MATH 635 WINTER 2024
HOMEWORK 2
DUE JANUARY 26, 2024

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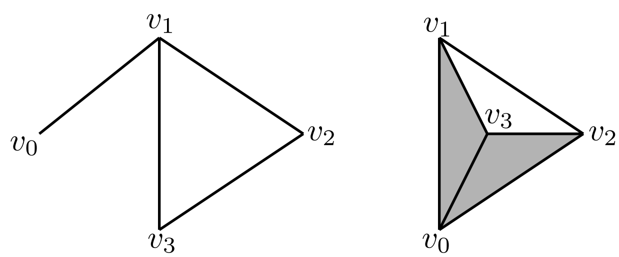
Required problems:

(1) Give abstract simplicial complexes whose geometric realizations are the following simplicial complexes. Compute the simplicial homology and cohomology groups of these complexes over $\mathbb{Z}$. (Note: the shaded triangles are 2-simplices.)



(a) (b)

(2) Find a simplicial complex homeomorphic to $\mathbb{R}P^2$.

(3) Suppose $X = \{X_n\}$ and $Y = \{Y_n\}$ are abstract simplicial complexes. A *simplicial map* $f\colon X \to Y$ is a map $f\colon X_0 \to Y_0$ so that if $\{v_1, \dots, v_n\} \in X_n$ then $\{f(v_0), \dots, f(v_n)\} \in Y_k$ (where k is the cardinality of $\{f(v_0), \dots, f(v_n)\}$). Given a simplicial map $f\colon X \to Y$, define a map $f_\#\colon C_n(X; \mathbb{Z}/(2)) \to C_n(Y; \mathbb{Z}/(2))$ by

$$f_\#(\{v_0, \dots, v_n\}) = \begin{cases} \{f(v_0), \dots, f(v_n)\} & |\{f(v_0), \dots, f(v_n)\}| = n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the maps $f_\#$ form a chain map, so induce a map f_* on simplicial homology and a map f^* on simplicial cohomology.

(Note: the fact that we are using $\mathbb{Z}/(2)$-coefficients means you do not have to keep track of signs. With an appropriate sign in the formula for $f_\#$, the result also holds over $\mathbb{Z}$, but the signs are a little tedious.)

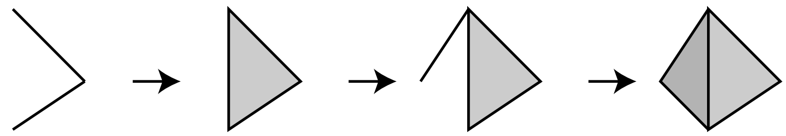
(4) Let $X_\bullet$ be an abstract simplicial complex and $v_0, \dots, v_n \in X_0$ be vertices of $X_\bullet$ and suppose that there is a $0 \le k \le n$ so that $\{v_0, \dots, \widehat{v_k}, \dots, v_n\} \notin X_{n-1}$ but $\{v_0, \dots, \widehat{v_i}, \dots, v_n\} \in X_{n-1}$ for all $i \neq k$. Then we can form a new simplicial complex $Y_\bullet$ with

(a) $Y_m = X_m$ for $m \notin \{n-1, n\}$,
(b) $Y_{n-1} = X_{n-1} \cup \{\{v_0, \dots, \widehat{v_k}, \dots, v_n\}\}$, and
(c) $Y_n = X_n \cup \{\{v_0, \dots, v_k, \dots, v_n\}\}$.

(In the special case $n = 1$, $Y_\bullet$ is obtained by adding a single new vertex w to X_0 and an edge from some vertex v_0 to w.) We say that $Y_\bullet$ is obtained from $X_\bullet$ by an

n-dimensional elementary expansion and $X_\bullet$ is obtained from $Y_\bullet$ by an *n-dimensional elementary collapse*. More generally, given abstract simplicial complexes $X_\bullet$ and $Y_\bullet$, we say that $X_\bullet$ is *simple homotopy equivalent* to $Y_\bullet$ if you can get from $X_\bullet$ to $Y_\bullet$ by a sequence of elementary expansions and elementary collapses of any dimension (and re-ordering vertices).

For example, here is a simple homotopy equivalence between two complexes:



(a) Sketch a proof that if $X_\bullet$ is simple homotopy equivalent to $Y_\bullet$ then their geometric realizations are homotopy equivalent. (The converse is false.)

(b) If $Y_\bullet$ is obtained from $X_\bullet$ is obtained from $X_\bullet$ by an elementary expansion, there is an inclusion map $i_m\colon C_m(X_\bullet;\mathbb{Z}/(2)) \hookrightarrow C_m(Y_\bullet;\mathbb{Z}/(2))$. Construct a quotient map $q_m\colon C_m(Y_\bullet;\mathbb{Z}/(2)) \to C_m(X_\bullet;\mathbb{Z}/(2))$ so that
   (i) the q_m form a chain map,
   (ii) $q_m \circ i_m = \mathbb{I}_{C_m(X_\bullet)}$ and
   (iii) there are maps $h_m\colon C_m(Y_\bullet) \to C_{m+1}(Y_\bullet)$ with
$$i_m \circ q_m - \mathbb{I}_{C_m(Y_\bullet)} = \partial \circ h_m + h_{m-1} \circ \partial.$$
   (Hint: for most simplices σ, you will probably define $h_m(\sigma) = 0$.)
   To save time, it's okay if you only consider the case of an n-dimensional elementary expansion for $n > 1$: the 1-dimensional case is similar, but maybe the notation is a bit different.
   (Again, the $\mathbb{Z}/(2)$-coefficients is so you don't have to keep track of signs; the result also holds over $\mathbb{Z}$.)

(c) Conclude that $H_m(Y_\bullet;\mathbb{Z}/(2)) \cong H_m(X_\bullet;\mathbb{Z}/(2))$ for each m and, consequently, that simple homotopy equivalent simplicial complexes have isomorphic homology groups.

(5) Show that if X is path connected then its singular cohomology satisfies $H^0(X) \cong \mathbb{Z}$.

(6) Let $\gamma_1, \gamma_2\colon S^1 \to X$ be homotopic loops in X. Regarding S^1 as the quotient space $\Delta^1/(\partial\Delta^1)$, we can think of each γ_i as a singular 1-chain in X. Prove that these 1-chains are cycles, so induce elements of $H_1(X)$, and moreover $[\gamma_1] = [\gamma_2] \in H_1(X)$.

Optional problems:
Think about these, but you don't have to turn them in.

(7) Let X be a simplicial complex and $A \subset X$ a subcomplex.
   (a) Define the relatively *simplicial* homology and cohomology groups $H_n^{\text{simp}}(X,A)$ and $H^n_{\text{simp}}(X,A)$.
   (b) Compute the simplicial homology and cohomology groups for the pair (X,A) where X is a 2-simplex (viewed as a simplicial complex with 3 0-simplices, 3 1-simplices, and 1 2-simplex) and A is:
      (i) A single vertex.
      (ii) Two vertices.
      (iii) An edge (and the two vertices at its ends).

(iv) The entire boundary of the 2-simplex.

(8) Give a simplicial complex homeomorphic to the n-sphere and compute its simplicial homology and cohomology groups over $\mathbb{Z}$.

(9) Give a simplicial complex homeomorphic to $\mathbb{R}P^3$ and compute its simplicial homology and cohomology.

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