# REGULARITY OF HAMILTONIAN STATIONARY EQUATIONS IN SYMPLECTIC MANIFOLDS 

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#### Abstract

In this paper, we prove that any $C^{1}$-regular Hamiltonian stationary Lagrangian submanifold in a symplectic manifold is smooth. More broadly, we develop a regularity theory for a class of fourth order nonlinear elliptic equations with two distributional derivatives. Our fourth order regularity theory originates in the geometrically motivated variational problem for the volume functional, but should have applications beyond.


## 1. Introduction

The main purpose of this paper is to prove the assertion: Any $C^{1}$-regular Hamiltonian stationary Lagrangian submanifold in a symplectic manifold is smooth.

We achieve this by developing a regularity theory for a class of fourth order nonlinear equations of double divergence form

$$
\begin{equation*}
\partial_{x_{l}} \partial_{x_{j}} F^{j l}\left(x, D u, D^{2} u\right)=\partial_{x_{k}} a^{k}\left(x, D u, D^{2} u\right)-b\left(x, D u, D^{2} u\right) . \tag{1.1}
\end{equation*}
$$

The coefficient functions $F^{j l}, a^{k}, b$ are smooth in the entries ( $x, D u, D^{2} u$ ) over a convex region $U \subset \mathbb{R}^{n} \times \mathbb{R}^{n} \times S^{n \times n}$, and the Legendre ellipticity condition holds: for a constant $\Lambda>0$

$$
\begin{equation*}
\frac{\partial F^{j l}}{\partial u_{i k}}(\xi) \sigma_{i j} \sigma_{k l} \geq \Lambda\|\sigma\|^{2}, \forall \sigma \in S^{n \times n} \text { and } \xi \in U . \tag{1.2}
\end{equation*}
$$

A function $u \in W^{2, \infty}$ is said to be a weak solution to the double divergence equation (1.1) if each of the derivatives $\partial_{x_{i}}$ presented in (1.1) are taken in a distributional sense, as in (2.1). For non-classical solutions to nonlinear partial differential equations, especially of order beyond two, attention needs to be paid even for the meaning of solutions, due to the fact that no uniform theory exists. In our case, the double divergence structure on the matrix-valued operator $F$, which involves $D^{2} u$ itself, permits us to define solutions, possibly in the weakest form, by flipping derivatives on $F$ and the lower order terms, to test functions via integration by parts as traditionally done for distributional solutions, but now only for half of the total order.

[^0]Equations in divergence form occupy an important place in the second order PDE theory. In fourth order, the most natural counterpart is an equation, linear or nonlinear, with a double divergence structure. Many well-known equations enjoy the structure such as for the bi-harmonic functions, extremal Kähler metrics, the Willmore surface, and the Hamiltonian stationary Lagrangian equations which are closely linked to elastic mechanics. We find that the double divergence structure, a less explored area, shares similar features, as second order equations in divergence form, toward a regularity theory. We demonstrate that when (1.2) holds, any weak solution $u$ to (1.1) is smooth, provided that the oscillation of $D_{q} F\left(x, D u, D^{2} u\right)$ can be bounded locally (in $x$ ) by a small positive constant.

The above fourth order nonlinear elliptic equation originates in the variational problem for volume of Lagrangian submanifolds under Hamiltonian variations in a symplectic manifold $(M, \omega)$ with a Riemannian metric $g$ compatible with $\omega$ in the sense that $\omega(X, Y)=g(J X, Y)$ for an almost complex structure $J$ on $M$.

A Lagrangian submanifold $L$ is Hamiltonian stationary if its mean curvature 1form $\omega(H, \cdot)$ is closed and coclosed, i.e. a harmonic 1-form on $L$ w.r.t. the induced metric from $(M, g)$ (cf. Oh [Oh93] also see [JLS11, p.1071-1072]). In a Calabi-Yau manifold $(M, \omega, \Omega)$ of complex dimension $n$, this is further equivalent to a scalar equation: the Lagrangian phase function $\Theta$ is harmonic. Here the holomorphic $n$ form $\Omega$ satisfies $\Omega \wedge \bar{\Omega}=\omega^{n} / n$ ! and defines $\Theta$ by $\left.\Omega\right|_{L}=e^{\sqrt{-1} \Theta} d \mu_{L}$. The scalar equation follows from the relation $H=J \nabla \Theta$ ([HL82], [Oh93], [SW01] $)$.

In $\mathbb{C}^{n}$ with the standard Kähler structure, a particular expression for $\Theta$ is available, namely, it is a sum of arctan of the eigenvalues of the Hessian of the potential function $u$ for a local graphical representation $L=(x, D u)$. This decomposition feature of the fourth order operator into two second order elliptic operators is essential in the work of Chen-Warren [CW19b] in which it is shown that a $C^{1}$-regular Hamiltonian stationary Lagrangian submanifold in $\mathbb{C}^{n}$ is real analytic. However, the same strategy for a Calabi-Yau other than $\mathbb{C}^{n}$ encounters difficulties for the reason that $\Theta$, still well-defined by $\Omega$ at least locally, now is no longer written in a clean form as sum of arctan functions, when representing $L$ as a gradient graph in a Darboux coordinate chart.

To overcome the obstacle presented above in the Calabi-Yau case, we find that, in a more general standpoint, the Riemannian picture without referring to a symplectic structure is helpful: dealing directly with the stationary point of the volume of $L=(x, D u)$ in an open ball $B \subset \mathbb{R}^{2 n}$ equipped with a Riemannian metric among nearby competing gradient graphs $L_{t}=(x, D u+t D \eta)$ for compactly supported smooth functions $\eta$. This leads us to study the fourth order nonlinear equation (1.1) with (1.2).

We now outline our approach to the regularity problem. Given a $W^{2, \infty}$ weak solution $u$ of (1.1) that satisfies the Legendre ellipticity condition (1.2), we show, in Proposition 2.1, that the difference quotient $[u(x)-u(x-h)] /|h|$ can be bounded
in $W^{2,2}$ uniformly in $h$. Letting $h \rightarrow 0$ asserts $u \in W^{3,2}$ with estimates controlled by $\|u\|_{W^{2, \omega}}$. This boosted regularity is then used to bound the $C^{1, \alpha}$ norm of the difference quotient uniformly in $h$ in Proposition 2.2, leading to a $C^{2, \alpha}$ bound on $u$. The key ingredient for this step is a closeness assumption, given by (2.13): this ensures that the operator is in fact close to a constant coefficient operator, given by its linearization at the origin, that leads to a uniform $C^{1, \alpha}$ bound on the difference quotient. Note that reaching $C^{2, \alpha}$ is a crucial step in proving smoothness since once $C^{2, \alpha}$ is achieved the functions $\frac{\partial F^{j}}{\partial u_{i} k} \frac{\partial F^{j l}}{\partial u_{k}}, \frac{\partial F^{j l}}{\partial x_{p}}$, which were barely measurable, are now all Hölder continuous in $x$, and this is sufficient to prove higher regularity for the equation satisfied by the difference quotient. The enhanced regularity alone improves the bound on the difference between the actual operator and its linearization by a factor of a power of $r$, which in turn ultimately leads to $u \in C^{3, \alpha}$. Moving from $C^{3, \alpha}$ to $C^{\infty}$ involves a similar bootstrapping procedure employed in [BW19] by considering the difference quotient.

For the general fourth order nonlinear equation, our main result is the following. Theorem 1.1. Suppose that $u \in W^{2, \infty}\left(B_{1}\right)$ is a weak solution of (1.1) that satisfies condition (1.2) on the unit ball $B_{1}$ in $\mathbb{R}^{n}$. There is an $\varepsilon_{0}(\Lambda, n)>0$ such that if

$$
\begin{equation*}
\left|\frac{\partial F^{j l}}{\partial u_{i k}}\left(x, D u, D^{2} u\right)-\frac{\partial F^{j l}}{\partial u_{i k}}(\xi)\right|<\varepsilon_{0} \tag{1.3}
\end{equation*}
$$

for some $\xi \in U$ and all $x \in B_{1}$, then $u$ is smooth in $B_{1}$.
This regularity statement suffices for answering affirmatively the motivating geometric question on smoothness of a $C^{1}$-regular critical point under Hamiltonian deformations in a symplectic manifold. The transition, from the general theory in euclidean space to the specific symplectic setting, is done in a Darboux coordinate chart with estimates on the Riemannian metric within the special coordinates. This is given by [JLS11, Prop. 3.2 and Prop. 3.4]. Our main result is the following.
Theorem 1.2. Let $(M, \omega)$ be a compact symplectic manifold with a Riemannian metric $g$ compatible with $\omega$ and some almost complex structure $J$ on M. Let L be a Hamiltonian stationary Lagrangian $C^{1}$-regular submanifold in $M$ with respect to $\omega, g$. Then L is smooth.

As critical points of the volume functional on submanifolds, Theorem 1.2 may be compared to some of the classical statements for minimal submanifolds. For minimal submanifolds (stationary for all smooth variations with compact support), a classical theorem of Morrey states: $C^{1}$-regular minimal submanifolds are smooth [Mor66, Theorem 10.7.1]. On the other hand, Lawson-Osserman [LO77] constructed enlightening examples demonstrating existence of Lipschitz minimal submanifolds (even graphical) that are not $C^{1}$. More generally, the regularity theory developed in [GM13] for second order elliptic systems does not seem to have direct impact on our single equation of higher order on a scalar function.

Convergence of a sequence of Hamiltonian stationary Lagrangian submanifolds was studied by Chen-Warren in [CW19a], the analysis therein, especially the smoothness estimates and $\varepsilon$-regularity, requires decomposing the fourth order operator into the form dealt with in [CW19b], therefore only established for $\mathbb{C}^{n}$. For a general Kähler background, techniques special to surfaces, such as conformality and bubble tree convergence with roots in the development of minimal surfaces, harmonic maps and $J$-holomorphic curves, were used to prove compactness statements in Chen-Ma [CM21] and Schoen-Wolfson [SW03]. In light of the new treatment about regularity in this paper, we will investigate the compactness question in a Kähler manifold of any dimension in a forthcoming paper.

The organization of the paper is as follows: in section 2, we introduce in detail the class of fourth order nonlinear equations and develop a regularity theory. In section 3, we derive the Euler-Lagrange equations for the variational problem on a Riemannian ball and show that it takes the form of the fourth order equation discussed in section 2. Finally, in section 4, we prove regularity for Hamiltonian stationary Lagrangian submanifolds in a symplectic manifold.

Notations. Through out this paper, we use $B_{r}$ to denote a ball with radius $r$ and center at the origin in $\mathbb{R}^{n}$, unless specified otherwise.

## 2. Fourth order elliptic theory

2.1. Preliminaries. We consider the following fourth order equation, written in double divergence form:

$$
\begin{equation*}
\int_{B_{1}}\left[F^{j l}\left(x, D u, D^{2} u\right) \eta_{j l}+a^{k}\left(x, D u, D^{2} u\right) \eta_{k}+b\left(x, D u, D^{2} u\right) \eta\right] d x=0 \tag{2.1}
\end{equation*}
$$

for all $\eta \in C_{c}^{\infty}\left(B_{1}\right)$ where $B_{1}$ is the unit ball in $\mathbb{R}^{n}$. The coefficients are smooth in the entries ( $x, D u, D^{2} u$ ) over a given convex region $U \subset \mathbb{R}^{n} \times \mathbb{R}^{n} \times S^{n \times n}$. Lower indices on a function stand for partial derivatives, e.g. $\eta_{j l}, \eta_{k}$, and summation convention is assumed.

