# ON THE REGULARITY OF HAMILTONIAN STATIONARY LAGRANGIAN SUBMANIFOLDS 

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#### Abstract

We prove a Morrey-type theorem for Hamiltonian stationary Lagrangian submanifolds of $\mathbb{C}^{n}$ : If a $C^{1}$ Lagrangian submanifold is a critical point of the volume functional under Hamiltonian variations, then it must be real analytic. Locally, a Hamiltonian stationary Lagrangian submanifold is determined geometrically by harmonicity of its Lagrangian phase function, or variationally by a nonlinear fourth order elliptic equation of the potential function whose gradient graph defines the Hamiltonian stationary Lagrangian submanifolds locally. Our result shows that Morrey's theorem for minimal submanifolds admits a complete fourth order analogue. We establish full regularity and removability of singular sets of capacity zero for weak solutions to the fourth order equation with $C^{1,1}$ norm below a dimensional constant, and to $C^{1,1}$ potential functions, under certain convexity conditions, whose Lagrangian phase functions are weakly harmonic.


## 1. Introduction

In this paper, we study regularity of Hamiltonian stationary Lagrangian submanifolds of the complex Euclidean space $\mathbb{C}^{n}$. These are critical points of the volume functional under Hamiltonian variations, and locally they are governed by a fourth order nonlinear elliptic equation. We show, among other results, that when a Hamiltonian stationary Lagrangian submanifold is $C^{1}$ then it must be real analytic. For minimal submanifolds, a classical theorem of Morrey states: If a minimal submanifold of Euclidean space is $C^{1}$, then it is real analytic [Mor66]. Our approach to the fourth order equation is very different from Morrey's for the second order minimal surface equations. We also establish results on removable singularities when the fourth order equation is satisfied away from a compact set of capacity zero by a weak solution with $C^{1,1}$ norm below a dimensional constant. This echoes the extendibility results for minimal surface systems in [HL75, Theorem 1.2], where it is shown that Lipschitz continuous weak solutions to the system of minimal surface equations on a domain $\Omega-A$ in $\mathbb{R}^{n}$, for a closed set $A$ of zero ( $n-1$ )-dimensional Hausdorff measure, extend to weak solutions on $\Omega$, and earlier results on removability of singularities for minimal hypersurfaces [Ber51], [DGS65].

We now describe the analytic setup of the geometric variational problem. For a fixed bounded domain $\Omega \subset \mathbb{R}^{n}$, let $u: \Omega \rightarrow \mathbb{R}$ be a smooth function. The gradient graph $\Gamma_{u}=\{(x, D u(x)): x \in \Omega\}$ is a Lagrangian $n$-dimensional submanifold in $\mathbb{C}^{n}$, with respect to the complex structure $J$ defined by the complex coordinates $z_{j}=x_{j}+\sqrt{-1} y_{j}$ for $j=1, \cdots, n$. The volume of $\Gamma_{u}$ is given by

$$
F_{\Omega}(u)=\int_{\Omega} \sqrt{\operatorname{det}\left(I+\left(D^{2} u\right)^{T} D^{2} u\right)} d x .
$$

[^0]A twice differentiable function $u$ is critical for $F_{\Omega}(u)$ under compactly supported variations of the scalar function $u$ if and only if $u$ satisfies the Euler-Lagrange equation

$$
\begin{equation*}
\int_{\Omega} \sqrt{\operatorname{det} g} g^{i j} \delta^{k l} u_{i k} \eta_{j l} d x=0 \quad \text { for all } \eta \in C_{c}^{\infty}(\Omega) \tag{1.1}
\end{equation*}
$$

Here, summation convention is applied over repeated indices, $\delta^{k l}$ is the Kronecker delta, and $g$ is the induced metric from the Euclidean metric on $\mathbb{R}^{2 n}$, which can be written as

$$
g=I+\left(D^{2} u\right)^{T} D^{2} u
$$

We can define the volume $F_{\Omega}(u)$ whenever $u \in W^{2, n}(\Omega)$, so the Sobolev space $W^{2, n}(\Omega)$ is a natural space on which we seek critical points. We shaill call (1.1) the variational Hamiltonian stationary equation. A function $u \in W^{2, n}(\Omega)$ is called a weak solution the variational Hamiltonian equation if $D^{2} u$ exists almost everywhere and (1.1) holds.

If the potential $u$ is in $C^{4}(\Omega)$, the equation (1.1) is equivalent to the following geometric Hamiltonian stationary equation

$$
\begin{equation*}
\Delta_{g} \theta=0 \tag{1.2}
\end{equation*}
$$

where $\Delta_{g}$ is the Laplace-Beltrami operator on $\Gamma_{u}$ for the induced metric $g$ (cf. [Oh93], [SW03, Proposition 2.2]). The function $\theta$ is called the Lagrangian phase function for the gradient graph $\Gamma_{u}$ and is defined by

$$
\theta=\operatorname{Im} \log \operatorname{det}\left(I+\sqrt{-1} D^{2} u\right)
$$

or equivalently,

$$
\begin{equation*}
\theta=\sum_{i=1}^{n} \arctan \lambda_{i} \tag{1.3}
\end{equation*}
$$

for $\lambda_{i}$ the eigenvalues of $D^{2} u$. The mean curvature vector along $\Gamma_{u}$ can be written

$$
\vec{H}=-J \nabla \theta
$$

where $\nabla$ is the gradient operator of $\Gamma_{u}$ for the metric $g$, see ([HL82, 2.19]). We say a function $u$ is a weak solution of (1.2) if
(1) $u \in W^{2, n}(\Omega)$.
(2) The quantity $\theta$ in (1.3) is in $W^{1,2}(\Omega)$.
(3) For all $\eta \in C_{c}^{\infty}(\Omega)$

$$
\begin{equation*}
\int_{\Gamma_{u}}\langle\nabla \theta, \nabla \eta\rangle d \mu_{g}=0 . \tag{1.4}
\end{equation*}
$$

From an elliptic PDE point of view, the equation (1.2) is much preferred: The equation (1.2) is a second order operator upon a second order quantity, so we may use the full power of the well-developed second order nonlinear elliptic theory against the equation. Importantly, the function (1.3) is a concave quantity when $\theta$ falls in certain ranges, or when $u$ is convex. On the other hand, nonlinear double divergence equations of the form (1.1) are not as well understood. We shall compare the geometric settings of the two equations in more depth in Section 2. We point out that the unique expression for $\theta$ in (1.3) is valid in $\mathbb{C}^{n}$, unlikely to hold in a general Calabi-Yau $n$-fold even in a system of Darboux coordinates.

A smooth Lagrangian submanifold $L \subset \mathbb{C}^{n}$ that solves (1.2) is called Hamiltonian stationary. Note that one can always define the Lagrangian phase function $\theta$, up to an additive constantt $2 k \pi$. In general, a Hamiltonian stationary Lagrangian submanifold in a symplectic manifold
$(M, J)$ is a critical point of the volume functional under Hamiltonian deformations, that is, the variations generated by $J \nabla \eta$ for some smooth compactly supported function $\eta$ on $M$. Recall that if $u$ satisfies the special Lagrangian equation [HL82]

$$
\begin{equation*}
\nabla \theta=0 \tag{1.5}
\end{equation*}
$$

i.e. $\vec{H} \equiv 0$, then the submanifold is critical for the volume functional under all compactly supported variations of the surface $\Gamma_{u}$. The special Lagrangians are Hamiltonian stationary. The Clifford torus in the complex plane is Hamitonian stationary but not special Lagrangain. There are non-flat cones that are Hamiltonian stationary but not special Lagrangian, and this regularity issue causes serious problems for constructing minimal Lagrangian surfaces in a KählerEinstein surface (see [SW03]).

Hamiltonian stationary Lagrangian submanifolds form an interesting class of Lagrangians in a symplectic manifold as critical points of the volume functional under Hamiltonian deformations. They generalize the minimal Lagrangian submanifolds in a Kähler-Einstein manifold, especially, the special Lagrangians in a Calabi-Yau manifold. The existence and stability problem has been studied by many people via different approaches (cf [Oh90], [CU98], [SW01], [HR02], [Anc03], [HR05], [JLS11], and references therein). Yet, a general theory for existence remains open.

Our first goal is to study the regularity of submanifolds that locally are described by potentials satisfying (1.1). In particular, we shall show that if $D^{2} u$ does not have large discontinuities then the potential $u$ must be smooth, hence solving both (1.1) and (1.2). We shall consider regularity for weak solutions that lie in the Sobolev space $W_{l o c}^{2, \infty}(\Omega)$.

Theorem 1.1. Let $\Omega$ be a domain in $\mathbb{R}^{n}$ and let $Q \subset \Omega$ be a compact subset (possible empty) with capacity zero. There is a $c(n)>0$ such that if $u \in C^{1,1}(\Omega \backslash Q)$ is a weak solution to (1.1) on $\Omega \backslash Q$ satisfying

$$
\|u\|_{C^{1,1}(\Omega \backslash Q)} \leq c(n)
$$

then $u$ is a smooth solution of both (1.2) and (1.1) on $\Omega$.
Recall that the capacity of a set $Q$ is defined as

$$
\operatorname{Cap}(Q)=\inf _{\substack{\phi \in C_{C}^{*}\left(\mathbb{R}^{n}\right), 0 \leq \Delta \leq 1, \phi=1 \text { near } Q}} \int|D \phi|^{2} d x .
$$

In particular, if the Hausdorff dimension of $Q$ is less than $n-2$ then $\operatorname{Cap}(Q)$ is zero.
We make several remarks: Firstly, by a rotation, one can choose a gradient graph representation of $\Gamma$ so that $D^{2} u(0)=0$, at any point where the tangent space is defined. Secondly, as there is no size restriction on $\Omega$, any continuity condition on the Hessian will suffice. More details are provided in section 3. Thirdly, this $c(n)$ is not obtained by a compactness argument, and can be made explicit. Finally, a point singularity may not be removed from a solution to the differential equation (1.2) even if it has high codimension but the potential function $u$ has only weak regularity, say Lipschitz continuous, for example $\Gamma_{u}$ is Hamiltonian stationary for $u(x)=|x|, n=3$ away from $x=0$ but it has a non-removable singularity at $x=0$ for $u$ as a solution to (1.2) although $\Gamma_{u}$ is a smooth submanifold of $\mathbb{C}^{n}$ (cf. [CW18r]).

Next we show that in certain cases where a (slightly weaker) Hessian bound is assumed, weak solutions to (1.2) enjoy full regularity.

Theorem 1.2. Suppose that $u \in C^{1,1}\left(\mathbb{B}_{1}(0)\right)$ and $u$ is a weak solution of (1.2). If either

$$
\begin{equation*}
\theta \geq \delta+\frac{\pi}{2}(n-2) \quad \text { a.e. } \tag{1.6}
\end{equation*}
$$

for some constant $\delta \in(0, \pi)$; or

$$
\begin{equation*}
u-\delta \frac{|x|^{2}}{2} \text { is convex } \tag{1.7}
\end{equation*}
$$

for some constant $\delta>0$; or

$$
\begin{equation*}
\|u\|_{C^{1,1}\left(\mathbb{B}_{1}(0)\right)} \leq 1-\delta \tag{1.8}
\end{equation*}
$$

for some constant $\delta \in(0,1)$, then for $k \geq 2$ we have

$$
\|u\|_{C^{k, \alpha}\left(\mathbb{B}_{1 / 2}(0)\right)} \leq C\left(k, n,\|u\|_{C^{1,1}\left(\mathbb{B}_{1}(0)\right)}, \delta\right) .
$$

The conclusion still holds if $\mathbb{B}_{1}(0)$ is replaced by $\mathbb{B}_{1}(0) \backslash Q$, where $Q$ is a compact subset of $\mathbb{B}_{1}(0)$ with capacity zero.

