Due Friday, April 18th.

**Problem 1** Let $\varphi$ be continuous function on $\partial B_1(0) \subset \mathbb{R}^2$. Let

$$\mathcal{F} = \{ v \in C(B_1(0)) : \Delta v \geq 0 \text{ in viscosity sense, and } v \leq \varphi \text{ on } \partial B_1(0) \}$$

Show that

$$u(x) = \sup_{v \in \mathcal{F}} v(x)$$

solves the Dirichlet problem (in a viscosity sense)

$$\Delta u = 0 \text{ on } B_1(0)$$

$$u = \varphi \text{ on } \partial B_1(0)$$

**Solution 2** Properties of taking supremum give that $u \leq \varphi$ on $\partial B_1(0)$ and that $\Delta u \geq 0$ in viscosity sense (this was shown in class.) We show that $\Delta u \geq 0$:

Suppose not. Then there exists a point $x_0$ such that a polynomial $P(x)$ touches $u$ from below at $x_0$ but $\Delta P = \varepsilon > 0$. To be precise there is a neighborhood $B_\delta(x_0)$ where $P \leq u$. It follows that the function

$$Q = P(x) - \frac{\varepsilon}{4n} |x - x_0|^2$$

also touches $u$ by below at $x_0$. As

$$Q = P(x) - \frac{\varepsilon}{4n} \delta^2 \leq u - \frac{\varepsilon}{4n} \delta^2 \text{ on } \partial B_\delta(x_0)$$

we have a strict subsolution

$$Q + \frac{\varepsilon}{8n} \delta^2$$

which lies strictly below $u$ on the boundary and strictly above $u$ at $x_0$. Now we may appeal to Proposition (2.8 in the book) which says that when $u \geq v$ on the boundary of a small ball we may extend to subsolution on the whole ball via

$$w = \begin{cases} 
  u \text{ outside } B_\delta(x_0) \\
  \sup \{u, v\} \text{ on } B_\delta(x_0) 
\end{cases}$$

Thus the function $w$ is in the class $\mathcal{F}$ but also satisfies $w(x_0) > u(x_0)$ which is a contradiction.

Next we show that $u$ takes the boundary values. Choose any $\varepsilon > 0$ and a point $z_0$ on the boundary. WLOG $z_0 = (1,0)$. We will call our variables $(x,y)$. Now there exists a $\delta > 0$ such that for all $z$ with $z \in \partial B_1(0)$

$$|z - z_0| < \delta \implies |\varphi(z) - \varphi(z_0)| \leq \varepsilon.$$

Consider the linear function

$$l(x,y) = A(x - 1) + \varphi(z_0) - \varepsilon.$$
Clearly \( l(z_0) = \varphi(z_0) - \varepsilon \).

Also \( l(z) \leq \varphi(z) \)

for \(|z - z_0| < \delta\), using the uniform continuity. Now

\[
\partial B_1(0) \cap \partial B_\delta((1,0)) = \{(1 - \delta_1, y_1), (1 - \delta_1, -y_1)\}
\]

for some positive value \( \delta_1 \). Now we can choose

\[
A = \frac{1}{\delta_1} \max_{z \in \partial B_1} (|\varphi(z_0) - \varphi|)
\]

Note that for \( x \leq (1 - \delta_1) \) we have

\[
l(x, y) \leq - \max_{z \in \partial B_1} (|\varphi(z_0) - \varphi|) + \varphi(z_0) - \varepsilon \leq \varphi(z)
\]

For \( x \geq (1 - \delta_1) \) we have \( l(z) \leq \varphi(z) \) by uniform continuity. Thus \( l \) lies below \( \varphi \) on the boundary. We make take a sup of such function to get the limit of \( \varphi(z_0) \).

**Problem 3** Given \( M \) symmetric matrix, and \( 0 < \lambda < \Lambda < \infty \). Suppose that \( M \) has eigenvalues \( \mu_i \). Define

\[
M^-(M, \lambda, \Lambda) = \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i
\]

\[
M^+(M, \lambda, \Lambda) = \Lambda \sum_{\mu_i > 0} \mu_i + \lambda \sum_{\mu_i < 0} \mu_i.
\]

Let

\[
A_{\lambda, \Lambda} = \{ A = A_{ij} \text{ symmetric matrix } \lambda |z|^2 \leq A_{ij} z_i z_j \leq \Lambda |z|^2 \}
\]

Show that

\[
M^-(M, \lambda, \Lambda) = \inf_{A \in A_{\lambda, \Lambda}} A_{ij} M_{ij}
\]

\[
M^+(M, \lambda, \Lambda) = \sup_{A \in A_{\lambda, \Lambda}} A_{ij} M_{ij}.
\]

**Solution 4** Diagonalize \( M \)

\[
M = O D O^T
\]

Note that

\[
M^-(D, \lambda, \Lambda) = M^-(M, \lambda, \Lambda)
\]

as \( M \) and \( D \) have the same eigenvalues. Also note that

\[
A_{ij} M_{ij} = \text{Tr}(AM) = \text{Tr}(O^T A O O^T M O) = \text{Tr}(O^T A O D) = \hat{A}_{ij} D_{ij}
\]
for some other $\tilde{A}^{ij}$ also in the same class $A_{\lambda,\Lambda}$, as the change of basis does not change the class $A_{\lambda,\Lambda}$. So suffice to show the statement is true for $D$.

First we show that

$$M^-(D, \lambda, \Lambda) \geq \inf_{A \in A_{\lambda,\Lambda}} A^{ii} D_{ii}$$

This follows easily by taking $A^{ii}$ to be a diagonal matrix with diagonal entries $\lambda, \Lambda$ corresponding to positive and negative entries of $D$, respectively. Clearly $A \in A_{\lambda,\Lambda}$.

Next we show that

$$M^-(D, \lambda, \Lambda) \leq \inf_{A \in A_{\lambda,\Lambda}} A^{ii} D_{ii}$$

Assume there is some other $A'$ such that

$$A'^{ii} D_{ii} < M^-(D, \lambda, \Lambda).$$

Then

$$\sum_{\mu_i > 0} \left(A'^{ii} - \lambda \right) \mu_i \sum_{\mu_i > 0} \left(A'^{ii} - \Lambda \right) \mu_i < 0$$

It follows that either $\left(A'^{ii} - \lambda \right) < 0$ or $\left(A'^{ii} - \Lambda \right)$ which contradicts the choice of $A \in A_{\lambda,\Lambda}$. 