We write $h_{p}=h e_{p}$ and denote the difference quotient of $u$ in the $e_{p}$ direction by $u^{h_{p}}$. We start by deriving a difference quotient expression from (2.1) in the direction $h_{p}$. Fixing a compactly supported function $\eta$ we can choose $h$ small enough so the function

$$
\begin{equation*}
\eta^{-h_{p}}(x)=\frac{\eta\left(x-h_{p}\right)-\eta(x)}{h} \tag{2.2}
\end{equation*}
$$

is a valid test function. Using a change of variables $x \rightarrow x+h_{p}$ on the first term of (2.2) with the first two terms of (2.1) and recombining, we get

$$
\begin{equation*}
\int_{B_{1}}\left(\left[F^{j l}\left(x, D u, D^{2} u\right)\right]^{h_{p}} \eta_{j l}+a^{k}\left(x, D u, D^{2} u\right) \eta_{k}^{-h_{p}}+b\left(x, D u, D^{2} u\right) \eta^{-h_{p}}\right) d x=0 . \tag{2.3}
\end{equation*}
$$

The function $F^{j l}$ is defined on open subsets of the vector space so for any fixed $x$ where $D^{2} u(x)$ is defined we can define

$$
\begin{aligned}
\xi_{0} & =\left(x, D u(x), D^{2} u(x)\right) \in \mathbb{R}^{n} \times \mathbb{R}^{n} \times S^{n \times n} \\
\xi_{h} & =\left(x+h_{p}, D u\left(x+h_{p}\right), D^{2} u\left(x+h_{p}\right)\right) \in \mathbb{R}^{n} \times \mathbb{R}^{n} \times S^{n \times n} \\
\vec{V} & =\xi_{h}-\xi_{0}
\end{aligned}
$$

in which case we have

$$
\begin{aligned}
& {\left[F^{j l}\left(x, D u, D^{2} u\right)\right]^{h_{p}}=\frac{1}{h}\left\{F^{j l}\left(\xi_{0}+\vec{V}\right)-F^{j l}\left(\xi_{0}\right)\right\}} \\
& \quad=\frac{1}{h} \int_{0}^{1} \frac{d}{d t} F^{j l}\left(\xi_{0}+t \vec{V}\right) d t \\
& \quad=\left.\frac{1}{h} \int_{0}^{1} D F^{j l}\right|_{\xi_{0}+t \vec{V}} \cdot \vec{V} d t \\
& \quad=\int_{0}^{1} \frac{\partial F^{j l}}{\partial u_{i k}}\left(\xi_{0}+t \vec{V}\right) \cdot u_{i k}^{h_{p}} d t+\int_{0}^{1}\left(\frac{\partial F^{j l}}{\partial u_{k}}\left(\xi_{0}+t \vec{V}\right) u_{k}^{h_{p}}+\frac{\partial F^{j l}}{\partial x_{p}}\left(\xi_{0}+t \vec{V}\right)\right) d t \\
& \quad=\left(\int_{0}^{1} \frac{\partial F^{j l}}{\partial u_{i k}}\left(\xi_{0}+t \vec{V}\right) d t\right) \cdot u_{i k}^{h_{p}}+\int_{0}^{1}\left(\frac{\partial F^{j l}}{\partial u_{k}}\left(\xi_{0}+t \vec{V}\right) u_{k}^{h_{p}}+\frac{\partial F^{j l}}{\partial x_{p}}\left(\xi_{0}+t \vec{V}\right)\right) d t \\
& \quad=\beta^{i, k l} \cdot u_{i k}^{h_{p}}+\gamma_{1}^{j l, k} u_{k}^{h_{p}}+\gamma_{2}^{j l}
\end{aligned}
$$

where we define

$$
\begin{equation*}
\beta^{i j, k l}(x)=\int_{0}^{1} \frac{\partial F^{j l}}{\partial u_{i k}}\left(\xi_{0}+t \vec{V}\right) d t \tag{2.4}
\end{equation*}
$$

and

$$
\begin{align*}
\gamma_{1}^{j l, k}(x) & =\int_{0}^{1} \frac{\partial F^{j l}}{\partial u_{k}}\left(\xi_{0}+t \vec{V}\right) d t  \tag{2.5}\\
\gamma_{2}^{j l}(x) & =\int_{0}^{1} \frac{\partial F^{j l}}{\partial x_{p}}\left(\xi_{0}+t \vec{V}\right) d t \tag{2.6}
\end{align*}
$$

Letting $f=u^{h_{p}}$ and

$$
\begin{align*}
\psi^{k}(x) & =a^{k}\left(x, D u, D^{2} u\right)  \tag{2.7}\\
\zeta(x) & =b\left(x, D u, D^{2} u\right), \tag{2.8}
\end{align*}
$$

we arrive the following equation by plugging the above expressions into (2.3) governing the difference quotients

$$
\int_{B_{1}}\left(\beta^{i j, k l} f_{i k} \eta_{j l}+\gamma_{1}^{j l, k} f_{k} \eta_{j l}+\gamma_{2}^{j l} \eta_{j l}+\psi^{k} \eta_{k}^{-h_{p}}+\zeta \eta^{-h_{p}}\right) d x=0
$$

This linearized equation, which holds true provided $\eta \in C_{c}^{\infty}\left(B_{1-h}\right)$ governs difference quotients for solutions to (2.1). Further simplifying notation we define

$$
\begin{equation*}
\gamma^{j l}(x)=\int_{0}^{1}\left(\frac{\partial F^{j l}}{\partial u_{k}}\left(\xi_{0}+t \vec{V}\right) f_{k}+\frac{\partial F^{j l}}{\partial x_{p}}\left(\xi_{0}+t \vec{V}\right)\right) d t \tag{2.9}
\end{equation*}
$$

to get

$$
\begin{equation*}
\int_{B_{1}}\left(\beta^{i j, k l} f_{i k} \eta_{j l}+\gamma^{j l} \eta_{j l}+\psi^{k} \eta_{k}^{-h_{p}}+\zeta \eta^{-h_{p}}\right) d x=0 \tag{2.10}
\end{equation*}
$$

Observe that since we do not start with a continuous Hessian, we leave the expressions for the above leading coefficients in their integral form.

Definition 1. We define the nonlinear fourth order equation (2.1) to be $\Lambda$-uniform on a convex neighborhood $U \subset \mathbb{R}^{n} \times \mathbb{R}^{n} \times S^{n \times n}$ if the standard Legendre ellipticity condition is satisfied for any $\xi \in U$

$$
\begin{equation*}
\frac{\partial F^{j l}}{\partial u_{i k}}(\xi) \sigma_{i j} \sigma_{k l} \geq \Lambda\|\sigma\|^{2}, \forall \sigma \in S^{n \times n} \tag{2.11}
\end{equation*}
$$

Remark 2.1. While this definition is tailored to equations of the form (2.1) it is important to note that it also applies to linear equations of the form (2.10), in which case

$$
F^{j l}(x)=\beta^{i j, k l}(x) f_{i k}+\gamma^{j l}(x)
$$

and

$$
\frac{\partial F^{j l}}{\partial u_{i k}}=\beta^{i j, k l}(x) .
$$

Thus when the nonlinear equation $(2.1)$ is $\Lambda$-uniform, then so is the linearized equation (2.10).

We will use the following results to prove higher regularity in section 2.2, We state the results here for the convenience of the reader.

Theorem 2.1. BW19, Theorem 2.1]. Suppose $w \in W^{2,2}\left(B_{r}\right)$ satisfies the $\Lambda$-uniform constant coefficient equation

$$
\int c_{0}^{i k, j l} w_{i k} \eta_{j l} d x=0, \quad \forall \eta \in C_{0}^{\infty}\left(B_{r}\right)
$$

Then for any $0<\rho \leq r$ there holds

$$
\begin{gathered}
\int_{B_{\rho}}\left|D^{2} w\right|^{2} \leq C_{1}\left(\frac{\rho}{r}\right)^{n}\left\|D^{2} w\right\|_{L^{2}\left(B_{r}\right)}^{2} \\
\int_{B_{\rho}}\left|D^{2} w-\left(D^{2} w\right)_{\rho}\right|^{2} \leq C_{2}\left(\frac{\rho}{r}\right)^{n+2} \int_{B_{r}}\left|D^{2} w-\left(D^{2} w\right)_{r}\right|^{2}
\end{gathered}
$$

where $C_{1}, C_{2}$ depend on the ellipticity constant $\Lambda$ and $\left(D^{2} w\right)_{\rho}$ is the average value of $D^{2} w$ on a ball of radius $\rho$.

Corollary 2.1. BW19, Corollary 2.2]. Suppose $w$ is as in the Theorem 2.1] Then for any $u \in W^{2,2}\left(B_{r}\right)$, and for any $0<\rho \leq r$, there holds

$$
\int_{B_{\rho}}\left|D^{2} u\right|^{2} \leq 4 C_{1}\left(\frac{\rho}{r}\right)^{n}\left\|D^{2} u\right\|_{L^{2}\left(B_{r}\right)}^{2}+\left(2+8 C_{1}\right)\left\|D^{2}(w-u)\right\|_{L^{2}\left(B_{r}\right)}^{2}
$$

and
$\int_{B_{\rho}}\left|D^{2} u-\left(D^{2} u\right)_{\rho}\right|^{2} \leq 4 C_{2}\left(\frac{\rho}{r}\right)^{n+2} \int_{B_{r}}\left|D^{2} u-\left(D^{2} u\right)_{r}\right|^{2}+\left(8+16 C_{2}\right) \int_{B_{r}}\left|D^{2}(u-w)\right|^{2}$ where $C_{1}, C_{2}$ depend on the ellipticity constant $\Lambda$.

Lemma 2.1. HL97, Lemma 3.4]. Let $\phi$ be a nonnegative and nondecreasing function on $[0, R]$. Suppose that

$$
\phi(\rho) \leq A\left[\left(\frac{\rho}{r}\right)^{\alpha}+\varepsilon\right] \phi(r)+B r^{\beta}
$$

for any $0<\rho \leq r \leq R$, with $A, B, \alpha, \beta$ nonnegative constants and $\beta<\alpha$. Then for any $\gamma \in(\beta, \alpha)$, there exists a constant $\varepsilon^{*}=\varepsilon^{*}(A, \alpha, \beta, \gamma)$ such that if $\varepsilon<\varepsilon^{*}$ we have for all $0<\rho \leq r \leq R$

$$
\phi(\rho) \leq c\left[\left(\frac{\rho}{r}\right)^{\gamma} \phi(r)+B r^{\beta}\right]
$$

where $c$ is a positive constant depending on $A, \alpha, \beta, \gamma$. In particular, we have for any $0<r \leq R$

$$
\phi(r) \leq c\left[\frac{\phi(R)}{R^{\gamma}} r^{\gamma}+B r^{\beta}\right] .
$$

The following boundary value problem existence result should come as no surprise, but is included for completeness.
Lemma 2.2. Suppose that $g \in W^{2,2}\left(B_{r}\right)$, and $c_{0}^{i j, k l}$ is as in Theorem 2.1] There exists a unique solution $w \in W^{2,2}\left(B_{r}\right)$ solving the following $B V P$

$$
\begin{aligned}
\int_{B_{r}} c_{0}^{i j, k l} w_{i k} \eta_{j l} d x & =0, \quad \forall \eta \in C_{0}^{\infty}\left(B_{r}\right) \\
w & =g, \quad D w=D g \quad \text { on } \partial B_{r}(y) .
\end{aligned}
$$

Proof. By [Fol95, Corollary 6.48, 6.49] the boundary condition is equivalent to $w-g \in H_{0}^{2}\left(B_{r}\right)$. The problem will be solved if we can find a function $v=w-g \in$ $H_{0}^{2}\left(B_{r}\right)$ such that

$$
\int_{B_{r}} c_{0}^{i j, k l}(w-g)_{i k} \eta_{j l} d x+\int_{B_{r}} c_{0}^{i j, k l} g_{i k} \eta_{j l} d x=0
$$

So it suffices to solve the problem

$$
\begin{aligned}
\int_{B_{r}} c_{0}^{i j, k l} v_{i k} \eta_{j l} d x & =-\int_{B_{r}} c_{0}^{i j, k l} g_{i k} \eta_{j l} d x \\
v & \in H_{0}^{2}\left(B_{r}\right) .
\end{aligned}
$$