Our strategy is as follows: For a weak solution $u$ to equation (1.2), if $\|u\|_{C^{1,1}}$ is strictly below 1, then the Lewy-Yuan rotation, adapted to the nonsmooth setting (see Proposition 4.1), converts the question to the case that a (new) potential function is uniformly convex, that is, (1.7), and then the machinery of viscosity solutions for concave operators applies. Note that the situation (1.6) can be dealt with using the same concave operator theory. Essentially, this is the Schauder theory for concave equations in [CC95] applied to the inhomogeneous equation of special Lagrangian type. For extending solutions across $Q$, we invoke a removable singularity theorem of Serrin [Ser64] for equations in divergence form. For a weak solution $u$ to (1.1) with small $C^{1,1}$ norm, we show that $u$ is in $W_{l o c}^{3,2}$. This is a key estimate in our analysis of equation (1.1) for it allows approximations by smooth functions in $W_{\text {loc }}^{3,2}$ norm and then leads to that $\theta$ (which is a priori merely $L^{\infty}$ ) satisfies (1.4), therefore, the full regularity obtained for equation (1.2) applies.

To prove our main geometric result, we combine the above two theorems as follows. Choosing an appropriate tangent plane, locally, we apply Theorem 1.1. Since the equation (1.2) is geometrically invariant (up to an immaterial additive constant), we may rotate the coordinates to where the quantity $\theta$ is concave, and apply Theorem 1.2 to obtain a description of smoothness of the same manifold. We have

Theorem 1.3. Any $C^{1}$ Hamiltonian stationary Lagrangian submanifold of $\mathbb{C}^{n}$ is real analytic. More generally, suppose $u \in W^{2, n}(\Omega)$, and $u$ satisfies equation (1.1) on $\Omega$. There is a constant $c_{0}(n)$ such that if the image of the tangent planes (where defined) of the gradient graph

$$
\Gamma_{u}=\{(x, D u(x)): x \in \Omega\}
$$

lies in a ball of radius $c_{0}(n)$ in the Grassmannian $G r(n, 2 n)$, then $\Gamma_{u}$ is a real analytic submanifold of $\mathbb{R}^{2 n}$.

In particular, if $D^{2} u$ is within distance $c(n)$ to a continuous function, then $u$ must be smooth, hence real analytic [Mor58, p.203]. For example, while we cannot rule out non-flat tangent cones occurring, we can rule out non-flat tangent cones that are nearly flat.

In two dimensions, regularity results have been obtained by Schoen and Wolfson [SW03, Theorem 4.7] in a general Kähler manifold setting, where singularities are known to occur. The examples of singularities are non-graphical over an open domain [SW01, Section 7]. On the other hand, the Euclidean case of [SW01, Proposition 4.6] states that $u$ solving (1.2) is smooth
whenever $u \in C^{2, \alpha}$. Theorem 1.3 is a generalization of this result when the ambient space is $\mathbb{C}^{n}$, see Corollary 5.1.

The rest of the paper is organized as follows. In section 2, we derive and compare the EulerLagrange equations, given mild regularity conditions on $u$. In section 3, we show that nonlinear divergence type fourth order equations enjoy a regularity boost from $W^{2, \infty}$ to $W^{3,2}$ given a condition on the nonlinearity, and from this prove Theorem 1.1. In section 4, we give details on the Lewy-Yuan rotation, as this will be necessary to prove the third part of Theorem 1.2. In section 5, we discuss and apply the Schauder theory for equations of special Lagrangian type, showing Schauder type results when the equation is concave. We then prove Theorem 1.2 under the first two conditions and combine this with the results from section 4 to give us the result in the third case. Theorem 1.3 will follow.

The results and methods in this paper will be important in our study of convergence of Hamiltonian stationary Lagrangian submanifolds [CW18c].

## 2. Derivation of the Euler-Lagrange equations

Consider the functional on the space of $C^{2}$ functions on a bounded domain $\Omega$ in $\mathbb{R}^{n}$

$$
\begin{equation*}
F_{\Omega}(u)=\int_{\Omega} \sqrt{\operatorname{det}\left(I+\left(D^{2} u\right)^{T} D^{2} u\right)} d x \tag{2.1}
\end{equation*}
$$

Note that for the gradient graph of a function $u$, we have the induced metric

$$
\begin{equation*}
g_{i j}=\delta_{i j}+u_{i k} \delta^{k l} u_{l j} \tag{2.2}
\end{equation*}
$$

in which case the above functional becomes

$$
\begin{equation*}
F_{\Omega}(u)=\int_{\Gamma_{u}} \sqrt{\operatorname{det} g} d x . \tag{2.3}
\end{equation*}
$$

Proposition 2.1. Suppose that $u \in C^{3}(\Omega)$. Then $u$ is a weak solution to (1.1) on $\Omega$ if and only if $u$ is a weak solution to (1.2) on $\Omega$, in which case (1.1) and (1.2) are each the Euler-Lagrange equation for the functional (2.1).

Proof. First we consider the case where $u$ solves (1.1). Take a variation generated by $\eta \in C_{c}^{\infty}(\Omega)$, which varies the manifold along the $y$-direction in $\mathbb{C}^{n}$. Computing the volume for the path of potentials

$$
\begin{equation*}
\gamma[t](x)=u(x)+t \eta(x) \tag{2.4}
\end{equation*}
$$

we get

$$
\begin{aligned}
\left.\frac{d}{d t} F_{\Omega}(\gamma[t])\right|_{t=0} & =\left.\int_{\Omega} \frac{1}{2} \sqrt{g[t]} g^{i j}[t] \frac{d}{d t} g_{i j}[t]\right|_{t=0} d x \\
& =\frac{1}{2} \int_{\Omega} \sqrt{g} g^{i j}\left(u_{i k} \delta^{k l} \eta_{l j}+\eta_{i k} \delta^{k l} u_{l j}\right) d x \\
& =\int_{\Omega} \sqrt{g} g^{i j} u_{i k} \delta^{k l} \eta_{l j} d x .
\end{aligned}
$$

Thus, the first variation of $F_{\Omega}$ at $u$ is given by

$$
\delta F_{\Omega}(\eta)=\int_{\Omega} \sqrt{g} g^{i j} u_{i k} \delta^{k l} \eta_{l j} d x
$$

We note that while defining $F_{\Omega}(u)$ requires only that $u \in W^{2, n}(\Omega)$.

On the other hand, we may compute the variation using the standard first variational formula for (2.3), when $u \in C^{3}$ :

$$
\left.\frac{d}{d t} F_{\Omega}(\gamma[t])\right|_{t=0}=\frac{d}{d t} \operatorname{Vol}\left(\Gamma_{u}\right)=-\int_{\Omega}\langle\vec{H}, V\rangle d \mu_{g}
$$

where $\vec{H}$ is the mean curvature vector, and $V$ is the variational field. Recall that the variation $V$ is Hamiltonian if $V=J D f$ for some compactly supported function $f$ in $\mathbb{C}^{n}$. For a Lagrangian submanifold, we also have [HL82, 2.19]

$$
\vec{H}=-J \nabla \theta
$$

Therefore, a $C^{2}$ Lagrangian submanifold is critical for the volume functional under Hamiltonian variations if and only if its Lagrangian phase is weakly harmonic.

In our case, namely, the gradient graph of $u \in C^{3}(\Omega)$, we have a vertical variational field that is Hamiltonian:

$$
\begin{equation*}
V(x)=\left.\frac{d}{d t}(x, D u(x)+t D \eta(x))\right|_{t=0}=(0, D \eta(x)) . \tag{2.5}
\end{equation*}
$$

We claim that $u$ is a weak solution to (1.2) is equivalent to that the gradient graph is critical for all vertical variations. In fact,

$$
\begin{aligned}
\delta F_{\Omega}(\eta) & =\int_{\Omega}\langle J \nabla \theta,(0, D \eta)\rangle d \mu_{g} \\
& =\int_{\Omega}\langle\nabla \theta,-J(0, D \eta)\rangle d \mu_{g} \\
& =\int_{\Omega}\langle\nabla \theta,(D \eta, 0)\rangle d \mu_{g} .
\end{aligned}
$$

with all inner products thus far being computed with respect to the ambient Euclidean metric. Now

$$
\nabla \theta=g^{i j} \theta_{i} \partial_{j}
$$

where

$$
\begin{aligned}
\partial_{1} & =\left(1,0, \ldots, 0, u_{11}, u_{21}, \ldots, u_{n 1}\right), \\
& \ldots \\
\partial_{n} & =\left(0,0, \ldots, 1, u_{1 n}, u_{2 n}, \ldots, u_{n n}\right),
\end{aligned}
$$

so we have

$$
\begin{aligned}
\delta F_{\Omega}(\eta) & =\int_{\Omega}\left\langle g^{i j} \theta_{i} \partial_{j},(D \eta, 0)\right\rangle d \mu_{g} \\
& =\int_{\Omega} g^{i j} \theta_{i} \eta_{j} d \mu_{g} \\
& =\int_{\Omega}\langle\nabla \theta, \nabla \eta\rangle_{g} d \mu_{g} .
\end{aligned}
$$

Thus we have

$$
\delta F_{\Omega}(\eta)=0 \quad \text { for all } \eta \in C_{0}^{\infty}(\Omega)
$$

if and only if

$$
\int_{\Omega}\langle\nabla \theta, \nabla \eta\rangle d \mu_{g}=0 \text { for all } \eta \in C_{c}^{\infty}(\Omega) .
$$

This equation has the weak form

$$
\int_{\Omega} \eta \Delta_{g} \theta d \mu_{g}=0 \text { for all } \eta \in C_{c}^{\infty}(\Omega)
$$

that is

$$
\begin{equation*}
\Delta_{g} \theta=0 . \tag{2.6}
\end{equation*}
$$

It follows that for $u \in C^{3}(\Omega)$, the volume (2.3) is stationary under Hamiltonian variations precisely when (1.2) is satisfied. Because (2.1) and (2.3) are the same functional, if follows that for $u \in C^{3}(\Omega)$, (1.1) and (1.2) are equivalent.

Observe that, for the gradient graph $\Gamma_{u}=\{(x, D u(x)): x \in \Omega\}$, the vertical variations constructed by (2.4) are in 1-1 correspondence with $C_{c}^{\infty}(\Omega)$. Note that one can also construct a variational field, $V=J \nabla \eta$ for each $\eta \in C_{c}^{\infty}\left(\Gamma_{u}\right)$. This is the traditional way of producing Hamiltonian variations along any Lagrangian submanifold, graphical or not. If the potential $u$ is smooth, then $C_{c}^{\infty}\left(\Gamma_{u}\right)=C_{c}^{\infty}(\Omega)$ where $\Omega$ is identified with $\Gamma_{u}$ by $F_{u}(x)=(x, D u(x))$, and the sets of variations are in 1-1 correspondence. One can then compute geometrically

$$
\begin{align*}
\left.\frac{d}{d t} F_{\Omega}(\gamma(t))\right|_{t=0} & =\int_{\Omega}\langle-\vec{H}, V\rangle d \mu_{g}  \tag{2.7}\\
& =\int_{\Omega}\langle J \nabla \theta, J \nabla \eta\rangle d \mu_{g} \\
& =\int_{\Omega}\langle\nabla \theta, \nabla \eta\rangle d \mu_{g} .
\end{align*}
$$

In particular, the first variational formula is the same.
When $u$ is not smooth, in general $C_{c}^{\infty}\left(\Gamma_{u}\right) \neq C_{c}^{\infty}(\Omega)$. For example if the submanifold $\Gamma_{u}$ is smooth but the gradient graph has vertical tangents (for instance, the curve $\Gamma_{u}=\left\{\left(x, x^{\frac{1}{3}}\right): x \in\right.$ $(-1,1)\}$ and $u=\frac{3}{4} x^{\frac{4}{3}}$ is the same smooth curve $\left(y^{3}, y\right)$ for $y \in(-1,1)$ ), one would expect some nearby Lagrangian manifolds that are not graphical over $x$ : These clearly cannot be reached through a path of vertical variations. In this case, we have strict containment

$$
C_{c}^{\infty}(\Omega) \subsetneq C_{c}^{\infty}\left(\Gamma_{u}\right) .
$$

Thus a Hamiltonian stationary Lagrangian submanifold, whose volume by definition is stationary under the larger set of variations, satisfies the equation (1.1) as well. In this sense, (1.1) is formally weaker than (1.2). It is worth asking when these equations are the same. We delve into this in the next section.

