# Math 251
# Final Exam - Winter 2013 - Answers

(1) (12 pts) Compute the following limits. In some cases the correct answer may be $\infty, -\infty$ or the limit may not exist.
You must justify your answer: no credit will be given if you just state the limit.

(a) $\lim_{x\to-\infty} \dfrac{2-7x+2x^2-4x^4}{5x^4+2x+1}$
**Solution:** We divide by $x^4$ which means we need to take the limit of

$$\frac{\frac{2}{x^4}-\frac{7}{x^3}+\frac{2}{x^2}-4}{5+\frac{2}{x^3}+\frac{1}{x^4}}$$

In the limit all terms are zero except the $-4$ and the 5, so the answer is $-4/5$.

(b) $\lim_{x\to\infty} \dfrac{x^2}{e^x}$
**Solution:** If we apply l'Hospital's rule twice, we get the limit of $2/e^x$ which is zero since the denominator approaches $+\infty$ and the numerator is constant.

(c) $\lim_{u\to 0} \dfrac{u^2}{\cos u}$
**Solution:** The limit is 0 since the numerator approaches 0 and the denominator approaches 1.

(d) $\lim_{t\to 1} \dfrac{5^t-5}{t-1}$
**Solution:** Using l'Hospital's rule,

$$\lim_{t\to 1}\frac{5^t-5}{t-1} = \lim_{t\to 1}\frac{5^t\cdot\ln(5)}{1} = 5\ln(5).$$

(2) (12 pts) Find the derivative of each function.

(a) $f(x) = \dfrac{\cos(x)}{\sqrt{x^2+1}}$
**Solution:**

$$\frac{-\sin(x)\sqrt{x^2+1}-\cos(x)\frac{1}{2}(x^2+1)^{-1/2}2x}{x^2+1}$$

(b) $g(x) = \ln(b(x)-x)$
Here $b(x)$ satisfies $b'(x) = -5b(x)+1$ and your answer may involve the function $b(x)$.
**Solution:**

$$g'(x) = \frac{b'(x)-1}{b(x)-x} = \frac{-5b(x)+1-1}{b(x)-x} = \frac{-5b(x)}{b(x)-x}.$$

(c) $h(x) = \arctan(kx-1)$
($k$ is constant.)
**Solution:**

$$h'(x) = \frac{1}{(kx-1)^2+1}\cdot k$$

(d) $A(u) = \dfrac{4}{3}\pi(u^2-6)^2+\pi^2$
**Solution:**

$$A'(u) - \frac{4}{3}\pi 2(u^2-6)\cdot 2u + 0 = \frac{16}{3}\pi u(u^2-6)$$

(3) (6 pts) Let $y = x^x$. Using logarithmic differentiation, find $\dfrac{dy}{dx}$ in terms of $x$.
**Solution:** Applying ln we get

$$\ln(y) = x\ln(x).$$

Taking derivatives with respect to $x$, we get

$$\frac{y'}{y} = \ln(x) + \frac{x}{x} = \ln(x) + 1.$$

Multiplying by $y$ we get

$$y' = y(\ln(x) + 1) = x^x(\ln(x) + 1).$$

Note that this can also be solved by implicit differentiation using the fact that $x^x = e^{x\ln(x)}$.

(4) (7 pts) A ballon is rising from the ground. When it is 30 meters above the ground, it is rising at a rate of 3 meters per second. An observer is on the ground 40 meters from the point where the balloon originated.

At what rate is the balloon moving away from the observer at this moment when the balloon is 30 meters above the ground? Include units in your answer.

**Solution:** Let $h(t)$ be the height. The distance to the balloon is

$$d(t) = \sqrt{h^2(t) + 40^2}$$

(in meters). So

$$d'(t) = \frac{1}{2}(h^2(t) + 40^2)^{-1/2} 2h(t)h'(t)$$

in meters per second. We evaluate this when $h(t) = 30$ meters and $h'(t) = 3\frac{m}{s}$. We get

$$\frac{1}{2}\frac{1}{50} 2 \cdot 30 \cdot 3\frac{m}{s} = \frac{9}{5}\frac{m}{s}.$$

(5) (6 pts) Find the equation of the tangent line to the curve $2x^2 + y^3 = 3$ at the point $(1, 1)$.

