MATH 692 SPRING 2024
HOMEWORK 1
DUE APRIL 27, 2024.

INSTRUCTOR: ROBERT LIPSHITZ

Solve any five of these problems. (Problems marked with stars are also available as mini-paper topics.)

(1) Prove that the group of self-homeomorphisms of D^2 which are the identity on the boundary (i.e., $Homeo^+(D^2, \partial D^2)$) is contractible. (This is the topological analogue of Smale's theorem.) (Hint: given a homeomorphism ϕ, consider applying a rescaled copy of ϕ to the disk of radius r inside D^2.)

(2) Prove that the smooth mapping class group of S^1 is trivial. What is the topology of $Diff^+(S^1)$?

(3) We proved that given non-separating simple closed curves $\alpha, \beta \subset \Sigma_g$ on the surface Σ_g of genus g, there is a diffeomorphism ϕ of Σ so that $\phi(\alpha) = \beta$. Suppose instead we have simple closed curves $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_k$ so that:
- α_i and α_j are disjoint for $i \neq j$,
- β_i and β_j are disjoint for $i \neq j$,
- $\Sigma \setminus (\alpha_1 \cup \cdots \cup \alpha_k)$ is connected, and
- $\Sigma \setminus (\beta_1 \cup \cdots \cup \beta_k)$ is connected.

Prove that there is a diffeomorphism ϕ so that $\phi(\alpha_i) = \beta_i$

(4) Let Σ be a surface with a single boundary component. We can consider the mapping class group $Mod(\Sigma)$ of diffeomorphisms of Σ or the mapping class group $Mod(\Sigma, \partial\Sigma)$ of diffeomorphisms which are the identity on the boundary. Prove there is an exact sequence

$$\mathbb{Z} \to Mod(\Sigma, \partial\Sigma) \to Mod(\Sigma) \to 1.$$

(The map $\mathbb{Z} \to Mod(\Sigma, \partial\Sigma)$ is, in fact, injective except when Σ is a disk.)

(5) Given a surface Σ and points $p_1, \dots, p_n \in \Sigma$ we can consider the group $Diff^+(\Sigma, \{p_1, \dots, p_n\})$ of diffeomorphisms of Σ taking $\{p_1, \dots, p_n\}$ to itself, and its group of path components $Mod(\Sigma, \{p_1, \dots, p_n\})$. (If Σ has boundary, we mean the maps which are the identity on the boundary.) Let $\mathrm{Conf}_n(\Sigma)$ denote the *configuration space of n points on Σ*, i.e., the space of n-tuples of disjoint points in Σ.

Suppose $\pi_1(Diff^+(\Sigma, \partial\Sigma)) = \{1\}$. Prove there is an exact sequence

$$1 \to \pi_1 \mathrm{Conf}_n(\Sigma) \to Mod(\Sigma, \{p_1, \dots, p_n\}) \to Mod(\Sigma, \partial\Sigma) \to 1.$$

(This is called the *Birman exact sequence*.) What does this say in the case $n = 1$?

(Hint: argue there is a fiber bundle $Diff^+(\Sigma, \{p_1, \dots, p_n\}) \to Diff^+(\Sigma, \partial\Sigma) \twoheadrightarrow \mathrm{Conf}_n(\Sigma)$.)

(6) Here are three equivalent definitions of the braid group B_n:

(a) The group of isotopy classes of smooth embeddings of n intervals into $[0,1] \times D^2$ so that the endpoints are $\{(0,0,0), (0,0,1/n), \dots, (0,0,(n-1)/n), (1,0,0), \dots, (1,0,(n-$

$1)/n)\}$ are the projection from each interval to $[0,1]$ is a diffeomorphism. The group operation is induced by concatenation in $[0,1]$ and rescaling.

(b) The group $\pi_1(\mathrm{Conf}_n(D^2, \{(0,0),(0,1/n),\dots,(0,(n-1)/n)\}))$ where Conf_n is as in the previous problem.

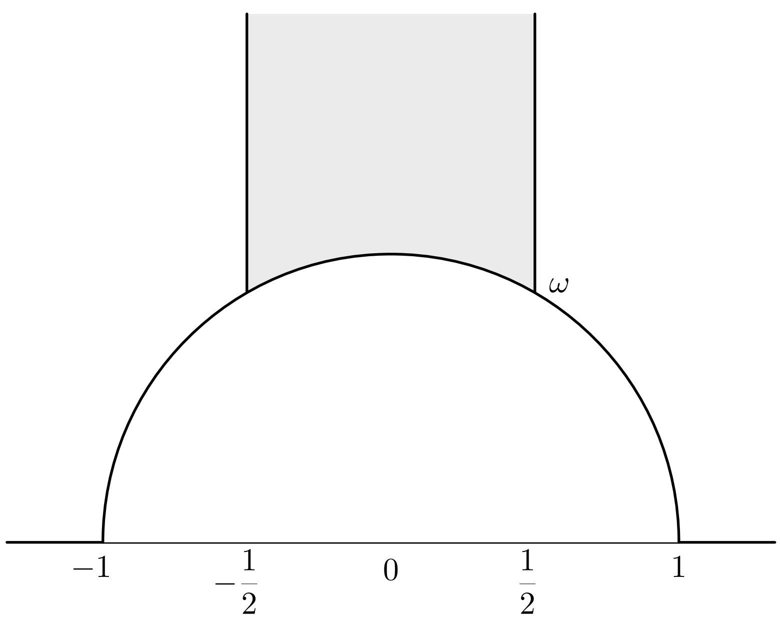
(c) The mapping class group $Mod(D^2, \{(0,0),\dots,(0,(n-1)/n)\})$, again with notation as in the previous problem.