First, we claim that

$$
\begin{equation*}
\langle\phi, \varphi\rangle=\int_{B_{r}} c_{0}^{i, k l} \phi_{i k} \varphi_{j l} d x \tag{2.12}
\end{equation*}
$$

defines a Hilbert space norm on the function space $H_{0}^{2}\left(B_{r}\right)$. In other words, the norm defined by (2.12) is equivalent to the $W_{0}^{2,2}\left(B_{r}\right)$ norm and the inner product is symmetric. First note that by the Legendre condition

$$
\langle\phi, \phi\rangle \geq \Lambda_{1} \int_{B_{r}}\left|D^{2} \phi\right|^{2}
$$

where $\Lambda_{1}$ depends on $\Lambda, n$, and because $c_{0}^{i j, k l}$ is bounded we have

$$
\langle\phi, \phi\rangle \leq \Lambda_{2} \int_{B_{r}}\left|D^{2} \phi\right|^{2}
$$

where $\Lambda_{2}$ depends on $n,\left\|c_{0}^{i, k}\right\|_{L^{\infty}}$ for $1 \leq i, j, k, l, \leq n$. Using the Poincaré inequality [GT01, (7.44)], for any $\phi \in W_{0}^{2,2}$ (hence $D \phi \in W_{0}^{1,2}$ )

$$
\frac{1}{C}\langle\phi, \phi\rangle \leq\|\phi\|_{W^{2,2}\left(B_{r}\right)}^{2} \leq C\langle\phi, \phi\rangle .
$$

Thus the norm $\langle\phi, \phi\rangle$ is continuous with respect to the $W^{2,2}$ norm.
Next we argue symmetry of (2.12): For $\phi, \varphi \in H_{0}^{2}\left(B_{r}\right)$ we may take $\phi_{m}, \varphi_{m} \in$ $C_{c}^{\infty}\left(B_{r}\right) \cap W^{2,2}\left(B_{r}\right)$, which converge respectively to $\phi, \varphi$ in $W^{2,2}$, as $m \rightarrow \infty$. We
have

$$
\begin{aligned}
\langle\phi, \varphi\rangle & =\lim _{m \rightarrow \infty}\left\langle\phi_{m}, \varphi_{m}\right\rangle \\
& =\lim _{m \rightarrow \infty} \int_{B_{r}} c_{0}^{i j, k l}\left(\phi_{m}\right)_{i k}\left(\varphi_{m}\right)_{j l} d x \\
& =(-1)^{2} \lim _{m \rightarrow \infty} \int_{B_{r}} c_{0}^{i j, k l}\left(\phi_{m}\right)_{i k j l}\left(\varphi_{m}\right) d x \\
& =(-1)^{4} \lim _{m \rightarrow \infty} \int_{B_{r}} c_{0}^{i j, k l}\left(\phi_{m}\right)_{j l}\left(\varphi_{m}\right)_{i k} d x \\
& =\lim _{m \rightarrow \infty}\left\langle\varphi_{m}, \phi_{m}\right\rangle \\
& =\langle\varphi, \phi\rangle .
\end{aligned}
$$

The linear operator

$$
f(\phi)=-\int_{B_{r}} c_{0}^{i j, k l} g_{i k} \phi_{j l} d x
$$

on $W_{0}^{2,2}\left(B_{r}\right)$ is bounded with respect to the norm defined by (2.12). To see this, take any $\phi$ in $H_{0}^{2}\left(B_{r}\right)$, then

$$
\begin{aligned}
f(\phi) & =-\int_{B_{r}} c_{0}^{i j, k l} g_{i k} \phi_{j l} d x \\
& \leq C_{1}\|g\|_{W^{2,2}\left(B_{r}\right)}\|\phi\|_{W^{2,2}\left(B_{r}\right)} \\
& \leq C_{1}\|g\|_{W^{2,2}\left(B_{r}\right)} C_{2}(\langle\phi, \phi\rangle)^{1 / 2} .
\end{aligned}
$$

By the Riesz representation theorem, there is a unique solution $v \in H_{0}^{2}\left(B_{r}\right)$ such that

$$
f(\eta)=\langle\eta, v\rangle=\int_{B_{r}} c_{0}^{i j, k l} v_{i k} \eta_{j l} d x
$$

that is

$$
-\int_{B_{r}} c_{0}^{i j, k l} g_{i k} \eta_{j l} d x=\int_{B_{r}} c_{0}^{i j, k l} v_{i k} \eta_{j l} d x
$$

Thus we can let

$$
w=v+g .
$$

This gives the solvability of the boundary value problem in $H_{0}^{2}\left(B_{r}\right)$.
2.2. Main regularity results. We will establish Theorem 1.1 by first proving the solution is $C^{2, \alpha}$ and then by bootstrapping for smoothness. We state our two main regularity boosting results below.

Theorem 2.2. Suppose that $u \in W^{2, \infty}\left(B_{1}\right)$ is a weak solution of the $\Lambda$-uniform equation (2.1) on $B_{1}$, such that

$$
\left\{\left(x, D u(x), D^{2} u(x)\right): x \in B_{1}\right\} \subset U .
$$

Fix $\alpha \in(0,1)$ and let $q=\frac{n}{2(1-\alpha)}$. There exists an $\varepsilon_{0}>0$, depending only on $\Lambda, \alpha$ and $n$ such that if the coefficients $\beta^{i j, k l}$ given by (2.4) satisfy

$$
\begin{equation*}
\left|\beta^{i j, k l}\left(x, D u, D^{2} u\right)-a_{0}^{i j, k l}\right|<\varepsilon_{0} \tag{2.13}
\end{equation*}
$$

where $a_{0}^{i, k l}=\frac{\partial F^{j l}}{\partial u_{i k}}(\xi)$ for some $\xi \in U$, then $u \in C^{2, \alpha}\left(B_{1}\right)$ with

$$
\left\|D^{2} u\right\|_{C^{\alpha}\left(B_{1 / 4}\right)} \leq C\left(\Lambda, \alpha,\|u\|_{W^{2, \infty}\left(B_{1}\right)},\|D F\|_{L^{\infty}(U)},\left\|a^{k}\right\|_{L^{\infty}(U)},\|b\|_{L^{\infty}(U)}\right)
$$

Theorem 2.3. Suppose that $u \in C^{2, \alpha}\left(B_{1}\right)$ satisfies the $\Lambda$-uniform equation (2.1) on $B_{1}$. Then $u$ is smooth in $B_{1}$.

Remark 2.2. The closeness condition (2.13) is not needed to reach $W^{3,2}$ from $W^{2, \infty}$. It is used to bootstrap to $C^{2, \alpha}$ from $W^{3,2}$, and $C^{2, \alpha}$ is enough to bootstrap further.
2.3. Proof of Theorem 2.2. To boost up regularity, we will work with equation (2.10) on the difference quotient $u_{p}^{h}$, rather than directly on (2.1) for $u$. Given a solution $f$ to (2.10), we begin with bounding its $W^{2,2}$ norm in terms of its $W^{1, \infty}$ norm in Proposition 2.1, then in Proposition 2.2, we show that the $C^{1, \alpha}$ norm of $f$ depends on its $W^{2,2}$ norm. This follows essentially the same arguments as in [CW19b, Lemma 3.1] and [BW19, Proposition 1.3].

Theorem [2.2 will then follow from Propositions 2.2 and 2.1, by taking $f=u_{p}^{h}$ therein.

Proposition 2.1. Suppose that $f \in W^{2, \infty}\left(B_{1}\right)$ satisfies the uniformly elliptic weak double divergence equation (2.10) on $B_{1}$. Then $f$ satisfies the following estimate:

$$
\begin{equation*}
\|f\|_{W^{2,2}\left(B_{1 / 2}\right)} \leq C\left(\Lambda,\|f\|_{W^{1, \infty}\left(B_{1}\right)},\|\psi\|_{L^{2}\left(B_{1}\right)},\|\zeta\|_{L^{2}\left(B_{1}\right)},\|\beta\|_{L^{\infty}\left(B_{1}\right)}\right) . \tag{2.14}
\end{equation*}
$$

Proof. Assuming $f \in W^{2, \infty}\left(B_{1}\right), f$ will be $W^{2,2}$ and the function $\tau^{4} f$ can be approximated by functions $\eta \in C_{c}^{\infty}\left(B_{3 / 4}\right)$ in $W^{2,2}$ norm for $\tau$ smooth compactly supported on $B_{3 / 4}$ which is 1 on $B_{1 / 2}$. Thus

$$
\int_{B_{1}}\left[\beta^{i, k l} f_{i k}\left(\tau^{4} f\right)_{j l}+\gamma^{j l}\left(\tau^{4} f\right)_{j l}+\psi^{k}\left(\tau^{4} f\right)_{k}^{-h_{p}}+\zeta\left(\tau^{4} f\right)^{-h_{p}}\right] d x=0
$$

Applying uniform ellipticity to the first term of the above expression, we get

$$
\begin{align*}
\Lambda \int_{B_{1}} \tau^{4}\left|D^{2} f\right|^{2} d x & \leq \int_{B_{1}}\left|\beta^{i, k l} f_{i k}\left(\left(\tau^{4}\right)_{j l} f+\left(\tau^{4}\right)_{l} f_{j}+\left(\tau^{4}\right)_{j} f_{l}\right)\right| d x  \tag{2.15}\\
& +\int_{B_{1}}\left(\left|\gamma^{j l}\left(\tau^{4} f\right)_{j l}\right|+\left|\psi^{k}\left(\tau^{4} f\right)_{k}^{-h_{p}}\right|+\left|\zeta\left(\tau^{4} f\right)^{-h_{p}}\right|\right) d x .
\end{align*}
$$

Straightforward use of inequalities gives

$$
\begin{aligned}
& \int_{B_{1}}\left|\beta^{i j, k l} f_{i k}\left(\left(\tau^{4}\right)_{j l} f+\left(\tau^{4}\right)_{l} f_{j}+\left(\tau^{4}\right)_{j} f_{l}\right)\right| d x \\
& \quad \leq C\left(D \tau, D^{2} \tau,\|f\|_{W^{1, \infty}},\|\beta\|_{L^{\infty}}\right) \int_{B_{1}} \tau^{2}\left|D^{2} f\right| d x \\
& \quad \leq C\left(D \tau, D^{2} \tau,\|f\|_{W^{1, \infty}},\|\beta\|_{L^{\infty}}\right)\left(\frac{1}{\varepsilon}+\varepsilon \int_{B_{1}} \tau^{4}\left|D^{2} f\right|^{2} d x\right) .
\end{aligned}
$$

Similarly

$$
\begin{equation*}
\int_{B_{1}}\left|\gamma^{j l}\left(\tau^{4} f\right)_{j l}\right| d x \leq C\left(D \tau, D^{2} \tau,\|f\|_{W^{1, \infty}},\|\beta\|_{L^{\infty}}\right)\left(\frac{1}{\varepsilon}+\varepsilon \int_{B_{1}} \tau^{4}\left|D^{2} f\right|^{2} d x\right) \tag{2.16}
\end{equation*}
$$

Now for

$$
\begin{equation*}
\int_{B_{1}}\left|\psi^{k}\left(\tau^{4} f\right)_{k}^{-h_{p}}\right| d x \tag{2.17}
\end{equation*}
$$

observe that

$$
\begin{aligned}
\int_{B_{1}}\left|\psi^{k} \frac{\left(\tau^{4} f\right)_{k}\left(x-h_{p}\right)-\left(\tau^{4} f\right)_{k}}{h}\right| d x & =\int_{B_{1}}\left|\psi^{k}\right|\left|\int_{0}^{1} D\left(\tau^{4} f\right)_{k}\left(x-t h_{p}\right) d t\right| d x \\
& \leq \int_{0}^{1} \int_{B_{1}}\left|\psi^{k}\right|\left|D\left(\tau^{4} f\right)_{k}\left(x-t h_{p}\right)\right| d x d t \\
& \leq \int_{0}^{1}\|\psi\|_{L^{2}\left(B_{1}\right)}\left\|D^{2}\left(\tau^{4} f\right)\right\|_{L^{2}\left(B_{1}\right)} d t \\
& =\|\psi\|_{L^{2}\left(B_{1}\right)}\left\|D^{2}\left(\tau^{4} f\right)\right\|_{L^{2}\left(B_{1}\right)}
\end{aligned}
$$

which can be treated as in (2.16)

$$
\int_{B_{1}}\left|\psi^{k}\left(\tau^{4} f\right)_{k}^{-h_{p}}\right| d x \leq C\left(D^{2} \tau,\|f\|_{W^{1, \infty}},\|\psi\|_{L^{2}\left(B_{1}\right)}\right)\left(\frac{1}{\varepsilon}+\varepsilon \int_{B_{1}} \tau^{4}\left|D^{2} f\right|^{2} d x\right) .
$$

Finally, treating the last term in (2.15) similarly as for (2.17), we can bound (2.15) in lower order terms of $f$.