**Solution:** We differentiate implicitly and get

$$4x + 3y^2y' = 0.$$

Using $(x, y) = (1, 1)$ we get $4 + 3y' = 0$. This tells us that $y' = -4/3$ at the point $(1, 1)$. We know have a point and a slope, so the line is given by

$$y - 1 = -\frac{4}{3}(x - 1).$$

(6) (8 pts) A farmer plans to build a rectangular pasture next to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions require the least amount of fencing if no fencing is needed along the river?

**Solution:** Let $v$ be the length of the side parallel to the river and $u$ be the length of the side perpendicular to the river. Then the total amount of fencing is

$$F = v + 2u. \text{ The relationship between } v \text{ and } u \text{ is } vu = 180,000.$$

So using the second equation to solve for $v$ in terms of $u$, we get $F = \frac{180,000}{u} + 2u$, and we'd like to find the $u$ that minimizes this. Clearly $u > 0$ for this to make sense, so we only analyze the situation $u \in (0, \infty)$.

$$F'(u) = -\frac{180,000}{u^2} + 2$$

This is 0 only when $2u^2 = 180,000$, or equivalently $u^2 = 90,000$. This corresponds to $u = 300$ (remember we are only considering $u > 0$). It is negative for $0 < u < 300$ and positive for $u > 300$. So the function decreases to a minimum at $u = 300$ and increases after that point. So the minimum value occurs at $u = 300$, $v = 600$. This corresponds to 1200 meters of fencing.

(7) (10 pts) Consider the function $f(x) = x^5 - 5x + 1$.

(a) Find the intervals on which $f$ is increasing or decreasing.

**Solution:** $f'(x) = 5x^4 - 5$. This is positive when $x^4 > 1$ and negative when $x^4 < 1$. So the function is increasing on $(-\infty, -1)$ and on $(1, \infty)$ but decreasing on $(-1, 1)$.

(b) Find the local maximum and minimum values of $f$.

**Solution:** Since the function is differentiable everywhere, the local extrema occur at critical points. In this case that is at $x = -1$ and $x = 1$. At $-1$ there is a local maximum since the function is increasing before $-1$ and decreasing on $(-1, 1)$. At 1 there is a local minimum since the function is decreasing on $(-1, 1)$ and increasing on $(1, \infty)$.

(c) How many solutions does $x^5 - 5x + 1 = 0$ have? Explain why your answer is correct.
**Solution:** On the interval $(-\infty, -1)$ the function is increasing from $-\infty$ to $f(-1) = 5$. So the function crosses the $x$-axis on that interval by the IVT. But only once because the function is increasing on the interval.
The function is decreasing on $(-1, 1)$ and $f(1) = -3$, so again the function crosses the $x$-axis once on that interval by the IVT and only once since it is decreasing along that interval.
On $[]1, \infty)$ the function is increasing from $-3$ to $\infty$, so the function crosses the $x$-axis once by the IVT, and only once since the function is increasing.
So the equation has three solutions, once in each interval.

(8) (9 pts) An ant is walking along the edge of a ruler. Its position in cm along the ruler at time $t = 1$ to $t = 4$ is given by

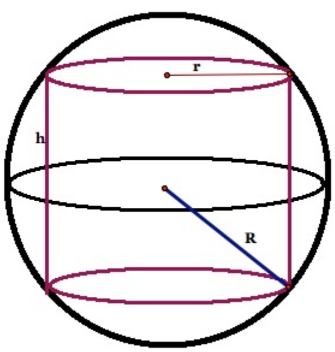
$$s(t) = t^3 - 6t^2 + 9t + 1$$

(a) At what moments of time is the ant at rest?
**Solution:** When $s'(t) = 0$. This is when $3t^2 - 12t + 9 = 0$. Dividing this equation by 3 gives $t^2 - 4t + 3 = 0$. This is true at $t = 1$ and $t = 3$ since the quadratic factors as $(t - 1)(t - 3)$.

(b) When is the ant the furthest along the ruler?
**Solution:** This occurs at the maximum of $s(t)$ on the interval $[1, 4]$. Since this is a differentiable function on a closed interval, the maximum occurs at an end point or a critical point. So we need to check $t = 1, 3, 4$. $s(1) = 5$. $s(3) = 1$. $s(4) - 64 - 6 \cdot 16 + 9 \cdot 4 + 1 = 5$.
So the ant is furthest along the ruler at $t = 1$ and $t = 5$.