We note, as it will become useful later, that if $D^{2} u$ is bounded by a fixed constant almost everywhere, then from (2.2) we see that the operator

$$
\Delta_{g}=\frac{1}{\sqrt{g}} \partial_{i}\left(\sqrt{g} g^{i j} \partial_{j}\right)
$$

is uniformly elliptic.

## 3. Proof of Theorem 1.1

First we shall consider a general fourth order Euler-Lagrange type equation of the form

$$
\begin{equation*}
\int a^{i j k l}\left(D^{2} u\right) u_{i k} \eta_{j l} d x=0 \tag{3.1}
\end{equation*}
$$

for all $\eta \in C_{c}^{\infty}$, where each $a^{i j k l}$ is a smooth function defined on the Hessian space, i.e. the space $S^{n \times n}$ of real symmetric $n \times n$ matrices. A function $u \in W^{2, \infty}(\Omega)$ is called a variational solution to (3.1) on $\Omega$, if (3.1) is satisfied for all $\eta \in C_{c}^{\infty}(\Omega)$. The choice of the space $W^{2, \infty}(\Omega)$ may not be the most general, however, it suffices for our purposes since we shall only be considering the case when $u \in C^{1,1}$.

The proof of the following lemma is based on the calculation in [Eva10, section 6.3]. Essentially, if we have a fourth order nonlinear elliptic equation of type (3.1) such that the nonlinearity $a^{i j k l}\left(D^{2} u\right)$ has either a mild or 'monotone' dependence on $D^{2} u$, we can prove increased regularity for solutions of the equation.

Lemma 3.1. Suppose that $u \in W^{2, \infty}(\Omega)$ is a weak solution to (3.1) on $\Omega$ for $n \geq 2$ and that there is a convex neighborhood $U \subset S^{n \times n}$ such that for all $M, M^{*}, M^{\prime} \in U$ and all $W \in S^{n \times n}$

$$
\begin{equation*}
\frac{\partial a^{i j k l}}{\partial u_{p q}}\left(M^{*}\right) M_{i k}^{\prime} W_{p q} W_{j l}+a^{i j k l}(M) W_{i k} W_{j l} \geq \beta \sum_{r, s} W_{r s}^{2} \tag{3.2}
\end{equation*}
$$

where $\beta$ is a positive constant. If $D^{2} u(\Omega) \subset U$, wherever $D^{2} u$ is defined, then $u \in W_{\text {loc }}^{3,2}(\Omega)$.

Proof. By approximation, the equation (3.1) must hold for compactly supported test functions in $W_{0}^{2, \infty}(\Omega)$; in particular, it must hold for the double difference quotient

$$
\eta=-\left[\zeta^{4} u^{\left(h_{m}\right)}\right]^{\left(-h_{m}\right)}
$$

where $\zeta \in C_{c}^{\infty}(\Omega)$ is a cutoff function that is 1 on some interior set, and the upper $\left(h_{m}\right)$ refers to the difference quotient

$$
f^{\left(h_{m}\right)}(x):=\frac{f\left(x+h e_{m}\right)-f(x)}{h}
$$

and we have chosen $h$ small enough (depending on $\zeta$ ) so that $\eta$ is well defined and compactly supported. We have

$$
\begin{equation*}
\int_{\Omega} a^{i j k l}\left(D^{2} u\right) u_{i k}\left(-\left[\zeta^{4} u^{\left(h_{m}\right)}\right]^{\left(-h_{m}\right)}\right)_{j l} d x=0 . \tag{3.3}
\end{equation*}
$$

For $h$ small, we can "integrate by parts" with respect to the difference quotient, i.e.

$$
\int_{\Omega}\left[a^{i j k l}\left(D^{2} u\right) u_{i k}\right]^{\left(h_{m}\right)}\left(\zeta^{4} u^{\left(h_{m}\right)}\right)_{j l} d x=0
$$

Now the "product rule" for difference quotients gives

$$
\begin{aligned}
{\left[a^{i j k l}\left(D^{2} u\right) u_{i k}\right]^{\left(h_{m}\right)}(x)=} & u_{i k}\left(x+h e_{m}\right) \frac{a^{i j k l}\left(D^{2} u\left(x+h e_{m}\right)\right)-a^{i j k l}\left(D^{2} u(x)\right)}{h} \\
& +a^{i j k l}\left(D^{2} u(x)\right) \frac{u_{i k}\left(x+h e_{m}\right)-u_{i k}(x)}{h} \\
= & u_{i k}\left(x+h e_{m}\right) \frac{1}{h} \int_{0}^{1} \frac{d}{d t} a^{i j k l}\left((1-t) D^{2} u(x)+t D^{2} u\left(x+h e_{m}\right)\right) d t \\
& +a^{i j k l}\left(D^{2} u(x)\right) \frac{u_{i k}\left(x+h e_{m}\right)-u_{i k}(x)}{h} \\
= & u_{i k}\left(x+h e_{m}\right) \int_{0}^{1} \frac{\partial a^{i j k l}}{\partial u_{p q}}\left((1-t) D^{2} u(x)+t D^{2} u\left(x+h e_{m}\right)\right) \frac{u_{p q}\left(x+h e_{m}\right)-u_{p q}(x)}{h} d t \\
& +a^{i j k l}\left(D^{2} u(x)\right) \frac{u_{i k}\left(x+h e_{m}\right)-u_{i k}(x)}{h} \\
= & A^{i j k l p q}(x) u_{i k}\left(x+h e_{m}\right) v_{p q}(x)+a^{i j k l}\left(D^{2} u(x)\right) v_{i k}(x)
\end{aligned}
$$

where

$$
v=u^{\left(h_{m}\right)}
$$

and

$$
\begin{aligned}
A^{i j k l, p q}(x) & =\int_{0}^{1} \frac{\partial a^{i j k l}}{\partial u_{p q}}\left((1-t) D^{2} u(x)+t D^{2} u\left(x+h e_{m}\right)\right) d t \\
& =\frac{\partial a^{i j k l}}{\partial u_{p q}}\left(M^{*}(x)\right)
\end{aligned}
$$

where

$$
M^{*}(x):=\left(1-t^{*}\right) D^{2} u(x)+t^{*} D^{2} u\left(x+h e_{m}\right)
$$

for some $t^{*}$ by the mean value theorem. (Note that for a fixed $h, D^{2} u$ exists at both $x$ and $x+h e_{m}$, almost everywhere, so all of the above quantities are defined almost everywhere.) So equation (3.3) becomes

$$
\int_{\Omega}\left(\frac{\partial a^{i j k l}}{\partial u_{p q}}\left(M^{*}(x)\right) u_{i k}\left(x+h e_{m}\right) v_{p q}(x)+a^{i j k l}\left(D^{2} u(x)\right) v_{i k}(x)\right)\left(\zeta^{4} v(x)\right)_{j l} d x=0 .
$$

Now differentiating the second factor,

$$
\begin{equation*}
\int_{\Omega}\binom{\left(\frac{\partial j^{i j k l}}{\partial u_{p q}}\left(M^{*}(x)\right) u_{i k}\left(x+h e_{m}\right) v_{p q}(x)+a^{i j k l}\left(D^{2} u(x)\right) v_{i k}(x)\right)}{\times\left(\zeta^{4} v_{j l}+4 \zeta^{3} \zeta_{j} v_{l}+4 \zeta^{3} \zeta_{l} v_{j}+4 v\left(\zeta^{3} \zeta_{j l}+3 \zeta^{2} \zeta_{j} \zeta_{l}\right)\right)(x)} d x=0 \tag{3.4}
\end{equation*}
$$

By the condition (3.2) in the hypothesis we have that

$$
\int_{\Omega}\left(\frac{\partial a^{i j k l}}{\partial u_{p q}}\left(M^{*}(x)\right) u_{i k}\left(x+h e_{m}\right) v_{p q}(x)+a^{i j k l}\left(D^{2} u(x)\right) v_{i k}(x)\right) \zeta^{4} v_{j l} d x \geq \beta \int_{\Omega} \zeta^{4} \sum_{r, s} v_{r s}^{2} d x
$$

For the remaining terms, note that for the second term in the expansion of (3.4) we have by Young's inequality

$$
\begin{aligned}
& \left|\frac{\partial a^{i j k l}}{\partial u_{p q}}\left(M^{*}(x)\right) u_{i k}\left(x+h e_{m}\right) v_{p q}(x) 4 \zeta^{3}(x) \zeta_{j}(x) v_{l}(x)\right| \leq \\
& C(n) \frac{1}{\varepsilon}\left(\frac{\partial a^{i j k l}}{\partial u_{p q}}\left(M^{*}(x)\right)\right)^{2}\left(u_{i k}\left(x+h e_{m}\right)\right)^{2} \zeta^{2}(x)|D \zeta(x)|^{2}|D v(x)|^{2}+\varepsilon \zeta^{4}(x) v_{p q}^{2}(x) .
\end{aligned}
$$

A similar expression can be made for each of the terms. Noting that $D^{2} u$ is bounded and $v$ is the different quotient of $u$, we obtain

$$
\begin{aligned}
& \int_{\Omega}\binom{\left(\frac{\partial\left(a^{i j k l}\right.}{\partial u_{p q}}\left(M^{*}(x)\right) u_{i k}\left(x+h e_{m}\right) v_{p q}(x)+a^{i j k l}\left(D^{2} u(x)\right) v_{i k}(x)\right)}{\times\left(4 \zeta^{3} \zeta_{j} v_{l}+4 \zeta^{3} \zeta_{l} v_{j}+4 v\left(\zeta^{3} \zeta_{j l}+3 \zeta^{2} \zeta_{j} \zeta_{l}\right)\right)(x)} d x \\
& \leq C\left(|D u|,\left|D^{2} u\right|,|D \zeta|,\left|D^{2} \zeta\right|^{2}, \mid D a^{i j k l \mid}\right) \frac{1}{\varepsilon} \int_{\Omega}|D v|^{2} d x+\varepsilon \int_{\Omega} \sum_{r, s} \zeta^{4} v_{r s}^{2} d x
\end{aligned}
$$

where $\left|D a^{i j k l}\right|$ is a norm on the total derivative of the functions $a^{i j k l}$ on the space of symmetric matrices.

We conclude that by choosing $\varepsilon$ appropriately, we have

$$
\begin{aligned}
\frac{\beta}{2} \int_{\Omega} \zeta^{4} \sum_{r, s} v_{r s}^{2} d x & \leq C\left(|D u|,\left|D^{2} u\right|,|D \zeta|,\left|D^{2} \zeta\right|^{2}, \mid D a^{i j k l \mid}\right) \frac{1}{\varepsilon} \int_{\Omega}|D v|^{2} d x \\
& \leq C\|v\|_{W^{1,2}(\Omega)} \\
& \leq C\|u\|_{W^{2,2}(\Omega)}
\end{aligned}
$$

Thus

$$
\|v\|_{W^{2,2}(\{x \mid \zeta(x)=1\})} \leq C .
$$

Now this estimate is uniform in $h$ and direction $e_{m}$ so we conclude that the derivatives are in $W^{2,2}(\Omega)$ and thus $u \in W^{3,2}(\{x \mid \zeta(x)=1\})$.