Combining and using the appropriately chosen $\tau$, we choose $\varepsilon$ appropriately in the above equation and in (2.16), to get

$$
\frac{\Lambda}{2} \int_{B_{1 / 2}}\left|D^{2} f\right|^{2} d x \leq C\left(\|f\|_{W^{1, \infty}\left(B_{1}\right)},\|\psi\|_{L^{2}\left(B_{1}\right)},\|\zeta\|_{L^{2}\left(B_{1}\right)},\|\beta\|_{L^{\infty}\left(B_{1}\right)}\right)
$$

therefore complete the proof.
Our next result is key in achieving $C^{2, \alpha}$ regularity of $u$.

Proposition 2.2. For a fixed $h_{p}$ with $|h|<\frac{1}{100}$ suppose that $f \in W^{2,2}\left(B_{1}\right)$ satisfies the uniformly elliptic double divergence equation (2.10) weakly on $B_{3 / 4}(0)$. Suppose that $\gamma^{j l}, \psi^{k}, \zeta \in L^{2 q}$ with $q=\frac{n}{2-2 \alpha}, \alpha \in(0,1)$. Then, there is an $\varepsilon_{0}(n, \Lambda, \alpha)>0$, such that if (2.13) holds as in Theorem [2.2] then we have $D f \in C^{\alpha}\left(B_{1 / 4}\right)$ and the estimates:

$$
\begin{equation*}
\|D f\|_{C^{\alpha}\left(B_{1 / 4}\right)} \leq C\left(\Lambda, \alpha,\|f\|_{W^{2,2}\left(B_{1 / 2}\right)},\left\|\gamma^{j}\right\|_{L^{2 q}\left(B_{1}\right)}\left\|\psi^{k}\right\|_{L^{2 q}\left(B_{1}\right)},\|\zeta\|_{L^{2 q}\left(B_{1}\right)}\right) . \tag{2.18}
\end{equation*}
$$

Proof. Pick an arbitrary point $y \in B_{1 / 4}$. Then $B_{r}(y) \subset B_{3 / 4}$ for any fixed $r<1 / 2$.
We write $v=f-w$, where $w$ satisfies the following constant coefficient partial differential equation on $B_{r}(y) \subset B_{3 / 4}$ :

$$
\begin{array}{cl}
\int_{B_{r}(y)} a_{0}^{i j, k l} w_{i k} \eta_{j l} d x=0, & \forall \eta \in C_{0}^{\infty}\left(B_{r}(y)\right) \\
w=f, \quad D w=D f & \text { on } \partial B_{r}(y)
\end{array}
$$

Here $a_{0}^{i j, k l}$ is the symbol occurring in our assumption (2.13). This solution exists by Lemma 2.2 and is smooth on the interior of $B_{r}(y)$ [Fol95, Theorem 6.33].

We may extend $v$ to a function (still named $v$ ) on $B_{3 / 4}$ by defining $v=0$ on $B_{3 / 4} \backslash B_{r}(y)$. As the original $v \in H_{0}^{2}\left(B_{r}(y)\right)$ is the limit of $C_{c}^{\infty}\left(B_{r}(y)\right)$ functions $\eta^{(m)}$ it follows that the extended $v$ must also remain in $H_{0}^{2}\left(B_{3 / 4}\right)$.

Now because $v$ is the $W^{2,2}\left(B_{r}(y)\right)$ limit of functions $\eta^{(m)} \in C_{c}^{\infty}\left(B_{r}(y)\right) \subset C_{c}^{\infty}\left(B_{3 / 4}\right)$ we may also write

$$
\begin{align*}
\int_{B_{r}(y)} a_{0}^{i j, k l} v_{i k} v_{j l} d x & =\lim _{m \rightarrow \infty} \int_{B_{r}(y)} a_{0}^{i, k l} v_{i k}\left(\eta^{(m)}\right)_{j l} d x \\
& =\lim _{m \rightarrow \infty} \int_{B_{r}(y)} a_{0}^{i j, k l} f_{i k}\left(\eta^{(m)}\right)_{j l} d x \\
& =\lim _{m \rightarrow \infty} \int_{B_{3 / 4}} a_{0}^{i j, k l} f_{i k}\left(\eta^{(m)}\right)_{j l} d x \\
& =\int_{B_{3 / 4}} a_{0}^{i, k l} f_{i k} v_{j l} d x . \tag{2.19}
\end{align*}
$$

Now taking limits of (2.10) for $\eta^{(m)} \rightarrow v$ we conclude that

$$
\begin{equation*}
\int_{B_{3 / 4}}\left(\beta^{i j, k l} f_{i k} v_{j l}+\gamma^{j l} v_{j l}+\psi^{k} v_{k}^{-h_{p}}+\zeta v^{-h_{p}}\right) d x=0 . \tag{2.20}
\end{equation*}
$$

Now we subtract (2.20) from (2.19)

$$
\begin{align*}
\int_{B_{r}(y)} a_{0}^{i j, k l} v_{i k} v_{j l} d x & =\int_{B_{3 / 4}} a_{0}^{i j, k l} f_{i k} v_{j l} d x-\int_{B_{3 / 4}}\left(\beta^{i, k l} f_{i k} v_{j l}+\gamma^{j l} v_{j l}+\psi^{k} v_{k}^{-h_{p}}+\zeta v^{-h_{p}}\right) d x  \tag{2.21}\\
& =\int_{B_{3 / 4}}\left(a_{0}^{i j, k l}-\beta^{i j, k l}\right) f_{i k} v_{j l} d x-\int_{B_{3 / 4}} \gamma^{j l} v_{j l} d x-\int_{B_{3 / 4}}\left(\psi^{k} v_{k}^{-h_{p}}+\zeta v^{-h_{p}}\right) d x .
\end{align*}
$$

First we note that our condition (2.13), for an $\varepsilon_{0}$ yet to determined, gives us

$$
\begin{equation*}
\int_{B_{3 / 4}}\left|\left(a_{0}^{i, k l}-\beta^{i j, k l}\right) f_{i k} v_{j l}\right| d x \leq \varepsilon_{0}\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \tag{2.22}
\end{equation*}
$$

making use of the fact that $v$ is supported in $B_{r}(y)$. Next, by Hölder's inequality (2.23)
$\int_{B_{3 / 4}}\left|\gamma^{j l} v_{j l}\right| d x \leq C(n)\|\gamma\|_{L^{2}\left(B_{r}(y)\right)}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \leq C(n)\|\gamma\|_{L^{2 q\left(B_{r}(y)\right)}} r^{n-2+2 \alpha}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}$ where $q=\frac{n}{2(1-\alpha)}$.

For the third term

$$
\begin{align*}
& \int_{B_{3 / 4}}\left|\psi^{k} \frac{v_{k}\left(x-h_{p}\right)-v_{k}(x)}{h}\right| d x=\lim _{m \rightarrow \infty} \int_{B_{3 / 4}}\left|\psi^{k} \frac{\left(\eta^{(m)}\right)_{k}\left(x-h_{p}\right)-\left(\eta^{(m)}\right)_{k}(x)}{h}\right| d x \\
& =\lim _{m \rightarrow \infty} \int_{B_{3 / 4}}\left|\psi^{k} \int_{0}^{1}\left(-D_{p k} \eta^{(m)}\left(x-t h_{p}\right)\right) d t\right| d x \\
& \leq \lim _{m \rightarrow \infty} \int_{B_{3 / 4}}\left|\psi^{k}\right| \int_{0}^{1}\left|D_{p k} \eta^{(m)}\left(x-t h_{p}\right) d t\right| d x \\
& \leq \lim _{m \rightarrow \infty} \int_{0}^{1} \int_{B_{3 / 4}}\left|\psi^{k}\right|\left|D_{p k} \eta^{(m)}\left(x-t h_{p}\right)\right| d x d t \quad \text { (Tonelli’s Theorem) } \\
& \leq \int_{0}^{1} \int_{B_{3 / 4}}\left|\psi^{k}\right|\left|D^{2} v\left(x-t h_{p}\right)\right| d x d t \quad \text { (Fatou's Lemma) } \\
& =\int_{0}^{1} \int_{B_{r+h}(y)}\left|\psi^{k}\right|\left|D^{2} v\left(x-t h_{p}\right)\right| d x d t \quad\left(\operatorname{supp} v \subset B_{r}(y)\right) \\
& \leq\|\psi\|_{L^{2}\left(B_{r+h}(y)\right)}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \quad \text { (Cauchy-Schwarz inequality) } \\
& \leq C(n)\|\psi\|_{L^{2 q}\left(B_{r+h}(y)\right)} r^{\frac{n-2+2 \alpha}{2}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \text {. (Hölder's inequality) } \tag{2.24}
\end{align*}
$$

A similar computation yields

$$
\begin{align*}
\int_{B_{3 / 4}}\left|\zeta(x) v^{-h_{p}}(x) d x\right| & \leq\|\zeta\|_{L^{2}\left(B_{r+h}(y)\right)} \cdot\|D v\|_{L^{2}\left(B_{r}(y)\right)} \\
& \leq C(n)\|\zeta\|_{L^{2 q}\left(B_{r+h}(y)\right)} r^{\frac{n-2+2 x}{2}} \cdot C_{p}\left|B_{r}(y)\right|^{\frac{1}{n}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \tag{2.25}
\end{align*}
$$

where $C_{p}$ is from the Poincaré inequality [GT01, (7.44)].
Now since $a_{0}^{i j, k l}$ has an ellipticity constant $\Lambda$, plugging the bounds (2.22), (2.23), (2.24), (2.25) into (2.21), we have (collecting dimensional constants into a new
$C(n)$ )

$$
\begin{aligned}
\Lambda\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}^{2} & \leq \varepsilon_{0}\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}+C(n)\|\gamma\|_{L^{2 q}} r^{\frac{n-2+2 \alpha}{2}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \\
& +C(n)\|\psi\|_{L^{2 q}} r^{\frac{n-2+2 \alpha}{2}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}+C(n)\|\zeta\|_{L^{2 q}} r^{\frac{n-2+2 \alpha}{2}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}
\end{aligned}
$$

Dividing by $\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}$ and collecting

$$
\Lambda\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \leq \varepsilon_{0}\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}+C(n)\left(\|\gamma\|_{L^{2 q}}+\|\psi\|_{L^{2 q}}+\|\zeta\|_{L^{2 q}}\right) r^{\frac{n-2+2 \alpha}{2}} .
$$

That is

$$
\Lambda^{2}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}^{2} \leq 2 \varepsilon_{0}^{2}\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+K r^{n-2+2 \alpha}
$$

for (again modifying $C(n)$ )

$$
K=C(n)\left(\|\gamma\|_{L^{2 q}}^{2}+\|\psi\|_{L^{2 q}}^{2}+\|\zeta\|_{L^{2 q}}^{2}\right) .
$$

Recalling $f=v+w$ and Corollary 2.1

$$
\int_{B_{\rho}(y)}\left|D^{2} f\right|^{2} \leq 4 C_{1}\left(\frac{\rho}{r}\right)^{n}\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+\left(2+8 C_{1}\right)\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}
$$

for $C_{1}$ depending on the ellipticity of $a_{0}^{i, k l}$ we see

$$
\begin{equation*}
\int_{B_{\rho}(y)}\left|D^{2} f\right|^{2} \leq 4 C_{1}\left(\frac{\rho}{r}\right)^{n}\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}}\left(\varepsilon_{0}^{2}\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+K r^{n-2+2 \alpha}\right) \tag{2.26}
\end{equation*}
$$

Now, we would like to apply Lemma 2.1. To this end, let

$$
\begin{aligned}
\phi(\rho) & =\int_{B_{\rho}}\left|D^{2} f\right|^{2} \\
A & =4 C_{1} \\
\varepsilon & =\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} \varepsilon_{0}^{2} \\
B & =\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} K \\
\alpha & =n \\
\beta & =n-2+2 \alpha \\
\gamma & =n-1 \\
R & =\frac{1}{2} .
\end{aligned}
$$

To be clear, in order to avoid notational double-dipping, the notations appearing on the left hand side of expressions above refer to constants as they are named in Lemma 2.1, while the right hand side refers to constants as they appear previously
in this proof so far. We observe that (2.26) can be written using notation on the left side of the above table as

$$
\begin{equation*}
\phi(\rho) \leq A\left[\left(\frac{\rho}{r}\right)^{\alpha}+\varepsilon\right] \phi(r)+B r^{\beta} \tag{2.27}
\end{equation*}
$$

for all $0<\rho \leq r<\frac{1}{2}$. There exists a constant $\varepsilon^{*}(A, \alpha, \beta, \gamma)$ so that (2.27) allows us to conclude that there is a constant $C>0$ such that

$$
\phi(\rho) \leq C\left[\left(\frac{\rho}{r}\right)^{n-1} \phi(r)+B r^{n-2+2 \alpha}\right]
$$

whenever

$$
\begin{equation*}
\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} \varepsilon_{0}^{2} \leq \varepsilon^{*}(A, \alpha, \beta, \gamma) \tag{2.28}
\end{equation*}
$$

We pick one such $\varepsilon_{0}$. Thus

$$
\begin{aligned}
\phi(r) & \leq C\left[2^{n-1} r^{n-1} \phi\left(\frac{1}{2}\right)+B r^{n-2+2 \alpha}\right] \\
& \leq C^{\prime} r^{n-2+2 \alpha}
\end{aligned}
$$

where $C^{\prime}$ depends on $\int_{B_{1 / 2}}\left|D^{2} f\right|^{2}, \Lambda, n, \alpha$, and $\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} K$.
We now have that

$$
\int_{B_{r}}\left|D^{2} f\right|^{2} \leq C^{\prime} r^{n-2+2 \alpha}
$$

Noting that we chose an arbitrary point in $B_{1 / 4}(0)$ we may apply Morrey's Lemma [Sim96, Lemma 3, page 8] to $D f$ to get the desired conclusion.

