# MATH 635 WINTER 2024
# HOMEWORK 3
# DUE FEBRUARY 5, 2024

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**Required problems**:

(1) Let $C_*$ and $D_*$ be chain complexes of $R$-modules, and $M$ any $R$-module. Prove that if $f, g\colon C_* \to D_*$ are chain homotopic chain maps then the induced maps $f^T, g^T\colon \mathrm{Hom}(D_*, M) \to \mathrm{Hom}(C_*, M)$ are chain homotopic.

(2) (Homotopy invariance of cohomology) Prove that if $f, g\colon X \to Y$ are homotopic maps then $f^* = g^*\colon H^n(Y) \to H^n(X)$. Conclude that if $X \simeq Y$ then $H^n(X) \cong H^n(Y)$ for each $n$.

(3) (Excision for cohomology) Recall that for $A \subset X$,

$$C^n(X, A) = \{c \in C^n(X) \mid c(\sigma) = 0 \text{ for all } \sigma\colon \Delta^n \to A\}.$$

   (a) Prove that $C^*(X, A)$ is a subcomplex of $C^*(X)$. So, we can define $H^n(X, A) = \ker(\delta|_{C^n(X,A)})/\mathrm{Im}(\delta|_{C^{n-1}(X,A)})$.

   (b) Suppose $Z \subset A \subset X$ with the closure of $Z$ contained in the interior of $A$. Prove that $H^n(X, A) \cong H^n(X \setminus Z, A \setminus Z)$. (This isomorphism is induced by the inclusion map $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$, though you don't have to prove that if you don't want to.)

(4) Hatcher 2.1.17 (p. 132).

(5) Hatcher 2.1.20 (p. 132). (The first half of this is an important observation.)

(6) Hatcher 2.1.27 (p. 133).

(7) Imitate our computation of $H_*(S^n)$ to compute the singular cohomology groups $H^*(S^n)$.

**Optional problems**:

Some homological algebra. This is optional for now. Next week I will do some of this in class, and the rest will probably be non-optional homework.

(1) Given a chain map $f\colon C_* \to D_*$, the *mapping cone* of $f$ is the chain complex $\mathrm{Cone}(f)$ with $\mathrm{Cone}(f)_n = C_{n-1} \oplus D_n$ and differential $\partial(c, d) = (-\partial(c), f(c) + \partial(d))$.

   (a) Suppose $X$ and $Y$ are simplicial complexes and $f\colon X \to Y$ is a simplicial map. Explain how the mapping cone of $f$, $\mathrm{Cone}(f)$, is a simplicial complex and show that the simplicial chain complex of $\mathrm{Cone}(f)$ is isomorphic to $\mathrm{Cone}(f_\#\colon C_*^{simp}(X) \to C_*^{simp}(X))$.

   (b) Let $C_*$ and $D_*$ be chain complexes, and $f$ a collection of homomorphisms $f_n\colon C_n \to D_n$. Even if $f_n$ is not a chain map, we can still form $\mathrm{Cone}(f)$ as above. Show that the differential on $\mathrm{Cone}(f)$ satisfies $\partial^2 = 0$ if and only if $f$ is a chain map.

   (c) Show that the chain map $f\colon C_* \to D_*$ induces an isomorphism on homology if and only if $\mathrm{Cone}(f)$ is acyclic (i.e., has trivial homology). (Hint: there is an obvious short exact sequence involving $C_*$, $D_*$, and $\mathrm{Cone}(f)$.)

(2) Let $C_*$ and $D_*$ be a chain complexes with $C_n = D_n = 0$ for $n < 0$, $C_n$ free for all $n$, and $H_*(D) = 0$. Let $f\colon C_* \to D_*$ be a chain map. Prove that $f$ is nullhomotopic. (Hint: construct a nullhomotopy $h$ of $f$ inductively, starting with $h_0\colon C_0 \to D_1$.)

(3) Let $C_*$ and $D_*$ be chain complexes with $C_n = D_n = 0$ for $n < 0$. Assume further that each $D_n$ is a free $R$-module. Let $f\colon C_* \to D_*$ be a chain map so that the induced map $f_*\colon H_*(C) \to H_*(D)$ on homology is an isomorphism. Prove that there is a map $g\colon D_* \to P_*$ so that $g \circ f$ is chain homotopic to the identity map. (Hint: consider $\mathrm{Cone}(f)$. By the previous problem, the canonical map $D_* \hookrightarrow \mathrm{Cone}(f)$ is nullhomotopic. A nullhomotopy is a pair of maps $\alpha\colon D_* \to C_*$, $\beta\colon D_* \to D_*$. Unpack the fact that $(\alpha, \beta)$ is a chain map to obtain the result.)

(4) Let $C_*$ and $D_*$ be chain complexes over $R$ so that $C_n = D_n = 0$ for $n < 0$ and $C_n$ and $D_n$ are free $R$-modules for each $n$. Suppose $f\colon C_* \to D_*$ is a chain map which induces an isomorphism on homology. Prove that $f$ is a chain homotopy equivalence. (Hint: this should be easy from the previous problem.)

(5) Let $C_*$ and $D_*$ be chain complexes over $R$ so that $C_n = D_n = 0$ for $n < 0$, each $C_n$ is free, and $H_n(D) = 0$ for $n > 0$. Prove that if $f, g\colon C_* \to D_*$ induce the same map $H_0(C) \to H_0(D)$ then they are chain homotopic. (Hint: let $M = H_0(D)$. Consider the complex $0 \leftarrow M \leftarrow D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow \cdots$. Observe that $f - g$ induces a map from $C_*$ to this extended version of $D$ and use Problem 2.)

Some other optional problems; think about these, but you don't have to turn them in.

(6) Hatcher 2.1.26 (p. 133).

(7) Hatcher 2.1.29 (p. 133).

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