Proposition 3.2. There is a bound $c(n)$ such that if

$$
\|u\|_{C^{1,1}(\Omega)} \leq c(n)
$$

for a weak solution $u$ to the Hamiltonian stationary equation (1.1), then $u \in W_{l o c}^{3,2}(\Omega)$.
Proof. First recall (cf. [Eva10, section 5.8.2]) that the Hessian $D^{2} u$ is defined almost everywhere and bounded where it is defined in terms of the $C^{1,1}$ norm. Considering (1.1) in the notation of (3.1) we have

$$
a^{i j k l}=\sqrt{g} g^{i j} \delta^{k l} .
$$

Our goal is to show that the condition (3.2) is satisfied on the set

$$
U=\left\{M \in S^{n \times n}:\|M\|_{\infty} \leq c(n)\right\} .
$$

For simplicity, we shall write $|M|$ for $\|M\|_{\infty}$, especially when Hessian is involved.

Computing, we see

$$
\begin{align*}
\frac{\partial a^{i j k l}}{\partial u_{m p}} & =\frac{1}{2} \sqrt{g} g^{a b} \frac{\partial}{\partial u_{m p}} g_{a b} g^{i j} \delta^{k l}-\sqrt{g} g^{i a} g^{b j} \frac{\partial}{\partial u_{m p}} g_{a b} \delta^{k l}  \tag{3.5}\\
& =\left(\frac{1}{2} g^{a b} g^{i j} \delta^{k l}-g^{i a} g^{b j} \delta^{k l}\right) \sqrt{g} \frac{\partial}{\partial u_{m p}} g_{a b} \\
& =\left(\frac{1}{2} g^{a b} g^{i j} \delta^{k l}-g^{i a} g^{b j} \delta^{k l}\right) \sqrt{g} \frac{\partial}{\partial u_{m p}}\left(\delta_{a b}+u_{a c} \delta^{c d} u_{d b}\right) \\
& =\left(\frac{1}{2} g^{a b} g^{i j} \delta^{k l}-g^{i a} g^{b j} \delta^{k l}\right) \sqrt{g}\left(\delta_{m p, a c} \delta^{c d} u_{d b}+u_{a c} \delta^{c d} \delta_{m p, d b}\right) .
\end{align*}
$$

In particular,

$$
\begin{equation*}
\left|\frac{\partial a^{i j k l}}{\partial u_{p q}}\left(D^{2} u\right)\right| \leq C(n)\left|D^{2} u\right|\left(1+\left|D^{2} u\right|^{2}\right)^{n / 2} . \tag{3.6}
\end{equation*}
$$

Next, note that if we let

$$
G_{i j}=\sqrt{g} g^{i j},
$$

we can write

$$
\sqrt{g} g^{i j} \delta^{k l} W_{i k} W_{j l}=\operatorname{Trace}\left(G^{T} W I_{n} W^{T}\right)
$$

But $G$ can be diagonalized by an orthogonal matrix $O$ :

$$
G^{T}=O^{T} D O
$$

where

$$
D=\sqrt{g}\left(\begin{array}{ccc}
\frac{1}{1+\lambda_{1}^{2}} & & \\
& \ddots & \\
& & \frac{1}{1+\lambda_{n}^{2}}
\end{array}\right) .
$$

Then

$$
\begin{aligned}
\sqrt{g} g^{i j} \delta^{k l} W_{i k} W_{j l} & =\operatorname{Trace}\left(O^{T} D O W W^{T}\right) \\
& =\operatorname{Trace}\left(O O^{T} D O W W^{T} O^{T}\right) \\
& =\operatorname{Trace}\left(D(O W)(O W)^{T}\right) \\
& \geq \min _{i} D_{i i} \cdot \operatorname{Trace}\left((O W)(O W)^{T}\right) \\
& =\min _{i} D_{i i}\|O W\|_{H S}^{2} \\
& =\min _{i} D_{i i}\|W\|_{H S}^{2}
\end{aligned}
$$

where we are using the Hilbert-Schmidt norm on matrices. Thus

$$
\begin{equation*}
\sqrt{g} g^{i j} \delta^{k l} W_{i k} W_{j l} \geq \frac{1}{1+c(n)^{2}}\|W\|_{H S}^{2} \tag{3.7}
\end{equation*}
$$

Combining (3.6) and (3.7) and plugging this into (3.2) we see for $M^{*}, M^{\prime}$, and $M$ in $U$ we have

$$
\begin{aligned}
& \frac{\partial a^{i j k l}}{\partial u_{p q}}\left(M^{*}\right) M_{i k}^{\prime} W_{p q} W_{j l}+a^{i j k l}(M) W_{i k} W_{j l} \\
& \geq \frac{1}{1+c(n)^{2}}\|W\|_{H S}^{2}-C(n)|c(n)|^{2}\left(1+c(n)^{2}\right)^{n / 2}\|W\|_{\infty}^{2} \\
& \geq \beta\|W\|_{H S}^{2}
\end{aligned}
$$

for some $\beta>0$, using the equivalence of norms, when $c(n)$ is chosen small. The conclusion follows from Lemma 3.1.

To extend solutions across a small set in Theorem 1.1. we shall need the following theorem of Serrin (Theorem 2 in [Ser64]).

Theorem 3.1. (Serrin) Suppose $n \geq 2$ and that $f$ is a bounded continuous weak solution to a uniformly elliptic second order divergence equation with bounded measurable coefficients on $\Omega-Q$, for an open domain $\Omega$ and $Q$ a compact subset. If $Q$ has capacity zero, then $f$ may be extended to a weak solution across the domain $\Omega$.

We now proceed to prove Theorem 1.1.
Proof. First, let us consider the case when $Q$ is the empty set. Because $u \in W_{l o c}^{3,2}(\Omega) \cap C^{1,1}(\Omega)$ we may use a standard mollification construction, letting

$$
u^{\varepsilon}=\rho_{\varepsilon} * u
$$

for an appropriate function $\rho_{\varepsilon}$ as in [Eva10, Appendix C.4]. In particular (see [Eva10, Appendix C, Theorem 6])

$$
\lim _{\varepsilon \rightarrow 0}\left\|u^{\varepsilon}-u\right\|_{W_{l o c}^{3,2}(\Omega)}=0
$$

and each $u^{\varepsilon}$ is smooth.
Now we define functionals on $C_{c}^{\infty}(\Omega)$ by

$$
\begin{aligned}
F^{\varepsilon}(\eta) & =\int_{\Omega}\left[\sqrt{g} g^{i j} \delta^{k l} u_{i k}\right]^{\varepsilon} \eta_{j l} d x \\
F(\eta) & =\int_{\Omega} \sqrt{g} g^{i j} \delta^{k l} u_{i k} \eta_{j l} d x
\end{aligned}
$$

with the notation $\left[\sqrt{g} g^{i j} \delta^{k l} u_{i k}\right]^{\varepsilon}$ meaning "constructed from $u^{\varepsilon}$ using (2.2)," (in particular, this does not mean the mollification of the expression).

First we check that for each $\eta$,

$$
F(\eta)=\lim _{\varepsilon \rightarrow 0} F^{\varepsilon}(\eta)
$$

We have

$$
\begin{aligned}
F^{\varepsilon}(\eta)-F(\eta) & =\int_{\Omega}\left(\left[\sqrt{g} g^{i j} u_{i k}\right]^{\varepsilon}-\sqrt{g} g^{i j} u_{i k}\right) \delta^{k l} \eta_{j l} d x \\
& =\int_{\Omega}\left(\left[\sqrt{g} g^{i j} u_{i k}\right]^{\varepsilon}-\left[\sqrt{g} g^{i j}\right]^{\varepsilon} u_{i k}+\left[\sqrt{g} g^{i j}\right]^{\varepsilon} u_{i k}-\sqrt{g} g^{i j} u_{i k}\right) \delta^{k l} \eta_{j l} d x \\
& =\int_{\Omega}\left(\left[\sqrt{g} g^{i j}\right]^{\varepsilon}\left(u_{i k}^{\varepsilon}-u_{i k}\right)+\left(\left[\sqrt{g} g^{i j}\right]^{\varepsilon}-\sqrt{g} g^{i j}\right) g^{i j} u_{i k}\right) \delta^{k l} \eta_{j l} d x
\end{aligned}
$$

Now because $u \in C^{1,1}$ and $\eta_{j l}$ is bounded, we simply have to check that

$$
\begin{aligned}
u_{i k}^{\varepsilon}-u_{i k} & \rightarrow 0 \text { in } L_{l o c}^{1} \\
{\left[\sqrt{g} g^{i j}\right]^{\varepsilon}-\sqrt{g} g^{i j} } & \rightarrow 0 \text { in } L_{l o c}^{1} .
\end{aligned}
$$

The first assertion is clear as $u \in W_{l o c}^{3,2}(\Omega)$.
Next,

$$
\left|\left[\sqrt{g} g^{i j}\right]^{\varepsilon}-\sqrt{g} g^{i j}\right| \leq \sup _{i, j}\left|\frac{\partial\left(\sqrt{g} g^{i j}\right)}{\partial u_{a b}}\right|\left(u_{a b}^{\varepsilon}-u_{a b}\right) .
$$

Mimicking computations following (3.5) we see

$$
\left|\frac{\partial\left(\sqrt{g} g^{i j}\right)}{\partial u_{a b}}\right| \leq C(n)\left|D^{2} u\right|\left(1+\left|D^{2} u\right|^{2}\right)^{n / 2} \leq C
$$

Thus

$$
\begin{equation*}
\left|\left[\sqrt{g} g^{i j}\right]^{\varepsilon}-\sqrt{g} g^{i j}\right| \leq C\left|D^{2} u^{\varepsilon}-D^{2} u\right| \tag{3.8}
\end{equation*}
$$

and the second assertion then follows from the first.
We conclude that

$$
F(\eta)=\lim _{\varepsilon \rightarrow 0} F^{\varepsilon}(\eta) .
$$

Next, we define functionals

$$
\begin{aligned}
G^{\varepsilon}(\eta) & =\int_{\Omega}\left[\sqrt{g} g^{i j} \theta_{i}\right]^{\varepsilon} \eta_{j} d x \\
G(\eta) & =\int_{\Omega} \sqrt{g} g^{i j} \theta_{i} \eta_{j} d x=\int_{\Omega} \sqrt{g} g^{i j} g^{a b} u_{a b i} \eta_{j} d x
\end{aligned}
$$

recalling that

$$
\theta_{i}=\left(\operatorname{Im} \log \operatorname{det}\left(I+i D^{2} u\right)\right)_{i}=g^{a b} u_{a b i}
$$

and noting that since $u \in W_{l o c}^{3,2}(\Omega)$, the third order derivatives exist almost everywhere.
Applying the first variational formulae for smooth submanifolds in section 2 to the smooth $\Gamma_{u^{\varepsilon}}$, we see that

$$
\delta F_{\Omega}(\eta)=\int_{\Omega}\left[\sqrt{g} g^{i j} \delta^{k l} u_{i k}\right]^{\varepsilon} \eta_{j l} d x=\int_{\Omega}\left[\sqrt{g} g^{i j} \theta_{i}\right]^{\varepsilon} \eta_{j} d x
$$

that is

$$
G^{\varepsilon}(\eta)=F^{\varepsilon}(\eta)
$$

So clearly, from our observations on $F^{\varepsilon}(\eta)$ we see that

$$
\lim _{\varepsilon \rightarrow 0} G^{\varepsilon}(\eta)=0
$$

All that remains is to show that

$$
\lim _{\varepsilon \rightarrow 0} G^{\varepsilon}(\eta)=G(\eta)
$$

We follow the same procedure as above:

$$
\begin{aligned}
G^{\varepsilon}(\eta)-G(\eta) & =\int_{\Omega}\left(\left[\sqrt{g} g^{i j} \theta_{i}\right]^{\varepsilon}-\sqrt{g} g^{i j} \theta_{i}\right) \eta_{j} d x \\
& =\int_{\Omega}\left(\left[\sqrt{g} g^{i j} \theta_{i}\right]^{\varepsilon}-\left[\sqrt{g} g^{i j}\right]^{\varepsilon} \theta_{i}+\left[\sqrt{g} g^{i j}\right]^{\varepsilon} \theta_{i}-\sqrt{g} g^{i j} \theta_{i}\right) \eta_{j} d x \\
& =\int_{\Omega}\left(\left[\sqrt{g} g^{i j}\right]^{\varepsilon}\left([\theta]_{i}^{\varepsilon}-\theta_{i}\right)+\left(\left[\sqrt{g} g^{i j}\right]^{\varepsilon}-\sqrt{g} g^{i j}\right) \theta_{i}\right) \eta_{j} d x
\end{aligned}
$$