Proof of Theorem [2.2] Applying Proposition 2.1] we see that $u \in W^{3,2}$, with estimates controlled by $\|u\|_{W^{2, \infty}}$. The difference quotient $f=u_{p}^{h}$ satisfies (2.10) where now $f \in W^{2,2}$ with estimates. Using the supremum norms of $D F, a^{k}, b$ and that $u \in W^{2, \infty}$, the conditions on $\gamma^{j l}, \psi^{k}, \zeta$ in Proposition 2.2 are fulfilled, namely, they are in $L^{2 q}$. In light of Proposition 2.2 we conclude $u_{p}^{h} \in C^{1, \alpha}$ with the estimate (2.18) where we note that now

$$
\begin{aligned}
\|f\|_{W^{1, \infty}} & =\left\|\frac{u(x)-u\left(x-h_{p}\right)}{h}\right\|_{W^{1, \infty}} \\
& =\operatorname{ess} \sup \left(\left|\frac{u(x)-u\left(x-h_{p}\right)}{h}\right|+\left|\frac{D u(x)-D u\left(x-h_{p}\right)}{h}\right|\right) \\
& \leq \operatorname{Lip}(u)+\operatorname{Lip}(D u) \\
& \leq \operatorname{ess} \sup \left(|u|+|D u|+\left|D^{2} u\right|\right) \\
& =\|u\|_{W^{2, \infty}}
\end{aligned}
$$

Letting $h \rightarrow 0$ in (2.18) yields the estimate that holds on $B_{1 / 4}$. Now take any interior point $x_{0}$ and consider the equation

$$
\begin{equation*}
\partial_{y_{l}} \partial_{y_{j}} \tilde{F}^{j l}\left(y, D v, D^{2} v\right)=\partial_{y_{k}} \tilde{a}^{k}\left(y, D v, D^{2} v\right)-\tilde{b}\left(y, D v, D^{2} v\right) \tag{2.29}
\end{equation*}
$$

with

$$
\begin{aligned}
\tilde{F}^{j l}\left(y, D v, D^{2} v\right) & =F^{j l}\left(x_{0}+r y, r D v\left(x_{0}+r y\right), D^{2} v\left(x_{0}+r y\right)\right) \\
\tilde{a}^{k}\left(y, D v, D^{2} v\right) & =r a^{k}\left(x_{0}+r y, r D v\left(x_{0}+r y\right), D^{2} v\left(x_{0}+r y\right)\right. \\
\tilde{b}\left(x, D v, D^{2} v\right) & =r^{2} b\left(x_{0}+r y, r D v\left(x_{0}+r y\right), D^{2} v\left(x_{0}+r y\right) .\right.
\end{aligned}
$$

Suppose that

$$
B_{r}\left(x_{0}\right) \subset B_{1} .
$$

Define

$$
v(y)=\frac{u\left(x_{0}+r y\right)}{r^{2}} .
$$

One can check that $v$ satisfies (2.29) on $B_{1}$ whenever $u$ satisfies (2.1).
Noting that

$$
\frac{\partial \tilde{F}^{j l}}{\partial v_{i k}}\left(y, D v, D^{2} v\right)=\frac{\partial F^{j l}}{\partial u_{i k}}\left(x_{0}+r y, r D v\left(x_{0}+r y\right), D^{2} v\left(x_{0}+r y\right)\right)
$$

we see equation (2.29) and the solution $v$ will satisfy the closeness condition (2.13) as well. This rescaling argument allows us to claim an estimate holds at any interior point in $B_{1}$.
2.4. Proof of Theorem 2.3, We start by boosting regularity from $C^{2, \alpha}$ to $C^{3, \alpha}$.

Proposition 2.3. Suppose that $u \in C^{2, \alpha}\left(B_{1}\right)$ satisfies the $\Lambda$-uniform equation (2.1) on $B_{1}$, and let $0<\delta<\alpha$. Then $D^{3} u \in C^{\alpha-\delta / 2}\left(B_{1 / 5}\right)$ and satisfies the following estimate:

$$
\begin{equation*}
\left\|D^{3} u\right\|_{C^{\alpha-\delta / 2}\left(B_{1 / 5}\right)} \leq C\left(\|u\|_{W^{2, \infty}\left(B_{1}\right)}, \Lambda, \alpha, \delta\right) . \tag{2.30}
\end{equation*}
$$

Proof. We assume that $u$ enjoys uniform $C^{2, \alpha}$ estimates on $B_{9 / 10}$. As before we take a difference quotient of the solution $u$ to (2.1) to get (2.10) with $f=u^{h_{p}}$, for some $h<1 / 100$. Since $D^{2} u \in C^{\alpha}\left(\bar{B}_{9 / 10}\right)$, the measurable coefficients are now integrals of Hölder continuous functions, when defined for any $x \in B_{3 / 4}$ as follows:

$$
\begin{aligned}
\beta^{i j, k l}(x) & =\int_{0}^{1} \frac{\partial F^{j l}}{\partial u_{i k}}\left(\xi_{0}+t \vec{V}\right) d t \in C^{\alpha}\left(B_{3 / 4}\right) \\
\gamma^{j l}(x) & =\int_{0}^{1}\left(\frac{\partial F^{j l}}{\partial u_{k}}\left(\xi_{0}+t \vec{V}\right) u_{k}^{h_{p}}+\frac{\partial F^{j l}}{\partial x_{p}}\left(\xi_{0}+t \vec{V}\right)\right) d t \in C^{\alpha}\left(B_{3 / 4}\right) .
\end{aligned}
$$

Note also that $\psi^{k}(x) \in C^{\alpha}\left(B_{3 / 4}\right)$. In particular

$$
\left|\beta^{i, k l}(x)-\beta^{i, k l}(y)\right| \leq C_{3}|x-y|^{\alpha} .
$$

Again, fixing $y \in B_{1 / 4}$ for a fixed $r<\frac{1}{2}$ we let $w$ solve the boundary value problem

$$
\begin{aligned}
\int_{B_{r}(y)} \beta^{i, k l}(0) w_{i j} \eta_{k l} d x & =0, \quad \forall \eta \in C_{0}^{\infty}\left(B_{r}(y)\right) \\
w & =f, D w=D f \quad \text { on } \partial B_{r}(y)
\end{aligned}
$$

and repeat verbatim the steps leading to $(2.21)$, with $a_{0}^{i j, k l}$ being replaced by $\beta^{i, k l}(0)$, again taking $v=f-w \in H_{0}^{2}\left(B_{r}(y)\right)$. Thus by (2.10)

$$
\int_{B_{r}(y)} \beta^{i j, k l}(0) v_{i j} v_{k l} d x=\int_{B_{r}(y)}\left(\beta^{i j, k l}(0)-\beta^{i j, k l}(x)\right) f_{i k} v_{j l} d x-\int_{B_{r}(y)}\left(\gamma^{j l} v_{j l}+\psi^{k} v_{k}^{-h_{p}}+\zeta v^{-h_{p}}\right) d x
$$

Now this time, we define

$$
\begin{equation*}
\Upsilon(r)=\sup \left\{\left|\beta^{i j, k l}(x)-\beta^{i, k l}\left(x^{\prime}\right)\right| \mid x, x^{\prime} \in B_{r}(y)\right\} \tag{2.31}
\end{equation*}
$$

which enjoys an estimate from the Hölder estimate on $D^{2} u$ :

$$
\begin{equation*}
\Upsilon(r) \leq C_{4} r^{\alpha} \tag{2.32}
\end{equation*}
$$

Since $v \in H_{0}^{2}\left(B_{r}(y)\right)$, we have, via integration by parts, that

$$
\begin{aligned}
& \int_{B_{1}} \gamma^{j l}(y) v_{j l}(x) d x=0 \\
& \int_{B_{1}} \psi^{k}(y) v_{k}^{-h_{p}}(x) d x=0
\end{aligned}
$$

and

$$
\int_{B_{1}} \zeta(y) v^{-h_{p}}(x) d x=\zeta(y) \frac{1}{h}\left(\int_{B_{1}} v\left(x-h_{p}\right) d x-\int_{B_{1}} v(x) d x\right)=0
$$

so we may write

$$
\begin{aligned}
\int_{B_{1}} & \left(\gamma^{j l} v_{j l}+\psi^{k} v_{k}^{-h_{p}}+\zeta v^{-h_{p}}\right) d x \\
& =\int_{B_{1}}\left(\left[\gamma^{j l}(x)-\gamma^{j l}(y)\right] v_{j l}+\left[\psi^{k}(x)-\psi^{k}(y)\right] v_{k}^{-h_{p}}+[\zeta(x)-\zeta(y)] v^{-h_{p}}\right) d x
\end{aligned}
$$

Now

$$
\int_{B_{1}}\left|\left[\gamma^{j l}(x)-\gamma^{j l}(y)\right] v_{j l}\right| d x \leq\|\gamma(x)-\gamma(y)\|_{L^{2}\left(B_{r}(y)\right)}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \leq C_{5}\left(r^{2 \alpha} r^{n}\right)^{\frac{1}{2}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}\right)}
$$

and similarly,

$$
\int_{B_{1}}\left|\left[\psi^{k}(x)-\psi^{k}(y)\right] v_{k}^{-h_{p}}\right| d x \leq C_{6}\left(r^{2 \alpha} r^{n}\right)^{\frac{1}{2}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}
$$

$$
\begin{aligned}
\int_{B_{1 / 2}}\left|[\zeta(x)-\zeta(y)] v^{-h_{p}}(x)\right| d x & \leq C_{7}\left(r^{2 \alpha} r^{n}\right)^{\frac{1}{2}}\|D v\|_{L^{2}\left(B_{r}(y)\right)} \\
& \leq C_{7}\left(r^{2 \alpha} r^{n}\right)^{\frac{1}{2}} C_{p}\left|B_{r}\right|^{\frac{1}{n}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \\
& \leq C_{p}^{\prime} C_{7}\left(r^{2 \alpha} r^{n}\right)^{\frac{1}{2}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}
\end{aligned}
$$

where $C_{p}$ is from the Poincaré inequality [GT01, (7.44)], $C_{p}^{\prime}=C_{p}\left|B_{1}\right|$, and

$$
\begin{aligned}
|\gamma(x)-\gamma(y)| & \leq C_{5} r^{\alpha} \\
|\psi(x)-\psi(y)| & \leq C_{6} r^{\alpha} \\
|\zeta(x)-\zeta(y)| & \leq C_{7} r^{\alpha} .
\end{aligned}
$$

