Due Friday, April 18th.

Problem 1 *Let φ be continuous function on $\partial B_1(0) \subset \mathbb{R}^2$. Let*

$$\mathcal{F} = \{v \in C(B_1(0)) : \triangle v \geq 0 \text{ in viscosity sense, and } v \leq \varphi \text{ on } \partial B_1(0) \ \}$$

Show that

$$u(x) = \sup_{v \in \mathcal{F}} v(x)$$

solves the Dirichlet problem (in a viscosity sense)

$$\begin{aligned} \triangle u &= 0 \text{ on } B_1(0) \\ u &= \varphi \text{ on } \partial B_1(0) \end{aligned}$$

Solution 2 *Properties of taking supremum give that $u \leq \varphi$ on $\partial B_1(0)$ and that $\triangle u \geq 0$ in viscosity sense (this was shown in class.) We show that $\triangle u \geq 0$: Suppose not. Then there exists a point x_0 such that a polynomial $P(x)$ touches u from below at x_0 but $\triangle P = \varepsilon > 0$. To be precise there is a neighborhood $B_\delta(x_0)$ where $P \leq u$. It follows that the function*

$$Q = P(x) - \frac{\varepsilon}{4n} |x - x_0|^2$$

also touches u by below at x_0. As

$$Q = P(x) - \frac{\varepsilon}{4n}\delta^2 \leq u - \frac{\varepsilon}{4n}\delta^2 \text{ on } \partial B_\delta(x_0)$$

we have a strict subsolution

$$Q + \frac{\varepsilon}{8n}\delta^2$$

which lies strictly below u on the boundary and strictly above u at x_0. Now we may appeal to Proposition (2.8 in the book) which says that when $u \geq v$ on the boundary of a small ball we may extend to subsolution on the whole ball via

$$w = \left\{ \begin{array}{c} u \text{ outside } B_\delta(x_0) \\ \sup\{u, v\} \text{ on } B_\delta(x_0) \end{array} \right\}$$

Thus the function w is in the class $\mathcal{F}$ but also satisfies $w(x_0) > u(x_0)$ which is a contradiction.

Next we show that u takes the boundary values. Choose any $\varepsilon > 0$ and a point z_0 on the boundary. WLOG $z_0 = (1, 0)$. We will call our variables (x, y) Now there exists a $\delta > 0$ such that for all z with $z \in \partial B_1(0)$

$$|z - z_0| < \delta \Longrightarrow |\varphi(z) - \varphi(z_0)| \leq \varepsilon.$$

Consider the linear function

$$l(x, y) = A(x - 1) + \varphi(z_0) - \varepsilon.$$

Clearly

$$l(z_0) = \varphi(z_0) - \varepsilon.$$

Also

$$l(z) \leq \varphi(z)$$

for $|z - z_0| < \delta$, *using the uniform continuity. Now*

$$\partial B_1(0) \cap \partial B_\delta((1,0)) = \{(1-\delta_1, y_1), (1-\delta_1, -y_1)\}$$

for some positive value δ_1. *Now we can choose*

$$A = \frac{1}{\delta_1} \max_{z \in \partial B_1} \left(|\varphi(z_0) - \varphi|\right).$$

Note that for $x \leq (1-\delta_1)$ *we have*

$$l(x,y) \leq -\max_{z \in \partial B_1} \left(|\varphi(z_0) - \varphi|\right) + \varphi(z_0) - \varepsilon \leq \varphi(z)$$

For $x \geq (1-\delta_1)$ *we have* $l(z) \leq \varphi(z)$ *by uniform continuity. Thus* l *lies below* φ *on the boundary. We make take a sup of such function to get the limit of* $\varphi(z_0)$.

Problem 3 *Given* M *symmetric matrix, and* $0 < \lambda < \Lambda < \infty$. *Suppose that* M *has eigenvalues* μ_i. *Define*

$$\begin{aligned} M^-(M,\lambda,\Lambda) &= \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i \\ M^+(M,\lambda,\Lambda) &= \Lambda \sum_{\mu_i > 0} \mu_i + \lambda \sum_{\mu_i < 0} \mu_i. \end{aligned}$$

Let

$$A_{\lambda,\Lambda} = \left\{A = A^{ij} \text{ symmetric matrix } \lambda|z|^2 \leq A^{ij} z_i z_j \leq \Lambda |z|^2\right\}$$

Show that

$$\begin{aligned} M^-(M,\lambda,\Lambda) &= \inf_{A \in A_{\lambda,\Lambda}} A^{ij} M_{ij} \\ M^+(M,\lambda,\Lambda) &= \sup_{A \in A_{\lambda,\Lambda}} A^{ij} M_{ij}. \end{aligned}$$

Solution 4 *Diagonalize* M

$$M = ODO^T$$

Note that

$$M^-(D,\lambda,\Lambda) = M^-(M,\lambda,\Lambda)$$

as M *and* D *have the same eigenvalues. Also note that*

$$\begin{aligned} A^{ij} M_{ij} &= Tr(AM) = Tr(O^T A O O^T M O) = Tr(O^T A O D) \\ &= \tilde{A}^{ij} D_{ij} \end{aligned}$$

for some other $\tilde{A}^{ij}$ also in the same class $A_{\lambda,\Lambda}$, as the change of basis does not change the class $A_{\lambda,\Lambda}$. So suffice to show the statement is true for D.

First we show that

$$M^-(D,\lambda,\Lambda) \geq \inf_{A\in A_{\lambda,\Lambda}} A^{ii}D_{ii}$$

This follows easily by taking A^{ii} to be a diagonal matrix with diagonal entries λ,Λ corresponding to positive and negative entries of D, respectively. Clearly $A \in A_{\lambda,\Lambda}$.

Next we show that

$$M^-(D,\lambda,\Lambda) \leq \inf_{A\in A_{\lambda,\Lambda}} A^{ii}D_{ii}$$

Assume there is some other A' such that

$$A^{'ii}D_{ii} < M^-(D,\lambda,\Lambda).$$

Then

$$\sum_{\mu_i>0}\left(A^{'ii}-\lambda\right)\mu_i \sum_{\mu_i>0}\left(A^{'ii}-\Lambda\right)\mu_i < 0$$

It follows that either $\left(A^{'ii}-\lambda\right) < 0$ or $\left(A^{'ii}-\Lambda\right)$ which contradicts the choice of $A \in A_{\lambda,\Lambda}$.