where $[\theta]^{\varepsilon}$ stands for the angle function in (1.3) using $u^{\varepsilon}$. Now we have to be slightly more careful, but proceed as before: Starting with the last term, we use (3.8)

$$
\begin{aligned}
\int_{\Omega}\left(\left[\sqrt{g} g^{i j}\right]^{\varepsilon}-\sqrt{g} g^{i j}\right) \theta_{i} \eta_{j} d x & \leq\|D \theta\|_{L^{2}}\|D \eta\|_{L^{\infty}}\left\|\left[\sqrt{g} g^{i j}\right]^{\varepsilon}-\sqrt{g} g^{i j}\right\|_{L^{2}} \\
& \leq\|D \theta\|_{L^{2}}\|D \eta\|_{L^{\infty}} C\left\|D^{2} u^{\varepsilon}-D^{2} u\right\|_{L^{2}} \\
& \rightarrow 0
\end{aligned}
$$

as

$$
\|D \theta\|_{L^{2}(K)} \leq C\|u\|_{W^{3,2}(K)}
$$

for any $K$ compact inside $\Omega$. Next

$$
\begin{aligned}
& \int_{\Omega}\left[\sqrt{g} g^{i j}\right]^{\varepsilon}\left([\theta]_{i}^{\varepsilon}-\theta_{i}\right) \eta_{j} d x \\
& =\int_{\Omega}\left[\sqrt{g} g^{i j}\right]^{\varepsilon}\left(\left[g^{a b}\right]^{\varepsilon} u_{a b i}^{\varepsilon}-\left[g^{a b}\right]^{\varepsilon} u_{a b i}+\left[g^{a b}\right]^{\varepsilon} u_{a b i}-g^{a b} u_{a b i}\right) \eta_{j} d x \\
& \leq C\left(\|u\|_{C^{1,1}(\Omega)}\right)\|D \eta\|_{L^{\infty}}\left\{\left\|\left[g^{-1}\right]^{\varepsilon}\right\|_{L^{2}}\left\|D^{3} u^{\varepsilon}-D^{3} u\right\|_{L^{2}}+\left\|\left[g^{-1}\right]^{\varepsilon}-g^{-1}\right\|_{L^{2}}\left\|D^{3} u\right\|_{L^{2}}\right\}
\end{aligned}
$$

by noticing that $\left|D^{2} u^{\varepsilon}\right|$ is bounded by $\|u\|_{C^{1,1}}$ for the chosen mollifiers $\rho_{\varepsilon}$. Because $u^{\varepsilon} \rightarrow u$ in $W_{l o c}^{3,2}$, these terms go to zero.

We conclude that

$$
G(\eta)=\int_{\Omega} \sqrt{g} g^{i j} \theta_{i} \eta_{j} d x=0
$$

for all test functions $\eta$. It follows that $\theta$ is a weak solution of the uniformly elliptic equation (1.2).

When $Q$ is a compact subset in $\Omega$, because $\Omega \backslash Q$ is itself an open domain, the result established above asserts that $u \in W_{l o c}^{3,2}(\Omega \backslash Q)$ and $u$ is a weak solution to (1.2) on $\Omega \backslash Q$. This means that (1.4) holds for all $\eta$ supported in $\Omega$ away from $Q$. So $\theta$ is now in the setting of Serrin's Theorem: We can extend $\theta$ to a weak solution across the entire domain, so $u$ is a weak solution to (1.2) on $\Omega$. Next, we apply Theorem 1.2 (whose proof is independent of Theorem 1.1), where the condition (1.8) applies. We conclude that $u$ is smooth on $\Omega$. Thus, the first variation formulae yield equivalence of (1.2) and (1.1), so $u$ must be a solution of (1.1) on $\Omega$.

## 4. Lewy-Yuan rotations

In this section we discuss and motivate the Lewy-Yuan rotation. We risk giving extra descriptions here in order to give a clear motivation as to what the rotation is useful for. We also rigorously justify low regularity versions of the Lewy-Yuan rotation.

In the special Lagrangian setting, Yuan [Yua02] used the following unitary change of coordinates

$$
\begin{align*}
U & : \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}  \tag{4.1}\\
U(x+\sqrt{-1} y) & =e^{-\sqrt{-1} \pi / 4}(x+\sqrt{-1} y)
\end{align*}
$$

In this case, a surface $\Gamma$ that was the gradient graph of a convex function $u$ over the original $\mathbb{R}^{n}$-plane, is now represented as a gradient graph of a new function $\bar{u}$ over the new $\mathbb{R}^{n}$-plane, but this time with

$$
-I_{n} \leq D^{2} \bar{u} \leq I_{n}
$$

We call this a downward rotation by angle $\pi / 4$ : The word 'downward' refers to the fact that the argument of the complex number $e^{-\sqrt{-1} \pi / 4}$ (4.1) is negative. Any surface $\Gamma$ that is the gradient graph of a semi-convex function $u$ can be rotated downward ([Yua06]). If for $\beta \in(0, \pi / 2)$ we have

$$
D^{2} u \geq-\tan \beta I_{n}
$$

then we can rotate the graph downward by any positive angle $\alpha<\pi / 2-\beta$. More precisely, given

$$
\Gamma=\{(x, D u(x)), x \in \Omega\} \subset \mathbb{R}^{n}+\sqrt{-1} \mathbb{R}^{n}
$$

over $\Omega$, let

$$
\begin{equation*}
\bar{\Gamma}=U_{\alpha} \Gamma \tag{4.2}
\end{equation*}
$$

where

$$
U_{\alpha}=\left(\begin{array}{lll}
e^{-\sqrt{-1} \alpha} & &  \tag{4.3}\\
& \ddots & \\
& & e^{-\sqrt{-1} \alpha}
\end{array}\right)
$$

Clearly, $\bar{\Gamma}$ is isometric to $\Gamma$ via the unitary rotation. In coordinates, this is equivalent to the following map:

$$
\begin{align*}
& \bar{x}=\cos (\alpha) x+\sin (\alpha) D u(x)  \tag{4.4}\\
& \bar{y}=-\sin (\alpha) x+\cos (\alpha) D u(x) .
\end{align*}
$$

Here $\bar{x}$ and $\bar{y}$ are simply the projections onto $\mathbb{R}^{n}$ and $\sqrt{-1} \mathbb{R}^{n}$ of $\bar{\Gamma}$, respectively.

Considering the functions $\bar{x}(x), \bar{y}(x)$ we may compute the differential form

$$
\begin{aligned}
\sum_{i} \bar{y}^{i} d \bar{x}^{i}= & \sum_{i}\left(-\sin (\alpha) x^{i}+\cos (\alpha) u_{i}(x)\right)\left(\cos (\alpha) d x^{i}+\sin (\alpha) u_{i j}(x) d x^{j}\right) \\
= & \sum_{i}\binom{-\sin (\alpha) \cos (\alpha) x^{i} d x^{i}+\cos ^{2}(\alpha) u_{i}(x) d x^{i}}{-\sin ^{2}(\alpha) x^{i} u_{i j}(x) d x^{j}+\cos (\alpha) \sin (\alpha) u_{i}(x) u_{i j}(x) d x^{j}} \\
= & -\sin (\alpha) \cos (\alpha) D \frac{|x|^{2}}{2}+\cos ^{2}(\alpha) D u(x) \\
& -\sin ^{2}(\alpha)(D(x \cdot D u(x))-D u(x))+\cos (\alpha) \sin (\alpha) D \frac{|D u(x)|^{2}}{2} \\
= & D u+\sin (\alpha) \cos (\alpha) D \frac{|D u(x)|^{2}-|x|^{2}}{2}-\sin ^{2}(\alpha)(D(x \cdot D u)) \\
= & D\left(u(x)+\sin (\alpha) \cos (\alpha) \frac{|D u(x)|^{2}-|x|^{2}}{2}-\sin ^{2}(\alpha)((x \cdot D u(x)))\right)
\end{aligned}
$$

We see that the 1 -form $\sum_{i} \bar{y}^{i} d \bar{x}^{i}$ is exact (regardless of cohomological conditions) as we can exhibit $\bar{u}(\bar{x})=\bar{u}(\bar{x}(x))$ solving $D_{\bar{x}} \bar{u}=\bar{y} d \bar{x}^{i}$. It follows that

$$
(\bar{x}, \bar{y})=\left(\bar{x}, D_{\bar{x}} \bar{u}(\bar{x})\right)
$$

for some function $\bar{u}(\bar{x})$. The potential $\bar{u}$ is given explicitly, however, the explicit formula is only given in terms of the $x$ coordinates. Fortunately, $\bar{x}(x)$ is a change of coordinates (this follows from the semi-convexity, see Proposition 4.1 below) and is invertible.

To summarize, we have exhibited $\bar{\Gamma}$ both as the gradient graph of a function $\bar{u}$ and as an isometric image of $\Gamma$. The result will be a new graph with a potential whose Hessian satisfies (see [War16, (1.5) and (1.6)])

$$
-\tan (\beta+\alpha) I_{n} \leq D^{2} \bar{u} \leq \tan (\pi / 2-\alpha) I_{n} .
$$

The takeaway is that any semi-convexity guarantees that the graph has a representation of bounded geometry. Also note that there is nothing sacred about downward rotations: A function with a Hessian upper bound may always be rotated upwards to obtain a representation with a Hessian lower bound as well.
4.1. When $\Gamma$ is not smooth. In the above computation, we referenced the second derivatives of $u$, despite the fact that the rotation itself is actually a map on first derivatives. Our goal in this section is to rigorously show that the Lewy-Yuan rotation can be performed in some low regularity settings where the second derivatives need not exist everywhere, as long as some semi-convexity is satisfied.

For a constant $K \in \mathbb{R}$, we say that $u$ is $K$-convex on $\Omega$ if

$$
u(x)-K \frac{|x|^{2}}{2} \text { is convex. }
$$

For $u \in C^{1}$ this is equivalent to the condition that, for all $x_{0}, x_{1} \in \Omega$

$$
\begin{equation*}
\left\langle D u\left(x_{1}\right)-D u\left(x_{0}\right), x_{1}-x_{0}\right\rangle \geq K\left|x_{1}-x_{0}\right|^{2} . \tag{4.5}
\end{equation*}
$$

Proposition 4.1. Suppose that $\Gamma=(x, D u(x))$ is a Lagrangian graph in $\Omega+\sqrt{-1} \mathbb{R}^{n} \subset \mathbb{C}^{n}$ with Du continuous. Suppose that

$$
\begin{equation*}
u+(\cot (\sigma)-\varepsilon) \frac{|x|^{2}}{2} \text { is convex } \tag{4.6}
\end{equation*}
$$

for some $\varepsilon>0, \sigma>0$. Consider the function

$$
\bar{u}(x)=u(x)+\sin (\sigma) \cos (\sigma) \frac{|D u(x)|^{2}-|x|^{2}}{2}-\sin ^{2}(\sigma) D u(x) \cdot x
$$

and the function $\bar{x}: \Omega \rightarrow \bar{\Omega} \subset \mathbb{R}^{n}$ given by

$$
\begin{equation*}
\bar{x}(x)=\cos (\sigma) x+\sin (\sigma) D u(x) \tag{4.7}
\end{equation*}
$$

Then
(1) The coordinate change (4.7) is invertible with Lipschitz continuous inverse,
(2) The derivative of $\bar{u}$ in $\bar{x}$ coordinates $\frac{D \bar{u}}{d \bar{x}}$ exists everywhere, and
(3) The gradient graph $\bar{\Gamma}=(\bar{x}, D \bar{u}(\bar{x})) \subset \bar{\Omega}+\sqrt{-1} \mathbb{R}^{n} \subset \mathbb{C}^{n}$ is the isometric image of $\Gamma$ under the rotation through $\sigma$ as in (4.2).