(Recall the components of these functions are smooth as functions of $D^{2} u$ so these will be Hölder continuous now as $D^{2} u$ is Hölder continuous.) Note that, for $\Lambda$ the ellipticity constant for $\beta$ we have
$\Lambda\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}^{2} \leq \Upsilon(r)\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}+\left(C_{5}+C_{6}+C_{p}^{\prime} C_{7}\right)\left(r^{2 \alpha} r^{n}\right)^{\frac{1}{2}}\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}$.
That is

$$
\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)} \leq \frac{1}{\Lambda}\left\{\Upsilon(r)\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}+\left(C_{5}+C_{6}+C_{p}^{\prime} C_{7}\right)\left(r^{2 \alpha} r^{n}\right)^{\frac{1}{2}}\right\}
$$

or

$$
\left\|D^{2} v\right\|_{L^{2}\left(B_{r}(y)\right)}^{2} \leq \frac{2}{\Lambda^{2}}\left\{\Upsilon^{2}(r)\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+\left(C_{5}+C_{6}+C_{p}^{\prime} C_{7}\right)^{2} r^{2 \alpha} r^{n}\right\} .
$$

Using Corollary 2.1, for any $0<\rho \leq r$ we get

$$
\begin{equation*}
\int_{B_{\rho}(y)}\left|D^{2} f-\left(D^{2} f\right)_{\rho}\right|^{2} \leq 4 C_{2}\left(\frac{\rho}{r}\right)^{n+2} \int_{B_{r}(y)}\left|D^{2} f-\left(D^{2} f\right)_{r}\right|^{2}+\left(8+16 C_{2}\right) \int_{B_{r}(y)}\left|D^{2} v\right|^{2} \tag{2.33}
\end{equation*}
$$

$$
\leq 4 C_{2}\left(\frac{\rho}{r}\right)^{n+2} \int_{B_{r}(y)}\left|D^{2} f-\left(D^{2} f\right)_{r}\right|^{2}+\frac{2}{\Lambda}\left\{\Upsilon^{2}(r)\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+\left(C_{5}+C_{6}+C_{p}^{\prime} C_{7}\right)^{2} r^{2 \alpha} r^{n}\right\}
$$

Next, to get decay on the $\left\|D^{2} f\right\|_{L^{2}}^{2}$ factor, we will find an $r_{0}<1 / 2$ to be determined, such that for $r<r_{0}$ we have

$$
\int_{B_{\rho}(y)}\left|D^{2} f\right|^{2} \leq C_{9} \rho^{n-\delta}
$$

where $\delta=1-\tilde{\alpha}<\alpha$. In order to do this, first observe (2.26). We may replace $\varepsilon_{0}$ by $\Upsilon^{2}(r)$ by virtue of (2.31). We let $\tilde{\alpha}=1-\delta$, which will result in a different value $\tilde{q}$ in the derivation leading up to (2.26). By repeating the derivation of (2.26) replacing
only $\varepsilon_{0}$ by $\Upsilon^{2}(r), \alpha$ by $\tilde{\alpha}=1-\delta$, and $K$ by a $\tilde{K}$ determined by the different norms arising from now the exponent $\tilde{q}=n /(2-2 \tilde{\alpha})$, we get

$$
\begin{aligned}
\int_{B_{\rho}(y)}\left|D^{2} f\right|^{2} & \leq 4 C_{1}\left(\frac{\rho}{r}\right)^{n}\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}}\left(\Upsilon^{2}(r)\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+\tilde{K} r^{n-2+2 \tilde{\alpha}}\right) \\
& =\left(4 C_{1}\left(\frac{\rho}{r}\right)^{n}+\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} \Upsilon^{2}(r)\right)\left\|D^{2} f\right\|_{L^{2}\left(B_{r}(y)\right)}^{2}+\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} \tilde{K} r^{n-2+2 \tilde{\alpha}} .
\end{aligned}
$$

As before, denote

$$
\begin{aligned}
\phi(\rho) & =\int_{B_{\rho}(y)}\left|D^{2} f\right|^{2} \\
A & =4 C_{1} \\
\varepsilon & =\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} \Upsilon^{2}\left(r_{0}\right) \\
B & =\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} \tilde{K} \\
\alpha & =n \\
\beta & =n-2 \delta \\
\gamma & =n-\delta .
\end{aligned}
$$

Now by (2.32) and Lemma 2.1, there exists $r_{0}$ small enough such that

$$
\frac{2\left(2+8 C_{1}\right)}{\Lambda^{2}} \Upsilon^{2}\left(r_{0}\right) \leq \varepsilon^{*}(A, \alpha, \beta, \gamma)
$$

for the $\varepsilon^{*}$ provided by Lemma2.1, and we have for $\rho<r_{0}$

$$
\begin{aligned}
\phi(\rho) & \leq C_{8}\left\{\left(\frac{\rho}{r}\right)^{n-\delta} \phi(r)+B r^{n-2 \delta}\right\} \\
& \leq C_{8} \frac{1}{r_{0}^{n-\delta}} \rho^{n-\delta}\left\|D^{2} f\right\|_{L^{2}\left(B_{r_{0}}\right)}+B \rho^{n-2 \delta} \\
& \leq C_{9} \rho^{n-\delta}
\end{aligned}
$$

Turning back to (2.33), we now have, for $r<r_{0}$

$$
\begin{aligned}
\int_{B_{\rho}(y)}\left|D^{2} f-\left(D^{2} f\right)_{\rho}\right|^{2} & \leq 4 C_{2}\left(\frac{\rho}{r}\right)^{n+2} \int_{B_{r}(y)}\left|D^{2} f-\left(D^{2} f\right)_{r}\right|^{2} \\
& +\frac{2}{\Lambda}\left\{\Upsilon^{2}(r) C_{9} r^{n-\delta}+\left(C_{5}+C_{6}+C_{p}^{\prime} C_{7}\right)^{2} r^{2 \alpha} r^{n}\right\} \\
& \leq 4 C_{2}\left(\frac{\rho}{r}\right)^{n+2} \int_{B_{r}(y)}\left|D^{2} f-\left(D^{2} f\right)_{r}\right|^{2}+\frac{2}{\Lambda} C_{4} C_{9} r^{2 \alpha+n-\delta} \\
& +\frac{2}{\Lambda}\left(C_{5}+C_{6}+C_{p}^{\prime} C_{7}\right)^{2} r^{2 \alpha} r^{n}
\end{aligned}
$$

Now we can apply Lemma 2.1 yet again, this time with

$$
\begin{aligned}
\phi(\rho) & =\int_{B_{\rho}(y)}\left|D^{2} f-\left(D^{2} f\right)_{\rho}\right|^{2} \\
A & =4 C_{2} \\
\alpha & =n+2 \\
B & =\frac{2}{\Lambda}\left[C_{4} C_{9}+\left(C_{5}+C_{6}+C_{p}^{\prime} C_{7}\right)^{2}\right] \\
\beta & =n+2 \alpha-\delta \\
\gamma & =n+2 \alpha .
\end{aligned}
$$

We then conclude that

$$
\int_{B_{r}(y)}\left|D^{2} f-\left(D^{2} f\right)_{\rho}\right|^{2} \leq C_{10} r^{n+2 \alpha-\delta}
$$

for $r<r_{0}$ (and will be necessarily true for $r \in\left[r_{0}, \frac{1}{2}\right]$ as well, perhaps modifying $C_{10}$ ). Noting that this applies for any $y \in B_{1 / 4}$ we apply [HL97, Theorem 3.1] to $D^{2} f$ to conclude that $D^{2} f \in C^{(2 \alpha-\delta) / 2}\left(B_{1 / 5}\right)$. Noting $f=u^{h_{p}}$ we may take a limit and conclude that $u$ must enjoy uniform $C^{3, \alpha}$ estimates on $B_{1 / 5}$.

We now apply the regularity bootstrapping procedure as in [BW19] to obtain smoothness.

Proof of Theorem [2.3. We may scale the estimate provided in Proposition 2.3 to get $u \in C^{3, \alpha}\left(B_{r}\right)$ for any $r<1$. Letting $f=u^{h_{p_{1}}}$ we may apply the dominated convergence theorem while passing the limit as $h \rightarrow 0$ to the equation (2.10) and conclude that, for $v=u_{p_{1}}$

$$
\int_{B_{1}}\left(\beta^{i j, k l} v_{i k} \eta_{j l}+\gamma^{j l} \eta_{j l}-\psi^{k} \eta_{k p_{1}}-\zeta \eta_{p_{1}}\right) d x=0
$$

where

$$
\begin{gathered}
\beta^{i j, k l}(x)=\frac{\partial F^{j l}}{\partial u_{i k}}\left(x, D u, D^{2} u\right) \in C^{1, \alpha}\left(B_{r}\right) \\
\left.\gamma^{j l}(x)=\frac{\partial F^{j l}}{\partial u_{k}}\left(x, D u, D^{2} u\right)\right) f_{k}(x)+\frac{\partial F^{j l}}{\partial x_{p_{1}}}\left(x, D u, D^{2} u\right) \in C^{1, \alpha}\left(B_{r}\right) .
\end{gathered}
$$

Noting that the functions $\psi^{k}, \zeta$ are $C^{1, \alpha}$ when $u$ in $C^{3, \alpha}$, we can integrate by parts in the last two terms to get

$$
\int_{B_{1}}\left(\beta^{i j, k l} v_{i k} \eta_{j l}+\gamma^{j l} \eta_{j l}+\partial_{x_{p_{1}}} \psi^{k} \eta_{k}+\partial_{x_{p_{1}}} \zeta \eta\right) d x=0
$$

Following the difference quotient procedure leading to (2.10), this time in the direction $p_{2}$

$$
\int_{B_{1}}\left(\left[\beta^{i j, k l} v_{i k}+\gamma^{j l}\right]^{h_{p}} \eta_{j l}+\partial_{x_{p_{1}}} \psi^{k} \eta_{k}^{-h_{p_{2}}}+\partial_{x_{p_{1}}} \zeta \eta^{-h_{p_{2}}}\right) d x=0 .
$$

Expanding

$$
\left.\int_{B_{1}}\left(\left(\beta^{i, k l}\right)^{h_{p_{2}}} v_{i k}+\left(\gamma^{j l}\right)^{h_{p_{2}}}+\left(\beta^{i j, k l}\right) v_{i k}^{h_{p_{2}}}\right) \eta_{j l}+\partial_{x_{p_{1}}} \psi^{k} \eta_{k}^{-h_{p_{2}}}+\partial_{x_{p_{1}}} \zeta \eta^{-h_{p_{2}}}\right) d x=0
$$

Observe that each of the terms $\left(\beta^{i j, k l}\right)^{h_{p_{2}}} v_{i k},\left(\gamma^{j l}\right)^{h_{p_{2}}}, \partial_{x_{p_{1}}} \psi^{k}, \partial_{x_{p_{1}}} \zeta$ are $C^{\alpha}$ with uniform estimates on $B_{r}$.

So letting

$$
\begin{aligned}
\tilde{\gamma}^{j l} & =\left(\beta^{i j, k l}\right)^{h_{p_{2}}} v_{i k}+\left(\gamma^{j l}\right)^{h_{p_{2}}} \\
\tilde{\psi}^{k} & =\partial_{x_{p_{1}}} \psi^{k} \\
\tilde{\zeta} & =\partial_{x_{p_{1}}} \zeta
\end{aligned}
$$

we see that $\tilde{v}=v^{h_{p_{2}}}$ satisfies

$$
\begin{equation*}
\int_{B_{1}}\left(\beta^{i j, k l} \tilde{v}_{i k} \eta_{j l}+\tilde{\gamma}^{j l} \eta_{j l}+\tilde{\psi}^{k} \eta_{k}^{-h_{p_{2}}}+\tilde{\zeta} \eta^{-h_{p_{2}}}\right) d x=0 \tag{2.34}
\end{equation*}
$$

which is of identical form as equation (2.10). By our $\Lambda$-uniform assumption on (2.1), the above equation is uniformly elliptic, as $\beta$ has not changed. Now we apply verbatim the proof of Proposition 2.3, noting that all coefficients in sight are Hölder continuous, we get $D^{2} \tilde{v} \in C^{\alpha^{\prime}}$. Since $\tilde{v}$ is the difference quotient of a derivative of $u$, we may take $h \rightarrow 0$ and conclude that $u_{p_{1} p_{2}} \in C^{2, \alpha^{\prime}}\left(\boldsymbol{B}_{r}\right)$ with estimates for any $\alpha^{\prime}<\alpha$, for $r<1$, thus $u \in C^{4, \alpha^{\prime}}\left(B_{r}\right)$.