Prove these definitions are equivalent.

(7) Prove that the mapping class group of $T^2 \setminus D^2$ (not fixing the boundary pointwise) is isomorphic to $SL(2,\mathbb{Z})$. (Hint: imitate our proof for T^2.) How is this consistent with Problem (5)?

(8) Use Macbeath's Theorem to find generators and relations for $\mathbb{Z}^2$, using the fact that $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by translations. (This is, of course, silly.)

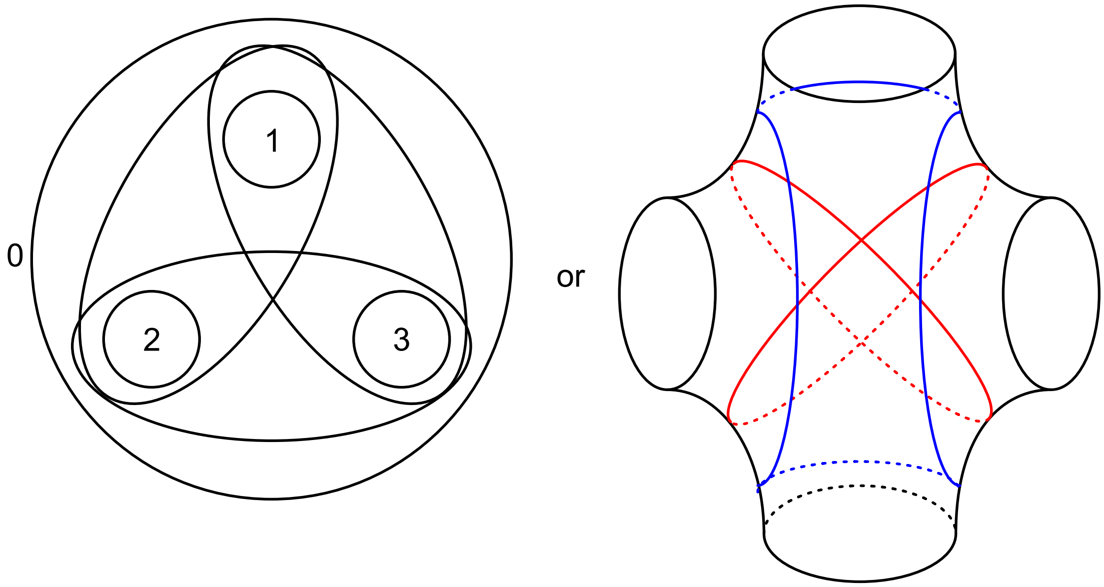
(9) The group $SL(2,\mathbb{Z})$ acts on the upper half plane by fractional linear transformations; you may have seen the following picture of the fundamental domain:



Use this action and Macbeath's Theorem to find generators and relations for $SL(2,\mathbb{Z})$.

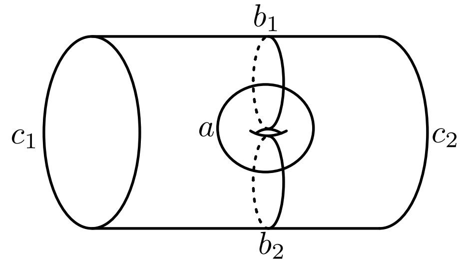
(10) The mapping class group Mod_g of Σ_g acts on $H_1(\Sigma_g)$. For $g = 1$ we saw that this action is faithful. Prove that for $g \geq 2$ it is not, by explicitly finding a nontrivial mapping class which acts trivially on $H_1(\Sigma)$. (The kernel of the action is called the *Torelli group*.)

(11) *Prove the lantern relation $\tau_0\tau_1\tau_2\tau_3 = \tau_{12}\tau_{13}\tau_{23}$, where the curves are the indicated circles

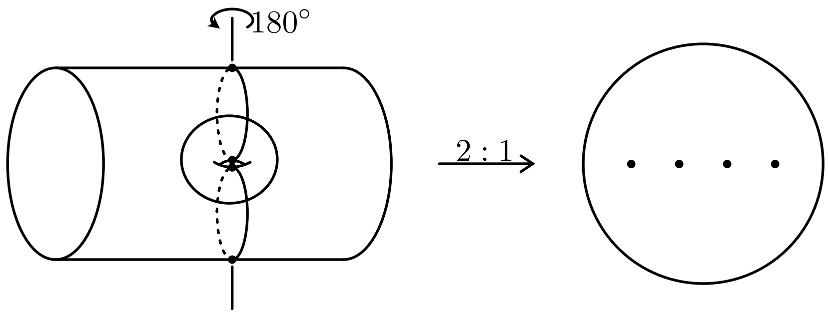


as follows. Choose a collection of arcs in the surface so that their complement is a disk. Verify that the images of those arcs under $\tau_0\tau_1\tau_2\tau_3$ and $\tau_{12}\tau_{13}\tau_{23}$ are isotopic. Then use Smale's theorem to conclude the result. (In principle, this strategy works to prove any relation in the mapping class group, though it may not be efficient or enlightening.)