Proof. Note that the convexity condition can be written as, for any two points $x_{0}, x_{1} \in \Omega$,

$$
\left\langle D u\left(x_{1}\right)-D u\left(x_{0}\right)+(\cot (\sigma)-\varepsilon)\left(x_{1}-x_{0}\right), x_{1}-x_{0}\right\rangle \geq 0 .
$$

This leads to

$$
\begin{equation*}
\left\langle\frac{D u\left(x_{1}\right)-D u\left(x_{0}\right)}{\left|x_{1}-x_{0}\right|}, \frac{x_{1}-x_{0}}{\left|x_{1}-x_{0}\right|}\right\rangle \geq-\cot (\sigma)+\varepsilon \tag{4.8}
\end{equation*}
$$

It then follows, for $x_{1} \neq x_{0}$, that

$$
\begin{align*}
\left|\frac{\bar{x}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)}{\left|x_{1}-x_{0}\right|}\right| & \geq\left\langle\frac{\bar{x}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)}{\left|x_{1}-x_{0}\right|}, \frac{x_{1}-x_{0}}{\left|x_{1}-x_{0}\right|}\right\rangle  \tag{4.9}\\
& =\left\langle\frac{\cos (\sigma)\left(x_{1}-x_{0}\right)+\sin (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)}{\left|x_{1}-x_{0}\right|}, \frac{x_{1}-x_{0}}{\left|x_{1}-x_{0}\right|}\right\rangle \\
& =\cos (\sigma)+\sin (\sigma)\left\langle\frac{D u\left(x_{1}\right)-D u\left(x_{0}\right)}{\left|x_{1}-x_{0}\right|}, \frac{x_{1}-x_{0}}{\left|x_{1}-x_{0}\right|}\right\rangle \\
& \geq \cos (\sigma)-\cot (\sigma) \sin (\sigma)+\sin (\sigma) \varepsilon \\
& =\sin (\sigma) \varepsilon
\end{align*}
$$

using (4.8). Therefore the continuous map $\bar{x}$ is invertible and its inverse is Lipschitz continuous with a Lipschitz constant $1 /(\sin (\sigma) \varepsilon)$.

Next, for the gradient of $\bar{u}$ in terms of $\bar{x}$, we shall compute a difference quotient

$$
\bar{u}_{\bar{j}}\left(\bar{x}_{0}\right)=\lim _{h \rightarrow 0} \frac{\bar{u}\left(\bar{x}_{0}+h \bar{e}_{j}\right)-\bar{u}\left(\bar{x}_{0}\right)}{h} .
$$

Since $\bar{x}$ is invertible, for $\bar{x}_{0} \in \bar{\Omega}$ we may solve, for small fixed $h$

$$
\begin{aligned}
& \bar{x}\left(x_{0}\right)=\bar{x}_{0} \\
& \bar{x}\left(x_{h}\right)=\bar{x}_{0}+h \bar{e}_{j}
\end{aligned}
$$

that is

$$
\begin{aligned}
& \cos (\sigma) x_{0}+\sin (\sigma) D u\left(x_{0}\right)=\bar{x}_{0} \\
& \cos (\sigma) x_{h}+\sin (\sigma) D u\left(x_{h}\right)=\bar{x}_{h}=\bar{x}_{0}+h \bar{e}_{j} .
\end{aligned}
$$

Let

$$
\vec{v}=x_{h}-x_{0} .
$$

Then $\vec{v}$ will satisfy

$$
\begin{equation*}
\cos (\sigma) \vec{v}+\sin (\sigma)\left[D u\left(x_{h}\right)-D u\left(x_{0}\right)\right]=h \bar{e}_{j} . \tag{4.10}
\end{equation*}
$$

Since $\vec{v} \neq 0$ for $h \neq 0$, there is a unique $\vec{V}$ with

$$
\vec{v}=h \vec{V}
$$

while the vector $\vec{V}$ depends on $h$, we suppress this dependence. Observe that

$$
|\vec{V}|=\frac{|\vec{v}|}{h}=\frac{\left|x_{h}-x_{0}\right|}{\left|\bar{x}\left(x_{h}\right)-\bar{x}\left(x_{0}\right)\right|} \leq \frac{1}{\varepsilon \sin \sigma}
$$

by (4.9). In particular, $\vec{V}$ is a bounded vector. The function $\bar{u}$ is given in term of $x$ coordinates, so in order to evaluate it, we have to use the change of coordinates, that is

$$
\bar{u}\left(\bar{x}_{0}\right)=\bar{u}\left(\bar{x}^{-1}\left(\bar{x}_{0}\right)\right)=\bar{u}\left(x_{0}\right) .
$$

So we may compute the difference quotient of $\bar{u}$ in terms of $x$

$$
\begin{aligned}
\frac{\bar{u}\left(\bar{x}_{h}\right)-\bar{u}\left(\bar{x}_{0}\right)}{h}= & \frac{\bar{u}\left(\bar{x}^{-1}\left(\bar{x}_{h}\right)\right)-\bar{u}\left(\bar{x}^{-1}\left(\bar{x}_{0}\right)\right)}{h} \\
= & \frac{u\left(x_{h}\right)-u\left(x_{0}\right)}{h}+\sin (\sigma) \cos (\sigma) \frac{\left|D u\left(x_{h}\right)\right|^{2}-\left|D u\left(x_{0}\right)\right|^{2}-\left|x_{h}\right|^{2}+\left|x_{0}\right|^{2}}{2 h} \\
& -\frac{1}{h} \sin ^{2}(\sigma)\left(D u\left(x_{h}\right)-D u\left(x_{0}\right)\right) \cdot\left(x_{0}+h \vec{V}\right)-\frac{1}{h} \sin ^{2}(\sigma) D u\left(x_{0}\right) \cdot\left(\left(x_{0}+h \vec{V}\right)-x_{0}\right) \\
= & \frac{u\left(x_{0}+h \vec{V}\right)-u\left(x_{0}\right)}{h}-\sin ^{2}(\sigma) D u\left(x_{0}\right) \cdot \vec{V} \\
& +\cos (\sigma) \frac{\left[\sin (\sigma)\left(D u\left(x_{0}+h \vec{V}\right)-D u\left(x_{0}\right)\right)\right]\left[D u\left(x_{0}+h \vec{V}\right)+D u\left(x_{0}\right)\right]}{2 h} \\
& -\sin (\sigma) \cos (\sigma)\left(x_{0} \cdot \vec{V}+\frac{h}{2}|\vec{V}|^{2}\right) \\
& -\frac{1}{h} \sin (\sigma)\left[\sin (\sigma)\left(D u\left(x_{0}+h \vec{V}\right)-D u\left(x_{0}\right)\right)\right] \cdot\left(x_{0}+h \vec{V}\right) .
\end{aligned}
$$

Rewriting (4.10) as

$$
\begin{equation*}
\sin (\sigma)\left[D u\left(x_{h}\right)-D u\left(x_{0}\right)\right]=h \bar{e}_{j}-\cos (\sigma) h \vec{V} \tag{4.11}
\end{equation*}
$$

we see

$$
\begin{aligned}
\frac{\bar{u}\left(\bar{x}_{h}\right)-\bar{u}\left(\bar{x}_{0}\right)}{h} & =\frac{u\left(x_{0}+h \vec{V}\right)-u\left(x_{0}\right)}{h}-\sin ^{2}(\sigma) D u\left(x_{0}\right) \cdot \vec{V} \\
& +\cos (\sigma) \frac{\left[h \bar{e}_{j}-\cos (\sigma) h \vec{V}\right]\left[D u\left(x_{0}+h \vec{V}\right)+D u\left(x_{0}\right)\right]}{2 h} \\
& -\sin (\sigma) \cos (\sigma)\left(x_{0} \cdot \vec{V}+\frac{h}{2}|\vec{V}|^{2}\right)-\frac{1}{h} \sin (\sigma)\left[h \bar{e}_{j}-\cos (\sigma) h \vec{V}\right] \cdot\left(x_{0}+h \vec{V}\right) \\
& =\frac{u\left(x_{0}+h \vec{V}\right)-u\left(x_{0}\right)}{h}-\sin ^{2}(\sigma) D u\left(x_{0}\right) \cdot \vec{V} \\
& +\cos (\sigma) \frac{1}{2}\left[\bar{e}_{j}-\cos (\sigma) \vec{V}\right]\left[2 D u\left(x_{0}\right)+\frac{h \bar{e}_{j}-\cos (\sigma) h \vec{V}}{\sin (\sigma)}\right] \\
& -\sin (\sigma) \cos (\sigma)\left(x_{0} \cdot \vec{V}+\frac{h}{2}|\vec{V}|^{2}\right)-\sin (\sigma)\left[\bar{e}_{j}-\cos (\sigma) \vec{V}\right] \cdot\left(x_{0}+h \vec{V}\right) \\
& =\frac{u\left(x_{0}+h \vec{V}\right)-u\left(x_{0}\right)}{h}-\sin ^{2}(\sigma) D u\left(x_{0}\right) \cdot \vec{V} \\
& +\cos (\sigma)\left[\bar{e}_{j}-\cos (\sigma) \vec{V}\right] \cdot D u\left(x_{0}\right)+\frac{h}{2} \frac{\cos (\sigma)}{\sin (\sigma)}\left|\bar{e}_{j}-\cos (\sigma) \vec{V}\right|^{2} \\
& -\sin (\sigma) \cos (\sigma) x_{0} \cdot \vec{V}-\sin (\sigma) \cos (\sigma) \frac{h}{2}|\vec{V}|^{2}-\sin (\sigma) \bar{e}_{j} \cdot x_{0}-h \sin (\sigma) \bar{e}_{j} \cdot \vec{V} \\
& +\sin (\sigma) \cos (\sigma) x_{0} \cdot \vec{V}+h \sin (\sigma) \cos (\sigma)|\vec{V}|^{2} \\
& =\frac{u\left(x_{0}+h \vec{V}\right)-u\left(x_{0}\right)}{h}-\sin ^{2}(\sigma) D u\left(x_{0}\right) \cdot \vec{V} \\
& +\cos (\sigma) \bar{e}_{j} \cdot D u\left(x_{0}\right)-\cos ^{2}(\sigma) D u\left(x_{0}\right) \cdot \vec{V}-\sin (\sigma) \bar{e}_{j} \cdot x_{0} \\
& +h\left[\begin{array}{r}
\left.\frac{\cos (\sigma)}{\sin (\sigma)} \frac{1}{2}\left|\bar{e}_{j}-\cos (\sigma) \vec{V}\right|^{2}-\sin (\sigma) \cos (\sigma) \frac{1}{2}|\vec{V}|^{2}\right] \\
\\
\end{array}+\operatorname{Du(x^{*})\cdot V-Du(x_{0})\cdot \vec {V}+\operatorname {sin}(\sigma )\overline {e}_{j}\cdot \vec {V}+\operatorname {sin}(\sigma )\operatorname {cos}(\sigma )|\vec {V}|^{2}}\right] \\
& +\left[\frac{\left.\cos (\sigma) \frac{1}{\sin (\sigma)} \frac{2}{2}\left|\bar{e}_{j}-\cos (\sigma) \vec{V}\right|^{2}-\sin (\sigma) \cos (\sigma) \frac{1}{2}|\vec{V}|^{2}\right]}{-\sin (\sigma) \bar{e}_{j} \cdot \vec{V}+\sin (\sigma) \cos (\sigma)|\vec{V}|^{2}}\right]
\end{aligned}
$$

where $x^{*}$ is some value between $x_{0}+h \vec{V}$ and $x_{0}$ obtained by the mean value theorem. Now we may take a limit with $h$ vanishing. Because $\vec{V}$ (which a priori can point in many directions) is bounded, the $h$-term vanishes in the limit. Because $D u$ is continuous, and $x(\bar{x})$ is Lipschitz, we also have that

$$
\lim _{h \rightarrow 0}\left|\left(D u\left(x^{*}\right)-D u\left(x_{0}\right)\right) \cdot V\right| \leq \lim _{h \rightarrow 0} \sup \left|D u\left(x^{*}\right)-D u\left(x_{0}\right)\right||V|=0 .
$$

We are left with

$$
\begin{equation*}
\lim _{h \rightarrow 0} \frac{\bar{u}\left(\bar{x}_{0}+h \bar{e}_{j}\right)-\bar{u}\left(\bar{x}_{0}\right)}{h}=\cos (\sigma) u_{j}\left(x_{0}\right)-\sin (\sigma) x_{0}^{j} . \tag{4.12}
\end{equation*}
$$

This is precisely the $\bar{y}$-component of the image of the rotation (4.4). It follows that the gradient graph of $\bar{u}$ exists everywhere and is isometric to the gradient graph of $u$.