Note that when bootstrapping from $C^{m-1, \alpha}$ to $C^{m, \alpha^{\prime}}$ via (2.34) for $\tilde{v}=u_{p_{1} p_{2} \ldots p_{m-3}}^{p_{m-2}}$ we may take the limit of (2.34) to get

$$
\int_{B_{1}}\left(\beta^{i j, k l} \tilde{v}_{p_{m-2} i k} \eta_{j l}+\tilde{\gamma}^{j l} \eta_{j l}-\tilde{\psi}^{k} \eta_{k p_{m-2}}-\tilde{\zeta} \eta_{p_{m-2}}\right) d x=0
$$

but now $\tilde{\psi}^{k}, \tilde{\zeta} \in C^{1, \alpha}$ so we may integrate by parts and take another difference quotient in another direction $p_{m-1}$ to obtain another expression very similar to (2.34), again with Hölder regularity holding for all the coefficients and one higher order of derivative arising in $\tilde{v}$. Repeating the proof of Proposition [2.3, we conclude $u_{p_{1} p_{2} \ldots p_{m-1}} \in C^{2, \alpha}\left(B_{r}\right)$. In this way we can obtain estimates of any order.

Proof of Theorem [1.1] Observing that condition (1.3) is equivalent to condition (2.13), the result follows immediately from Theorems 2.2 and 2.3,

## 3. Derivation of the Euler-Lagrange equations on a Riemannian ball

We start by deriving the equation for a manifold that is volume stationary among gradient graphs.
Definition 2. Let $\Gamma$ be the set of gradient graphs of functions $u \in C^{1,1}\left(B_{1}\right)$ with $D u(0)=0$ and $\|D u\|_{L^{\infty}} \leq 1$, where $B_{1} \subset \mathbb{R}^{n}$, and

$$
\Gamma(u)=\left\{(x, D u(x)): x \in B_{1}\right\} \subset B_{2}^{2 n} .
$$

Let $h$ be a Riemannian metric on the euclidean ball $B_{2}^{2 n}$ in $\mathbb{R}^{2 n}$ with $h(0)=\delta_{0}$. We say that $\Gamma(u)$ is volume stationary in $\left(B_{1}, h\right)$ among gradient graphs in $\Gamma$, if

$$
\left.\frac{d}{d t} \operatorname{Vol}_{h}(\Gamma(u+t \eta))\right|_{t=0}=0, \quad \forall \eta \in C_{c}^{\infty}\left(B_{1}\right)
$$

where $\mathrm{Vol}_{h}$ is volume measured in $h$.
The volume functional $\mathrm{Vol}_{h}$ acting on $\Gamma$ is given by

$$
\operatorname{Vol}_{h}(\Gamma(u))=\int_{B_{1}} \sqrt{\operatorname{det}\left(g_{i j}(x)\right)} d x
$$

where, in the standard euclidean basis $\left\{e_{1}, \ldots, e_{n}, e_{1+n}, \ldots, e_{2 n}\right\}$ of $\mathbb{R}^{2 n}=\mathbb{R}^{n} \times \mathbb{R}^{n}$, the induced metric $g$ from $h$ on $\Gamma(u) \subset \mathbb{R}^{n} \times \mathbb{R}^{n}$ is

$$
\begin{align*}
g_{i j} & =h\left(e_{i}+\sum_{k} u_{k i} e_{k+n}, e_{j}+\sum_{l} u_{l j} e_{l+n}\right)  \tag{3.1}\\
& =h_{i j}+\sum_{k} u_{k i} h_{k+n, j}+\sum_{l} u_{l j} h_{l+n, i}+\sum_{k, l} u_{k i} u_{l j} h_{l+n, k+n}
\end{align*}
$$

with $1 \leq i, j \leq n$. We may write

$$
\begin{aligned}
h_{i j}(x, D u(x)) & =\delta_{i j}+\mathcal{A}_{i j}(x, D u(x)) \\
h_{l+n, k+n}(x, D u(x)) & =\delta_{k l}+\mathcal{B}_{k l}(x, D u(x)) \\
h_{k+n, j}(x, D u(x)) & =C_{k j}(x, D u(x)) .
\end{aligned}
$$

Note that $\mathcal{C}$ need not be symmetric, while $\mathcal{A}$ and $\mathcal{B}$ are symmetric. In block diagonal form of matrices we have

$$
h=\left(\begin{array}{ll}
I & 0  \tag{3.2}\\
0 & I
\end{array}\right)+\left(\begin{array}{cc}
\mathcal{A} & C \\
\mathcal{C}^{T} & \mathcal{B}
\end{array}\right) .
$$

Now we have

$$
\begin{equation*}
g_{i j}=\delta_{i j}+u_{i k} \delta^{k l} u_{l j}+\mathcal{A}_{i j}+u_{i m} u_{p j} \delta^{m k} \delta^{p l} \mathcal{B}_{k l}+u_{k i} \delta^{k l} C_{l j}+u_{k j} \delta^{k l} C_{l i} . \tag{3.3}
\end{equation*}
$$

Therefore, as a matrix-valued function defined on $(x, D u)$, the induced metric $g$ is quadratic in $D^{2} u$. In particular,

$$
\begin{equation*}
\left|\frac{\partial g_{i j}}{\partial u_{k l}}\right| \leq C\left(n,\|h\|_{C^{0}}\right) \sup \left|D^{2} u\right|+C(n) \sup _{i, j}\left|h_{i+n, j}\right|, \tag{3.4}
\end{equation*}
$$

where (and in sequel) we set

$$
\begin{equation*}
\|h\|_{C^{0}}=\sup _{B_{2}}\left\{\left|h_{p q}\right|, 1 \leq p, q \leq 2 n\right\} . \tag{3.5}
\end{equation*}
$$

Now, we compute the first variation of $\mathrm{Vol}_{h}$. Take a variation generated by $\eta \in$ $C_{c}^{\infty}\left(B_{1}\right)$ for the path

$$
\begin{equation*}
\gamma[t](x)=u(x)+\operatorname{t\eta }(x) \tag{3.6}
\end{equation*}
$$

which varies the manifold $\Gamma(u)$ along the $y$-direction in $B_{2}^{2 n}$. Denote the induced metric from $h$ on $\Gamma(u+t \eta)$ by $g(t)$. Straightforwardly,

$$
\begin{aligned}
g_{i j}(t)= & \delta_{i j}+\left(u_{i k}+t \eta_{i k}\right) \delta^{k l}\left(u_{l j}+t \eta_{l j}\right) \\
& +\mathcal{A}_{i j}(x, D u(x)+t D \eta(x)) \\
& +\left(u_{i m}+t \eta_{i m}\right)\left(u_{p j}+t \eta_{p j}\right) \delta^{m k} \delta^{p l} \mathcal{B}_{k l}(x, D u(x)+t D \eta(x)) \\
& +\left(u_{k i}+t \eta_{k i}\right) \delta^{k l} C_{l j}(x, D u(x)+t D \eta(x)) \\
& +\left(u_{k j}+t \eta_{k j}\right) \delta^{k l} C_{l i}(x, D u(x)+t D \eta(x)) .
\end{aligned}
$$

Next, we compute the derivative at $t=0$

$$
\begin{aligned}
\left.\frac{d}{d t} g_{i j}(t)\right|_{t=0}= & \left(u_{i k} \delta^{k l} \eta_{l j}+\eta_{i k} \delta^{k l} u_{l j}\right)+\left(u_{i m} \eta_{p j}+\eta_{i m} u_{p j}\right) \delta^{m k} \delta^{p l} \mathcal{B}_{k l}(x, D u(x)) \\
& +\eta_{k i} \delta^{k l} C_{l j}(x, D u(x))+\eta_{k j} \delta^{k l} C_{l i}(x, D u(x)) \\
& +\left\{\begin{array}{c}
D_{y} \mathcal{A}_{i j}(x, D u(x)) \\
+u_{k i} \delta^{k l} D_{y} C_{l j}(x, D u(x))+u_{k j} \delta^{k l} D_{y} C_{l i}(x, D u(x)) \\
+u_{i m} u_{p j} \delta^{m k} \delta^{l l} D_{y} \mathcal{B}_{k l}(x, D u(x))
\end{array}\right\} \cdot D \eta .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \left.\frac{d}{d t} \operatorname{Vol}_{h}(\gamma[t])\right|_{t=0}=\left.\int_{B_{1}} \frac{1}{2} \sqrt{g[t]} g^{i j}[t] \frac{d}{d t} g_{i j}[t] d x\right|_{t=0} \\
& =\frac{1}{2} \int_{B_{1}} \sqrt{g} g^{i j}\left(u_{i k} \delta^{k l} \eta_{l j}+\eta_{i k} \delta^{k l} u_{l j}+\left(u_{i m} \eta_{p j}+\eta_{i m} u_{p j}\right) \delta^{m k} \delta^{p l} \mathcal{B}_{k l}(x, D u(x))\right) d x \\
& +\frac{1}{2} \int_{B_{1}} \sqrt{g} g^{i j}\left(\eta_{k i} \delta^{k l} C_{l j}(x, D u(x))+\eta_{k j} \delta^{k l} C_{l i}(x, D u(x))\right) d x \\
& +\frac{1}{2} \int_{B_{1}} \sqrt{g} g^{i j}\left\{\begin{array}{c}
D_{y} \mathcal{A}_{i j}(x, D u(x)) \\
+u_{k i} \delta^{k l} D_{y} C_{l j}(x, D u(x))+u_{k j} \delta^{k l} D_{y} C_{l i}(x, D u(x)) \\
+u_{i m} u_{p j} \delta^{m k} \delta^{p l} D_{y} \mathcal{B}_{k l}(x, D u(x))
\end{array}\right\} \cdot D \eta d x .
\end{aligned}
$$

Dropping dependencies for easier presentation, and making use of symmetries

$$
\begin{aligned}
\left.\frac{d}{d t} \operatorname{Vol}_{h}(\gamma[t])\right|_{t=0} & =\int_{B_{1}} \sqrt{g} g^{i j}\left(u_{i k} \delta^{k l}+u_{i m} \delta^{m q} \delta^{l k} \mathcal{B}_{q k}\right) \eta_{l j} d x+\int_{B_{1}} \sqrt{g} g^{i j} \eta_{k j} \delta^{k l} C_{l i} d x \\
& +\frac{1}{2} \int_{B_{1}} \sqrt{g} g^{i j}\left\{D_{y} \mathcal{A}_{i j}+2 u_{i k} \delta^{k l} D_{y} C_{l j}+u_{i m} u_{p j} \delta^{m k} \delta^{p l} D_{y} \mathcal{B}_{k l}\right\} \cdot D \eta d x .
\end{aligned}
$$

Then we arrive at the Euler-Lagrange equation of $\mathrm{Vol}_{h}$ for variations in $\Gamma$ :
Lemma 3.1. For $1 \leq i, j, k, l \leq n$, let

$$
\begin{align*}
a^{i j, k l}\left(x, D u, D^{2} u\right) & =\sqrt{g} g^{i j} \delta^{k l}+\sqrt{g} g^{i j} \mathcal{B}_{l k}  \tag{3.7}\\
b^{j k}\left(x, D u, D^{2} u\right) & =\sqrt{g} g^{i j} C_{k i} \\
c^{k}\left(x, D u, D^{2} u\right) & =\frac{1}{2} \sqrt{g} g^{i j}\left(D_{y^{k}} \mathcal{A}_{i j}+2 u_{i k} D_{y^{k}} C_{k j}+u_{i k} u_{l j} D_{y^{k}} \mathcal{B}_{k l}\right) \\
F^{j l}\left(x, D u, D^{2} u\right) & =a^{i j, k l} u_{i k}+b^{j l} \tag{3.8}
\end{align*}
$$

Then the Euler-Lagrange equation of Vol $_{h}$ under variations in $\Gamma$ is

$$
\begin{equation*}
\int F^{j l} \eta_{j l}+c^{k} \eta_{k} d x=0, \quad \text { for all } \eta \in C_{c}^{\infty}\left(B_{1}\right) \tag{3.9}
\end{equation*}
$$

