal of this exercise is to prove that if $X$ admits a finite good cover $\mathcal{U} = \{U_1, \dots, U_k\}$ then the local cohomology of $X$ with coefficients in $E$ is the Čech cohomology of $\mathscr{F}$. (The condition that the good cover be finite is not needed, but the condition that $X$ admits a good cover is.)
  (1) Prove that $\mathscr{F}$ is a sheaf.
  (2) Show that if $U$ is contractible then $E|_U$ is a trivial bundle of groups.
  (3) Generalize the proof of the Mayer-Vietoris sequence to show that for each $n$ there is an exact sequence

$$0 \to C^n_{\mathcal{U}}(X;E) \to \bigoplus_i C^n(U_i;E|_{U_i}) \to \bigoplus_{i<j} C^n(U_i \cap U_j; E|_{U_i\cap U_j}) \to \bigoplus_{i<j<\ell} C^n(U_i\cap U_j\cap U_\ell; E|_{U_i\cap U_j\cap U_\ell}) \to \cdots .$$

  (4) Show that the maps in the exact sequence above are chain maps, with respect to the differential $\delta\colon C^n_{\mathcal{U}}(X;E) \to C^{n+1}_{\mathcal{U}}(X;E)$ and $\delta\colon C^n(U_{i_1} \cap \cdots \cap U_{i_j}; E|_{U_{i_1}\cap\cdots\cap U_{i_j}}) \to C^{n+1}(U_{i_1} \cap \cdots \cap U_{i_j}; E|_{U_{i_1}\cap\cdots\cap U_{i_j}})$.
  (5) Now, view

$$\bigoplus_i C^n(U_i;E|_{U_i}) \to \bigoplus_{i<j} C^n(U_i \cap U_j; E|_{U_i\cap U_j}) \to \bigoplus_{i<j<\ell} C^n(U_i\cap U_j\cap U_\ell; E|_{U_i\cap U_j\cap U_\ell}) \to \cdots$$

as a bicomplex, where the horizontal differential is the maps in the exact sequence and the vertical differential is the differential on the singular cochain complex (with local coefficients). Let $D^*$ be the associated total complex. Show that there is a chain map $C^n_{\mathcal{U}}(X;E) \to D^*$ inducing an isomorphism on homology.

(6) Show that the homology of $C^*(U_{i_1} \cap \cdots \cap U_{i_j}; E|_{U_{i_1} \cap \cdots \cap U_{i_j}})$ (with respect to the differential on the singular cochain complex) is:
  - 0 if $U_{i_1} \cap \cdots \cap U_{i_j} = \emptyset$.
  - 0 in gradings $* > 0$,
  - $G$ in grading 0 if $U_{i_1} \cap \cdots \cap U_{i_j} \neq \emptyset$.

(7) Prove that the homology of $D^*$ is the Čech cohomology of the sheaf $\mathcal{F}$. Since this is isomorphic to $H^n(X;E)$, this proves the result.

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