Corollary 4.2. An analogous result holds when $u$ is semi-concave, and $\sigma$ is negative. The rotations through $\sigma$ and $-\sigma$ are inverse operations where they are defined, up to an additive constant in the potential function.

Proof. While we could claim a proof that is formally the same as the proof of Proposition 4.1, we offer an alternative argument based on the fact that, whenever $u$ is semi-concave, $-u$ must be semi-convex. Starting with a semi-convex $-u$, we may rotate the graph $\Gamma_{-u}$ by a downward rotation through $-\sigma$, applying Proposition 4.1, and then take the complex conjugate of the result in $\mathbb{C}^{n}$. This follows from the fact that, as operators on $\mathbb{C}^{n}\left(\mathbb{R}\right.$-linear on $\left.\mathbb{R}^{2 n}\right)$ for any diagonal unitary matrix $U$ we have

$$
c \circ U \circ c=U^{-1}=U^{*}
$$

where $c$ is the $\mathbb{R}$-linear complex conjugation map on $\mathbb{R}^{2 n}$, that is

$$
c(x+\sqrt{-1} y)=x-\sqrt{-1} y
$$

In particular, taking $-\overline{(-u)}$ via rotation of $-u$ (not complex conjugation), we obtain the potential $\bar{u}$ for the graph rotated through a negative angle $-\sigma$.

The following technical result is useful when we approximate $u$ while keeping $K$-convexity.
Lemma 4.3. Let $u^{\varepsilon}$ be a standard mollification of $u$. If $u$ is $K$-convex on $\Omega$, then so is $u^{\varepsilon}$ on

$$
\begin{equation*}
\mathbf{\Omega}^{\varepsilon}=\{x: d(x, \partial \Omega)>\varepsilon\} . \tag{4.13}
\end{equation*}
$$

Proof. Consider a mollifier $\phi$ that is radial, supported in $B_{\varepsilon}(0)$ and has unit integral. Given a point $x \in \Omega^{\varepsilon}$,

$$
\begin{aligned}
u^{\varepsilon}(x) & =\int_{\Omega} \phi(x-y) u(y) d y \\
& =\int_{B_{\varepsilon}(x)} \phi(x-y) u(y) d y \\
& =\int_{B_{\varepsilon}(0)} \phi(z) u(x+z) d z
\end{aligned}
$$

so we have

$$
D u^{\varepsilon}(x)=\int_{B_{\varepsilon}(0)} \phi(z) D u(x+z) d z
$$

Now consider, for $x_{1}, x_{0} \in \Omega^{\varepsilon}$, the expression

$$
\begin{aligned}
& \left\langle D u^{\varepsilon}\left(x_{1}\right)-D u^{\varepsilon}\left(x_{0}\right), x_{1}-x_{0}\right\rangle \\
& =\left\langle\int_{B_{\varepsilon}(0)} \phi(z) D u\left(x_{1}+z\right) d z-\int_{B_{\varepsilon}(0)} \phi(z) D u\left(x_{0}+z\right) d z, x_{1}-x_{0}\right\rangle \\
& =\int_{B_{\varepsilon}(0)}\left\langle\phi(z)\left(D u\left(x_{1}+z\right)-D u\left(x_{0}+z\right)\right), x_{1}-x_{0}\right\rangle d z \\
& =\int_{B_{\varepsilon}(0)} \phi(z)\left\langle D u\left(x_{1}+z\right)-D u\left(x_{0}+z\right),\left(x_{1}+z\right)-\left(x_{0}+z\right)\right\rangle d z \\
& \geq \int_{B_{\varepsilon}(0)} \phi(z) K\left|x_{1}-x_{0}\right|^{2} d z \\
& =K\left|x_{1}-x_{0}\right|^{2} .
\end{aligned}
$$

Proposition 4.4. Suppose that $u$ is $\tan (\kappa)$-convex and $C^{1}$ and $\bar{u}$ is obtained as in Proposition 4.1. If $\kappa, \sigma, \kappa-\sigma \in(-\pi / 2, \pi / 2)$, then $\bar{u}$ is $\tan (\kappa-\sigma)$-convex.

Proof. We define the following functions

$$
\begin{aligned}
& \bar{x}_{\varepsilon}=\cos (\sigma) x+\sin (\sigma) D u^{\varepsilon}(x) \\
& \bar{y}_{\varepsilon}=-\sin (\sigma) x+\cos (\sigma) D u^{\varepsilon}(x)
\end{aligned}
$$

Note that, as before, the set

$$
\bar{\Gamma}_{\varepsilon}=\left\{\left(\bar{x}_{\varepsilon}(x), \bar{y}_{\varepsilon}(x)\right): x \in \Omega\right\}
$$

is the rotation of the gradient graph of $u^{\varepsilon}$ through angle $\sigma$. (To be clear, we are not taking the gradient graph of the mollified rotated function, rather we are rotating the gradient graph of the mollified function.)

Now $D u$ is continuous, so the mollified derivatives $D u^{\varepsilon}$ will converge locally uniformly to $D u$ as $\varepsilon \rightarrow 0$ (cf. [Eva10, Appendix C, Theorem 6]). It follows that the functions $\bar{x}_{\varepsilon}$ and $\bar{y}_{\varepsilon}$ will also converge locally uniformly, to $\bar{x}$ and $\bar{y}$ respectively, as functions of $x$, where

$$
\begin{aligned}
& \bar{x}=\cos (\sigma) x+\sin (\sigma) D u(x) \\
& \bar{y}=-\sin (\sigma) x+\cos (\sigma) D u(x) .
\end{aligned}
$$

We have seen in Proposition 4.1 that

$$
\bar{\Gamma}=\{(\bar{x}(x), \bar{y}(x)): x \in \Omega\}
$$

is precisely the gradient graph of the function $\bar{u}$ over $\bar{\Omega}$. The semi-convexity condition (4.5) on $\bar{u}$ that we are trying to show is

$$
\left\langle\bar{y}\left(x_{1}\right)-\bar{y}\left(x_{0}\right), \bar{x}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)\right\rangle \geq \tan (\kappa-\sigma)\left|\bar{x}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)\right|^{2} .
$$

We claim that

$$
\begin{equation*}
\left\langle\bar{y}_{\varepsilon}\left(x_{1}\right)-\bar{y}\left(x_{0}\right), \bar{x}_{\varepsilon}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)\right\rangle \geq \tan (\kappa-\sigma)\left|\bar{x}_{\varepsilon}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)\right|^{2} \tag{4.14}
\end{equation*}
$$

for all $\varepsilon>0$. The local uniform convergence of $\bar{x}_{\varepsilon}$ and $\bar{y}_{\varepsilon}$ will then give us the result. To show (4.14), we start by computing the Jacobian of the map $\bar{x}_{\varepsilon}$ :

Since $u^{\varepsilon}$ is smooth

$$
\frac{d \bar{x}_{\varepsilon}}{d x}=\cos (\sigma) I_{n}+\sin (\sigma) D^{2} u^{\varepsilon}(x) .
$$

By assumption, $u$ is $\tan (\kappa)$-convex, and hence so is $u^{\varepsilon}$, by Lemma 4.3, at least on $\Omega^{\varepsilon}$ (recall (4.13)). It follows that

$$
D^{2} u^{\varepsilon}(x) \geq \tan (\kappa) I_{n} .
$$

So

$$
\begin{aligned}
\frac{d \bar{x}_{\varepsilon}}{d x} & \geq \cos (\sigma) I_{n}+\sin (\sigma) \tan (\kappa) I_{n} \\
& =\frac{\cos (\sigma-\kappa)}{\cos (\kappa)} I_{n}>0
\end{aligned}
$$

since $\kappa$ and $\sigma-k \in(-\pi / 2, \pi / 2)$. The coordinate change is invertible and the Jacobian can be computed

$$
\frac{d x}{d \bar{x}_{\varepsilon}}=\left(\cos (\sigma) I_{n}+\sin (\sigma) D^{2} u^{\varepsilon}(x)\right)^{-1}
$$

Next

$$
D \bar{y}_{\varepsilon}=\left(-\sin (\sigma) I_{n}+\cos (\sigma) D^{2} u^{\varepsilon}(x)\right)
$$

Now each $\bar{\Gamma}_{\varepsilon}$ is the gradient graph of a function $\bar{u}_{\varepsilon}\left(\bar{x}_{\varepsilon}\right)$ on the region $\bar{x}_{\varepsilon}(\Omega)$. In order to compute the Hessian of $\bar{u}_{\varepsilon}$ in terms of $\bar{x}_{\varepsilon}$, we compute

$$
\begin{aligned}
D_{\bar{x}_{\varepsilon}}^{2} \bar{u}_{\varepsilon} & =D_{x} \bar{y}_{\varepsilon} \cdot \frac{d x}{d \bar{x}_{\varepsilon}}=D_{\bar{x}_{\varepsilon}} \bar{y}_{\varepsilon} \\
& =\left(-\sin (\sigma) I_{n}+\cos (\sigma) D^{2} u^{\varepsilon}(x)\right)\left(\cos (\sigma) I_{n}+\sin (\sigma) D^{2} u^{\varepsilon}(x)\right)^{-1}
\end{aligned}
$$

At any point, we may diagonalize the expression for $D_{\bar{x}^{\varepsilon}}^{2} \bar{u}_{\varepsilon}(\bar{x})$ by diagonalizing $D^{2} u^{\varepsilon}(x(\bar{x}))$ :

$$
D_{\bar{x}^{\varepsilon}}^{2} \bar{u}_{\varepsilon}=\left(\begin{array}{ccc}
\frac{-\sin (\sigma)+\cos (\sigma) \lambda_{1}}{\cos (\sigma)+\sin (\sigma) \lambda_{1}} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \frac{-\sin (\sigma)+\cos (\sigma) \lambda_{n}}{\cos (\sigma)+\sin (\sigma) \lambda_{n}}
\end{array}\right)=\left(\begin{array}{ccc}
\bar{\lambda}_{1} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \bar{\lambda}_{n}
\end{array}\right) .
$$

Now

$$
\bar{\lambda}_{j}=\frac{-\sin (\sigma)+\cos (\sigma) \lambda_{j}}{\cos (\sigma)+\sin (\sigma) \lambda_{j}}=\frac{-\frac{\sin (\sigma)}{\cos (\sigma)}+\lambda_{j}}{1+\frac{\sin (\sigma)}{\cos (\sigma)} \lambda_{j}}=\tan \left(-\sigma+\arctan \left(\lambda_{j}\right)\right)
$$