Lemma 3.2. For any $s>0$ there exists $\varepsilon_{1}(s, n)<1$ depending only on $s$ and $n$ such that if

$$
\begin{aligned}
h(0) & =I_{2 n} \\
\left\|D^{2} u\right\|_{L^{\infty}\left(B_{1}\right)} & \leq \varepsilon_{1} \\
\|D h\|_{L^{\infty}\left(B_{1}\right)} & \leq \varepsilon_{1}
\end{aligned}
$$

all hold we have

$$
\left\|a^{i j, k l}\left(x, D u(x), D^{2} u(x)\right)-\delta^{i j} \delta^{k l}\right\|_{L^{\infty}\left(B_{1}\right)}<s .
$$

(Here and below the norms $\|\cdot\|$ are defined as in (3.5).)
Proof. From (3.7)

$$
a^{i j, k l}=\delta^{i j} \delta^{k l}+\left(\sqrt{g} g^{i j}-\delta^{i j}\right) \delta^{k l}+\sqrt{g} g^{i j} \mathcal{B}_{l k}
$$

It will be convenient to define the following function

$$
\omega(z)=\sup _{M \in S^{n \times n},\|M\| \leq z}\left\|\sqrt{\operatorname{det}(I+M)}(1+M)^{i j}-\delta^{i j}\right\|
$$

which is clearly continuous for small values of $z$ and vanishes at $z=0$. This allows us to write

$$
\left\|a^{i j, k l}-\delta^{i j} \delta^{k l}\right\| \leq \omega\left(\left\|g-\delta_{i j}\right\|\right)+\left(1+\omega\left(\left\|g-\delta_{i j}\right\|\right) \mathcal{B}_{l k}\right.
$$

Noting from (3.2)

$$
\begin{aligned}
\mathcal{A}(0) & =0 \\
\mathcal{B}(0) & =0 \\
\mathcal{C}(0) & =0
\end{aligned}
$$

and

$$
\sup _{B_{2}^{2 n}}\{|\mathcal{A}|,|\mathcal{B}|,|C|\} \leq 2 \varepsilon_{1}
$$

we may inspect (3.3) and see that

$$
\left\|g_{i j}-\delta_{i j}\right\| \leq C(n)\left(\varepsilon_{1}+3 \varepsilon_{1}^{2}+\varepsilon_{1}^{3}\right)
$$

Then

$$
\left\|a^{i j, k l}-\delta^{i j} \delta^{k l}\right\| \leq \omega\left(C(n) \varepsilon_{1}\right)+\left(1+\omega\left(C(n) \varepsilon_{1}\right)\right) \varepsilon_{1}
$$

Because $\omega$ is continuous near 0 we choose an $\varepsilon_{1}$ such that

$$
\left\|a^{i, k l}-\delta^{i j} \delta^{k l}\right\|<s
$$

Theorem 3.1. Suppose that $u(x)$ is a $C^{1,1}$ function on $B_{1}$ such that $D u=0, D^{2} u(0)=$ 0 and

$$
\left\|D^{2} u\right\|_{L^{\infty}\left(B_{1}\right)} \leq \varepsilon_{1}\left(\varepsilon_{0}, n\right)
$$

for $\varepsilon_{1}$ determined by Lemma 3.2 and $\varepsilon_{0}\left(\frac{1}{2}, n\right)$ determined by (1.3). If $\Gamma(u)=\{(x, D u)\}$ is volume stationary among gradient graphs over the $x$-plane in for a Riemannian metric $h$ on the euclidean ball $B_{2}^{2 n}$ in $\mathbb{R}^{2 n}$, then $u$ is smooth in a neighborhood of 0 .
Proof. We start by perfoming a rescaling. Consider the map

$$
S: B_{2 R}^{2 n} \rightarrow B_{2}^{2 n}
$$

given by

$$
S(x, y)=\left(\frac{x}{R}, \frac{y}{R}\right) .
$$

This gives us a metric $\tilde{h}$ on $B_{2 R}^{2 n}$ via

$$
\tilde{h}=S^{*} h
$$

which satisfies

$$
\|D \tilde{h}\|=\frac{1}{R}\|D h\|
$$

In particular, by choosing $R$ large, we can scale so that

$$
\|D \tilde{h}\| \leq \varepsilon_{1}\left(\varepsilon_{0}\right)
$$

Notice that by letting

$$
\tilde{u}=R^{2} u\left(\frac{x}{R}\right) \text { on } B_{R}
$$

the gradient graph $\tilde{u}$ is precisely the pullback of the gradient graph of $u$ via the scaling $S$ :

$$
\begin{aligned}
S(x, D \tilde{u}(x)) & =\frac{1}{R} \cdot\left(x, R D u\left(\frac{x}{R}\right)\right) \\
& =\left(\frac{x}{R}, D u\left(\frac{x}{R}\right)\right) .
\end{aligned}
$$

Note also that

$$
\begin{equation*}
D^{2} \tilde{u}(x)=D^{2} u\left(\frac{x}{R}\right) \tag{3.10}
\end{equation*}
$$

will satisfy the same bounds. Now restricting $\tilde{h}$ to $B_{2}^{2 n}$ and $\tilde{u}$ to $B_{1}$ we can apply Lemma 3.2, observe (3.8) and conclude that the Euler-Lagrange equation (3.9) satisfies the condition in Theorem1.1. Thus $\tilde{u}$ is smooth inside $B_{1}$. Rescaling, we see that $u$ is smooth insde $B_{1 / R}$.

## 4. Hamiltonian stationary Lagrangian submanifolds in a symplectic manifold

Let $(M, \omega)$ be a symplectic $2 n$-manifold with a symplectic 2 -form $\omega$, and a Riemannian metric $h$ on $M$ compatible with $\omega$ and an almost complex structure $J$ on $M$, i.e. $\omega(X, Y)=h(J X, Y)$ for arbitrary smooth vector fields $X, Y$ on $M$. Suppose that $L$ is a $C^{1}$-regular submanifold of $M$ which is Lagrangian respect to $\omega$ and Hamiltonian stationary among all Hamiltonian variations of $L$ fixing the boundary of $L$ if it is non-empty.

In order to arrange for the setting of Theorem 3.1, we adapt a result from [JLS11, Prop. 3.2 and Prop. 3.4] on existence of Darboux coordinates with estimates on a symplectic manifold. Let $\pi: \mathcal{U} \rightarrow M$ be the $U(n)$ frame bundle of $M$. A point in $\mathcal{U}$ is a pair $(p, v)$ with $\pi(p, v)=p \in M$ and $v: \mathbb{R}^{2 n} \rightarrow T_{p} M$ an isomorphism satisfying $v^{*}\left(\omega_{p}\right)=\omega_{0}$ and $v^{*}\left(\left.h\right|_{p}\right)=h_{0}$ (the standard metric on $\mathbb{C}^{n}$ ). The right action of $U(n)$ on $\mathcal{U}$ is free: $\gamma(p, v)=(p, v \circ \gamma)$ for any $\gamma \in U(n)$.

Proposition 4.1 (Joyce-Lee-Schoen). Let $(M, \omega)$ be a real $2 n$-dimensional symplectic manifold without boundary, and a Riemannian metric $h$ compatible with $\omega$ and an almost complex structure $J$. Let $\mathcal{U}$ be the $U(n)$ frame bundle of $M$. Then for small $\varepsilon>0$ we can choose a family of embeddings $\Upsilon_{p, v}: B_{\varepsilon}^{2 n} \rightarrow M$ depending smoothly on $(p, v) \in U$, where $B_{\varepsilon}^{2 n}$ is the ball of radius $\varepsilon$ about 0 in $\mathbb{C}^{n}$, such that for all $(p, v) \in U$ we have
(1) $\Upsilon_{p, v}(0)=p$ and $\left.d \Upsilon_{p, v}\right|_{0}=v: \mathbb{C}^{n} \rightarrow T_{p} M$;
(2) $\Upsilon_{p, v \circ \gamma}(0) \equiv \Upsilon_{p, v} \circ \gamma$ for all $\gamma \in U(n)$;
(3) $\Upsilon_{p, v}^{*}(\omega) \equiv \omega_{0}=\frac{\sqrt{-1}}{2} \sum_{j=1}^{n} d z_{j} \wedge d \bar{z}_{j}$; and
(4) $\Upsilon_{p, v}^{*}(h)=h_{0}+O(|z|)$.

Moreover, for a dilation map $\mathbf{t}: B_{R}^{2 n} \rightarrow B_{\varepsilon}^{2 n}$ given by $\mathbf{t}(z)=t z$ where $t \leq \varepsilon / R$, set $h_{p, v}^{t}=t^{-2}\left(\Upsilon_{p, v} \circ \mathbf{t}\right)^{*} h$. Then it holds
(5) $\left\|h_{p, v}^{t}-h_{0}\right\|_{C^{0}} \leq C_{0} t$ and $\left\|\partial h_{p, v}^{t}\right\|_{C^{0}} \leq C_{1} t$,
where norms are taken w.r.t. $h_{0}$ and $\partial$ is the Levi-Civita connection of $h_{0}$.
The following is our main regularity result:
Theorem 4.1. Let $(M, \omega, h)$ be a symplectic manifold. Suppose that $L$ is a $C^{1}$ Hamiltonian stationary Lagrangian submanifold (possibly open but without boundary) embedded in M. Then L is smooth.

Proof. Fix an arbitrary point $p$ in $L \subset M$. By Proposition 4.1, we can choose Darboux coordinates around $\Upsilon_{p, v}$ at $p$, choosing $v$ so that $d \Upsilon_{p,\left.v\right|_{0}}\left(\mathbb{R}^{n}\right)=T_{p} L$. Now the submanifold

$$
L_{0}=\Upsilon_{p, v}^{-1}\left(L \cap \Upsilon_{p, v}\left(B_{\varepsilon}^{2 n}\right)\right) \subset B_{\varepsilon}^{2 n} \subset \mathbb{C}^{n}
$$

is Lagrangian and Hamiltonian stationary in $\left(B_{\varepsilon}^{2 n}, \Upsilon_{p, v}^{*} h, \omega_{0}\right)$. As a Lagrangian submanifold tangent to $\mathbb{R}^{n}$ at $0, L_{0}$ must be represented in a neighborhood of 0 as the gradient graph of function $u$ satisfying $D u(0)=0$ and $D^{2} u(0)=0$. Because $L_{0}$ is $C^{1}$, the Hessian $D^{2} u$ is continuous: We can choose $0<\varepsilon_{2}<\varepsilon$ if necessary such that

$$
\left\|D^{2} u\right\|_{C^{0}\left(B_{\varepsilon_{2} / 2}\right)}<\varepsilon_{1}
$$

and so that the projection of $L_{0} \cap B_{\varepsilon}^{2 n}$ to $\mathbb{R}^{n}$ contains $B_{\varepsilon_{2} / 2}$, for the $\varepsilon_{1}$ provided by Theorem 3.1. Next we make use of the dilation map in Proposition4.1(5), choosing $t<\frac{1}{2} \varepsilon_{2}$, small enough so that

$$
\left\|\partial h_{p, v}^{t}\right\|_{C^{0}} \leq C_{1} t<\varepsilon_{1} .
$$

We now have the following: A rescaled submanifold $\tilde{L}_{0}$, still Lagrangian, and Hamiltonian stationary with respect to the metric $h_{p, v}^{t}$, which satisfies

$$
\left\|D h_{p, v}^{t}\right\|<\varepsilon_{1}
$$

so that the projection of $\tilde{L}_{0} \cap B_{2 \varepsilon}^{2 n}$ to $\mathbb{R}^{n}$ contains $B_{1}$. Noting that the scaling does not change the Hessian $D^{2} u$ (recall (3.10)), we see that we are in the setting of Theorem 3.1. Since $\omega_{0}$ is the standard symplectic form, the condition of being Lagrangian Hamiltonian stationary is equivalent to being critical for gradient graphs. Theorem 3.1 now gives us that $u$ is smooth in a neighborhood of 0 , so $L$ is smooth in a neighborhood of $p$. As $p$ was arbitrary, $L$ is smooth everywhere.

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[^0]:    Chen was partially supported by an NSERC Discovery Grant (22R80062).