(c) What is the total distance the ant walks between $t = 1$ and $t = 4$?
**Solution:** From $t = 1$ to $t = 3$ the ant walks from 5 back to 1. From $t = 3$ to $t = 4$, the ant walks from 1 back to 5. So the total distance walked is 8 cm.

(9) (12 pts) A cylinder is inscribed in a sphere of radius $R = 10$. Find the largest possible volume of such a cylinder.




**Solution:** Let $h$ be the height of the cylinder. Note that $h \in [0, 20]$. $h$ is related to $r$ by $r^2 + (h/2)^2 = R^2 = 100$. Since $r$ is also non-negative, $r = \sqrt{100 - (h/2)^2}$.
The volume of the cylinder is

$$V = \pi r^2 h = \pi(100 - (h/2)^2)h = \pi(100h - h^3/4).$$

$V$ is maximized at an endpoint of $[0, 20]$ or at a critical point.

$$V'(h) = \pi(100 - 3h^2/4)$$

so we get critical points when $h^2 = 400/3$, which is when $h = 20/\sqrt{3}$. Since $V(0) = V(20) = 0$, the maximum volume is at the critical point.

$$V(20/\sqrt{3}) = \pi(100 - 100/3)20/\sqrt{3} = 4000/(3\sqrt{3}).$$

(10) (6 pts) Find the absolute maximum and minimum of $f(t) = 2t^3 - 9t^2 + 12t - 1$ on $[1, 3]$.

**Solution:** Since this is a differentiable function on a closed interval, the max and min are at endpoints or at critical points. So we start by finding the critical points:

$$f'(t) = 6t^2 - 18t + 12. \text{ This is 0 when } t^2 - 3t + 2 = 0.$$

By factoring, this is when $t = 1$ or $t = 2$. So we need to check $t = 1, 2, 3$.

$f(1) = 4$. $f(2) = 3$. $f(3) = 2 \cdot 27 - 9 \cdot 9 + 12 \cdot 3 - 1 = 8$.

So the absolute minimum is 3 and the absolute maximum is 8.

(11) (12 pts) Consider the function

$$f(x) = \frac{x^2 - 2x + 4}{x - 2}.$$

Sketch the graph $y = f(x)$ by finding all asymptotes, intervals of increasing and decreasing, local minima and maxima, intervals of concavity, and inflection points.

**Solution:** The function is undefined at $x = 2$, so it has an asymptote there.

As $x$ approaches 2 from below, the denominator is negative approaching zero, and the numerator approaches 4. So $\lim_{x\to 0^-} f(x) = -\infty$.

As $x$ approaches 2 from above, the denominator is positive approaching zero, and the numerator approaches 4. So $\lim_{x\to 0^+} f(x) = \infty$.

This describes the behavior near the asymptote.

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to-\infty} f(x) = -\infty$$

Next we need to analyze the derivative.

$$f'(x) = \frac{(2x-2)(x-2) - (x^2 - 2x + 4)}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2}$$

This is 0 at $x = 0, 4$. It is positive on $(-\infty, 0)$ and on $(4, \infty)$. It is negative on $(0, 2)$ and also on $(2, 4)$. So the function is increasing on the intervals $(-\infty, 0)$ and $(4, \infty)$ and decreasing on $(0, 2)$ and on $(2, 4)$.

We now have enough information to describe the basic shape. The function increasing from $-\infty$ to $f(0) = -2$ on $(-\infty, 0)$. The value -2 is a local max at 0. Then the function decreases to $-\infty$ on the interval $(0, 2)$. On $(2, 4)$ the function decreases from $\infty$ to $f(4) = 6$, then increases to $\infty$ on $(4, \infty)$.

To refine our sketch, we consider the second derivative.

$$f''(x) = \frac{(2x-4)(x-2)^2 - (x^2 - 4x)2(x-2)}{(x-2)^4} = \frac{8x - 16}{(x-2)^4}$$

This is never 0 (the numerator is 0 at $x = 2$, but so is the denominator, so the function is simply undefined at $x = 2$). So there are no points of inflection. It is positive for $x > 2$, so the function is concave up on the intervals $(2, \infty)$. It is negative when $x < 2$, so the function is concave down on $(-\infty, 2)$