MATH 635 WINTER 2024
HOMEWORK 1
DUE JANUARY 19, 2024.

INSTRUCTOR: ROBERT LIPSHITZ

Required problems:

(1) Let M be the cokernel of the map $\mathbb{Z}^3 \to \mathbb{Z}^3$ given by the matrix

$$A = \begin{bmatrix} 2 & 4 & 10 \\ -2 & 2 & 2 \\ 8 & -2 & 4 \end{bmatrix}$$

(So, A is a presentation matrix for M.) Decompose M as $\mathbb{Z}^k \oplus \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_\ell)$.

(2) Consider the chain complex C_* given by

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\begin{bmatrix} 2 & -4 & 2 \end{bmatrix}} \mathbb{Z}^3 \xleftarrow{\begin{bmatrix} 10 & 36 \\ 4 & 15 \\ -2 & -6 \end{bmatrix}} \mathbb{Z}^2 \longleftarrow 0$$

where the left-most $\mathbb{Z}$ is in grading 0. Compute the homology of C_*.

(3) Identify the following abelian groups ($\mathbb{Z}$-modules):
   (a) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(3), \mathbb{Z}/(4))$
   (b) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(3), \mathbb{Z}/(9))$
   (c) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(3), \mathbb{Z}/(15))$
   (d) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(3), \mathbb{Q})$
   (e) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}} \mathbb{Z}/(4)$
   (f) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}} \mathbb{Z}/(9)$
   (g) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}} \mathbb{Z}/(15)$
   (h) $\mathbb{Z}/(3) \otimes_{\mathbb{Z}} \mathbb{Q}$

(4) Prove that the Hom and tensor product operations distribute with finite direct sums. That is

$$\mathrm{Hom}_R(M, N_1 \oplus N_2) \cong \mathrm{Hom}_R(M, N_1) \oplus \mathrm{Hom}_R(M, N_2)$$
$$\mathrm{Hom}_R(M_1 \oplus M_2, N) \cong \mathrm{Hom}_R(M_1, N) \oplus \mathrm{Hom}_R(M_2, N)$$
$$M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2).$$

Which of these statements are true for infinite direct sums? (You don't have to prove your answer to this part, though you should be able to.)

(5) Prove that if $f\colon R^n \to R^m$ is represented by the $m \times n$ matrix A then $f^T\colon R^m \cong \mathrm{Hom}_R(R^m, R) \to \mathrm{Hom}_R(R^n, R) \cong R^n$ is represented by A^T. (Here, the outer isomorphisms are the ones constructed in the previous problem, say.)

(6) Let R be a ring and S an R-algebra. Suppose $f\colon R^n \to R^m$ is given by the $m \times n$ matrix A. Describe as explicitly as you can the map $(f \otimes \mathbb{I})\colon R^n \otimes S \to R^m \otimes S$.

(7) Let C_* be the chain complex

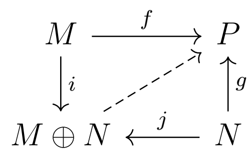
$$0 \longleftarrow \mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}} \mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 0 \\ 3 \end{bmatrix}} \mathbb{Z} \longleftarrow 0$$

where the left-most $\mathbb{Z}^2$ is in grading 0. Compute the (co)homology of:
(a) C_*.
(b) The dual cochain complex $\mathrm{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})$.
(c) The cochain complex $\mathrm{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z}/(2))$.
(d) The chain complex $C_* \otimes_{\mathbb{Z}} \mathbb{Z}/(2)$.
(e) The chain complex $C_* \otimes_{\mathbb{Z}} \mathbb{Q}$.
(f) The chain complex $C_* \otimes_{\mathbb{Z}} \mathbb{Z}/(3)$.

Optional problems:
Solve these if you have *not* seen this material before this class. (That is, if you don't know how to do it, do it.) You can turn them in or not, as you prefer.

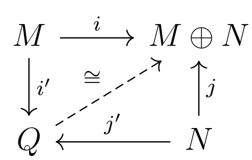
(8) Given an R-module M and a submodule N, recall that we defined M/N to be the abelian group M/N with R-action given by $r \cdot [m] = [r \cdot m]$. Prove that this action is well defined and does, in fact, make M/N into an R-module.

(9) Given R-modules M, N, and P and homomorphisms $f\colon M \to P$ and $g\colon N \to P$, prove that there is a unique homomorphism $M \oplus N \to P$ so that the following diagram commutes:



Here, the maps i and j are the inclusions $i(m) = (m, 0)$ and $j(n) = (0, n)$. (This is called the *universal property of coproducts*.)

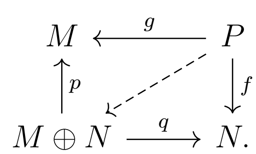
Moreover, prove that this property characterizes $M \oplus N$ up to (unique) isomorphism, in the sense that given any other module Q and maps $i'\colon M \to Q$ and $j'\colon N \to Q$ with the same property as $M \oplus N$, there is a (unique) isomorphism $M \oplus N \cong Q$ so that the diagram



commutes.

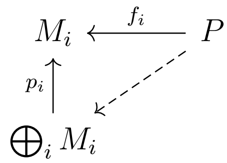
Generalize to direct sums of arbitrarily many modules.

(10) Given R-modules M and N and homomorphisms $f\colon P \to M$ and $g\colon P \to N$, prove that there is a unique homomorphism $P \to M \oplus N$ so that the following diagram commutes:



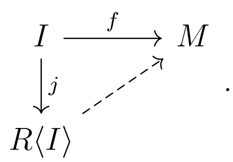
Here, the maps p and q are given by $p(m,n) = m$ and $q(m,n) = n$. Again, prove that this characterizes $M \oplus N$ up to (unique) isomorphism.

(This is the *universal property of products.*) Give an example showing, however, that this does not hold for infinite direct sums: it is not true that given R-modules M_i, $i \in \mathbb{N}$, and maps $f_i \colon P \to M_i$ there is necessarily a homomorphism $P \to \bigoplus_{i \in I} M_i$ so that for all i the diagram



commutes. (Hint: you can find a very easy example illustrating this.)

(11) Let $R\langle I\rangle$ denote the free module generated by a set I. Suppose M is any R-module, and $f \colon I \to M$ is a map of sets. Prove that there is a unique map of modules $R\langle I\rangle \to M$ so that the following diagram (of sets) commutes:



Here, j is the standard inclusion of I into $R\langle I\rangle$.

Moreover, prove that this property characterizes $R\langle I\rangle$ up to (unique) isomorphism.

Email address: `lipshitz@uoregon.edu`