(12) *In class, I asserted there is a relation $\tau_{c_1}\tau_{c_2} = (\tau_{b_1}\tau_a\tau_{b_2})^4$, where the curves a, b_1, b_2, c_1, and c_2 are as shown:

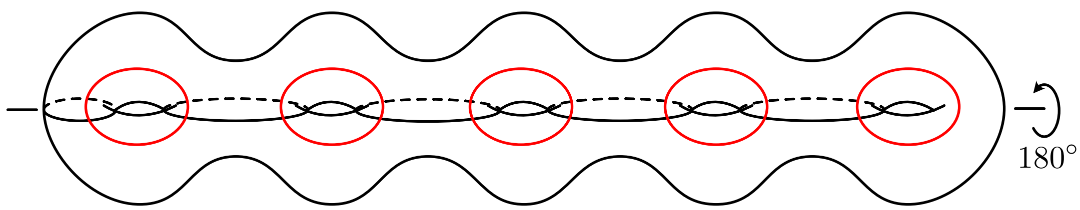


Prove this as follows. The surface shown is the double cover of a disk branched over four points; the covering involution is rotation by π around the skewer shown:



Given a 4-stranded braid, viewing that as a mapping class of the disk relative to the four branched points induces a mapping class of the branched double cover. Think about what braid $(\tau_{b_1}\tau_a\tau_{b_2})^4$ induces and use that to obtain the result.

(13) Prove that the Humphries generators minus one, i.e., the Dehn twists along the following curves,



do not generate the mapping class group, by showing that they all commute with the hyperelliptic involution (rotating the surface by π along the skewer shown) but not all mapping classes do.

(14) Prove that for a surface of genus $g \geq 3$ there are a pair of mapping classes ϕ, ψ conjugate to the hyperelliptic involution (see the previous problem) so that $\phi\psi$ has infinite order. (In particular, ϕ and ψ do not commute with each other.)

(15) Given a diffeomorphism $\phi\colon \Sigma \to \Sigma$, the *mapping torus* T_ϕ of ϕ is $[0,1] \times \Sigma/(1,p) \sim (0,\phi(p))$. Prove that if ϕ and ψ represent the same mapping class then the mapping tori of ϕ and ψ are diffeomorphic.

(16) Find a Heegaard splitting for T_ϕ and describe explicitly the gluing map between the two handlebodies, in terms of ϕ.

(17) For p and q relatively prime integers, the *lens space* $L(p,q)$ is the quotient of $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ by the action of $\mathbb{Z}/p$ generated by $(z,w) \mapsto (e^{2\pi i/p}z, e^{2\pi i q/p}w)$. Show that $L(p,q)$ admits a genus 1 Heegaard splitting and describe the gluing map explicitly.

(18) Give an explicit bijection between $Mod(T^2)/\left(\begin{smallmatrix}\pm 1 & 0\\ n & \pm 1\end{smallmatrix}\right)$ and $\mathbb{Q} \cup \{\infty\}$.

(19) Given a surface Σ and k pairwise disjoint, simple closed curves $\alpha_1, \dots, \alpha_k \subset \Sigma$, prove that $\Sigma \setminus (\alpha_1 \cup \cdots \cup \alpha_k)$ is connected if and only if $[\alpha_1], \dots, [\alpha_k] \in H_1(\Sigma)$ are linearly independent.

(20) Prove that every Heegaard splitting for a 3-manifold Y comes from some Morse function on Y.

(21) Given Morse functions $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$, prove that $h(x, y) = f(x) + g(y)$ is a Morse function on $M \times N$, and describe the corresponding handle decomposition of $M \times N$.

(22) Given Heegaard diagrams for Y_1 and Y_2, explain how to construct a Heegaard diagram for $Y_1 \# Y_2$.

(23) Explain how to compute the homology of Y from a Heegaard splitting of Y, both in terms of the gluing map and in terms of a Heegaard diagram.

(24) Show that for any g there is a 3-manifold Y so that any Heegaard splitting for Y has genus at least g.

(25) Let K be a knot (smoothly embedded circle) in S^3. Prove that there is a Heegaard splitting $S^3 = H \cup H'$ so that, with respect to an identification $H = (S^1 \times D^2) \natural \cdots \natural (S^1 \times D^2)$, K is identified with $S^1 \times \{0\}$. Find such an identification explicitly in the case that K is the figure-8 knot.

Email address: `lipshitz@uoregon.edu`