Because

$$
\arctan \left(\lambda_{j}\right) \geq \kappa
$$

we conclude that

$$
\bar{\lambda}_{j} \geq \tan (-\sigma+\kappa)
$$

and $D_{\bar{x}_{\varepsilon}}^{2} \bar{u}_{\varepsilon}$ is $\tan (-\sigma+\kappa)$-convex, that is

$$
\begin{equation*}
\left\langle D_{\bar{x}_{\varepsilon}} \bar{u}_{\varepsilon}\left(x_{1}\right)-D_{\bar{x}_{\varepsilon}} \bar{u}_{\varepsilon}\left(x_{0}\right), \bar{x}_{\varepsilon}\left(x_{1}\right)-\bar{x}_{\varepsilon}\left(x_{0}\right)\right\rangle \geq \tan (-\sigma+\kappa)\left|\bar{x}_{\varepsilon}\left(x_{1}\right)-\bar{x}_{\varepsilon}\left(x_{0}\right)\right|^{2} \tag{4.15}
\end{equation*}
$$

or

$$
\begin{equation*}
\left\langle\bar{y}_{\varepsilon}\left(x_{1}\right)-\bar{y}_{\varepsilon}\left(x_{0}\right), \bar{x}_{\varepsilon}\left(x_{1}\right)-\bar{x}_{\varepsilon}\left(x_{0}\right)\right\rangle \geq \tan (-\sigma+\kappa)\left|\bar{x}_{\varepsilon}\left(x_{1}\right)-\bar{x}_{\varepsilon}\left(x_{0}\right)\right|^{2} \tag{4.16}
\end{equation*}
$$

provided that $x_{1}$ and $x_{0}$ are at least $\varepsilon$ away from the boundary of $\Omega$. By the local uniform convergence, we conclude that

$$
\begin{equation*}
\left\langle\bar{y}\left(x_{1}\right)-\bar{y}\left(x_{0}\right), \bar{x}\left(x_{1}\right)-\bar{x}\left(x_{1}\right)\right\rangle \geq \tan (-\sigma+\kappa)\left|\bar{x}\left(x_{1}\right)-\bar{x}\left(x_{1}\right)\right|^{2} \tag{4.17}
\end{equation*}
$$

that is, $\bar{u}$ is $\tan (\kappa-\sigma)$-convex.

The following is an observation on how semi-convexity can lead to bounded geometry, even when the potential is not given as being twice differentiable.

Corollary 4.5. Suppose that $u \in C^{1}$ and is semi-convex. Then the gradient graph of $u$ is isometric to the gradient graph of a $C^{1,1}$ function.

Proof. Choose $\sigma \in(0, \pi / 2)$ and $\varepsilon>0$ for which (4.6) is satisfied. Now to control the $C^{1,1}$ norm of $\bar{u}$ we note that

$$
\begin{aligned}
\|\bar{u}\|_{C^{1,1}(\bar{\Omega})} & =\sup _{\bar{x}_{0}, \bar{x}_{1} \in \bar{\Omega}} \frac{\left|D \bar{u}\left(\bar{x}_{1}\right)-D \bar{u}\left(\bar{x}_{0}\right)\right|}{\left|\bar{x}_{1}-\bar{x}_{0}\right|} \\
& =\sup _{x_{0}, x_{1} \in \Omega} \frac{\left|\bar{y}\left(x_{1}\right)-\bar{y}\left(x_{0}\right)\right|}{\left|\bar{x}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)\right|} .
\end{aligned}
$$

So for any pair $x_{0}, x_{1} \in \Omega$

$$
\begin{aligned}
\frac{\left|\bar{y}\left(x_{1}\right)-\bar{y}\left(x_{0}\right)\right|}{\left|\bar{x}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)\right|} & =\frac{\left|\cos (\sigma) D u\left(x_{1}\right)-\sin (\sigma) x_{1}-\cos (\sigma) D u\left(x_{0}\right)+\sin (\sigma) x_{0}\right|}{\left|\cos (\sigma) x_{1}+\sin (\sigma) D u\left(x_{1}\right)-\cos (\sigma) x_{0}+\sin (\sigma) D u\left(x_{0}\right)\right|} \\
& =\frac{\left|\cos (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)-\sin (\sigma)\left(x_{1}-x_{0}\right)\right|}{\left|\cos (\sigma)\left(x_{1}-x_{0}\right)+\sin (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)\right|} .
\end{aligned}
$$

To show this is bounded, we explore two cases. Let $A=2 \cot (\sigma)>0$. The first case is when

$$
\begin{equation*}
\left|D u\left(x_{1}\right)-D u\left(x_{0}\right)\right| \leq A\left|x_{1}-x_{0}\right| . \tag{4.18}
\end{equation*}
$$

Recall $\sigma \in(0, \pi / 2)$, we have

$$
\frac{\left|\cos (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)-\sin (\sigma)\left(x_{1}-x_{0}\right)\right|}{\left|\cos (\sigma)\left(x_{1}-x_{0}\right)+\sin (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)\right|} \leq \frac{|\cos (\sigma) A| x_{1}-x_{0}|+\sin (\sigma)| x_{1}-x_{0}| |}{\left|\cos (\sigma)\left(x_{1}-x_{0}\right)+\sin (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)\right|}
$$ and

$$
\begin{aligned}
& \left\langle\cos (\sigma)\left(x_{1}-x_{0}\right)+\sin (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right), \frac{x_{1}-x_{0}}{\left|x_{1}-x_{0}\right|}\right\rangle \\
& =\cos (\sigma)\left|x_{1}-x_{0}\right|+\left\langle\sin (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right), \frac{x_{1}-x_{0}}{\left|x_{1}-x_{0}\right|}\right\rangle \\
& \geq \cos (\sigma)\left|x_{1}-x_{0}\right|+\sin (\sigma)\left|x_{1}-x_{0}\right|(-\cot (\sigma)+\varepsilon) \\
& =\sin (\sigma)\left|x_{1}-x_{0}\right| \varepsilon
\end{aligned}
$$

where we used (4.8) in the second line. Thus (4.18) leads to

$$
\frac{\left|\bar{y}\left(x_{1}\right)-\bar{y}\left(x_{0}\right)\right|}{\left|\bar{x}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)\right|} \leq\left|\frac{\cos (\sigma) A+\sin (\sigma)}{\sin (\sigma) \varepsilon}\right|=\frac{\cos ^{2}(\sigma)+1}{\sin ^{2}(\sigma)} \frac{1}{\varepsilon} .
$$

The next case is when

$$
\begin{equation*}
\left|D u\left(x_{1}\right)-D u\left(x_{0}\right)\right| \geq A\left|x_{1}-x_{0}\right| . \tag{4.19}
\end{equation*}
$$

Then by the triangle inequality and (4.19)

$$
\begin{aligned}
\left|\cos (\sigma)\left(x_{1}-x_{0}\right)+\sin (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)\right| & \geq \sin (\sigma)\left|D u\left(x_{1}\right)-D u\left(x_{0}\right)\right|-\cos (\sigma)\left|x_{1}-x_{0}\right| \\
& \geq\left(\sin (\sigma)-\frac{\cos (\sigma)}{A}\right)\left|D u\left(x_{1}\right)-D u\left(x_{0}\right)\right| \\
& =\frac{1}{2} \sin (\sigma)\left|D u\left(x_{1}\right)-D u\left(x_{0}\right)\right|
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\left|\cos (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)-\sin (\sigma)\left(x_{1}-x_{0}\right)\right|}{\left|\cos (\sigma)\left(x_{1}-x_{0}\right)+\sin (\sigma)\left(D u\left(x_{1}\right)-D u\left(x_{0}\right)\right)\right|} & \leq \frac{\cos (\sigma)\left|D u\left(x_{1}\right)-D u\left(x_{0}\right)\right|+\sin (\sigma) \frac{\left|D u\left(x_{1}\right)-D u\left(x_{0}\right)\right|}{A}}{\frac{1}{2} \sin (\sigma)\left|D u\left(x_{1}\right)-D u\left(x_{0}\right)\right|} \\
& =\frac{\cos ^{2}(\sigma)+1}{\sin (\sigma) \cos (\sigma)} .
\end{aligned}
$$

In either case, we have

$$
\frac{\left|\bar{y}\left(x_{1}\right)-\bar{y}\left(x_{0}\right)\right|}{\left|\bar{x}\left(x_{1}\right)-\bar{x}\left(x_{0}\right)\right|} \leq \max \left\{\frac{\cos ^{2}(\sigma)+1}{\sin ^{2}(\sigma)} \frac{1}{\varepsilon}, \frac{\cos ^{2}(\sigma)+1}{\sin (\sigma) \cos (\sigma)}\right\}=C
$$

and $\bar{u}$ is $C^{1,1}$.

The following corollary is immediate from the above by applying the De Giorgi-Nash theorem.

Corollary 4.6. Suppose that $u \in C^{1}$ is a semi-convex weak solution to (1.2). Then the phase $\theta$ enjoys interior Hölder estimates (with respect to the metric distances) on $\Gamma_{u}$.

Finally, we show that smoothness and strong semi-concavity estimates on the rotated potential can be used to conclude smoothness on $u$.

Proposition 4.7. Suppose that $u$ and $\bar{u}$ are as in Proposition 4.1 and $\bar{u} \in C^{2}(\bar{\Omega})$. Suppose also that for some constant $\epsilon>0$

$$
\begin{equation*}
D_{\bar{x}}^{2} \bar{u} \leq\left(\frac{\cos (\sigma)}{\sin (\sigma)}-\epsilon\right) I_{n} \tag{4.20}
\end{equation*}
$$

Then for any integer $k>1$

$$
\left\|D^{k} u\right\|_{L^{\infty}(\Omega)} \leq C(\sigma, \epsilon, n)\left(\left\|D^{k} \bar{u}\right\|_{L^{\infty}(\bar{\Omega})},\left\|D^{k-1} u\right\|_{L^{\infty}(\Omega)}\right) .
$$

Proof. The function $\bar{u}$ was obtained by a downward rotation of $\sigma$ from $u$, so $u$ may be obtained by the inverse rotation. In particular as $\bar{u} \in C^{2}(\bar{\Omega})$, the change of variable formulae hold on $\bar{\Omega}$ :

$$
\begin{aligned}
& x=\cos (\sigma) \bar{x}-\sin (\sigma) D_{\bar{x}} \bar{u}(\bar{x}) \\
& y=\sin (\sigma) \bar{x}+\cos (\sigma) D_{\bar{x}} \bar{u}(\bar{x}) .
\end{aligned}
$$

Differentiating the first formula leads to

$$
\frac{d x}{d \bar{x}}=\cos (\sigma) I_{n}-\sin (\sigma) D_{\bar{x}}^{2} \bar{u}(\bar{x})
$$

and noting that

$$
y=D_{x} u(x)=D_{x} u(x(\bar{x}))
$$

we have

$$
D_{x} u(\bar{x})=\sin (\sigma) \bar{x}+\cos (\sigma) D_{\bar{x}} \bar{u}(\bar{x}) .
$$

Now

$$
\begin{aligned}
D_{x}^{2} u & =D_{x} D_{x} u \\
& =D_{x}\left(\sin (\sigma) \bar{x}+\cos (\sigma) D_{\bar{x}} \bar{u}(\bar{x})\right) \\
& =\left(\sin (\sigma) I_{n}+\cos (\sigma) D_{\bar{x}}^{2} \bar{u}(\bar{x})\right) \frac{d \bar{x}}{d x} .
\end{aligned}
$$

Noting (4.20), we may invert (4.1) and conclude

$$
\begin{align*}
D_{x}^{2} u(\bar{x}) & =\left(\sin (\sigma) I+\cos (\sigma) D_{\bar{x}}^{2} \bar{u}(\bar{x})\right) \cdot\left(\cos (\sigma) I_{n}-\sin (\sigma) D_{\bar{x}}^{2} \bar{u}(\bar{x})\right)^{-1}  \tag{4.21}\\
& :=F_{\sigma}\left(D_{\bar{x}}^{2} \bar{u}(\bar{x}(x))\right) .
\end{align*}
$$

First, we shall show that if $D_{\bar{x}}^{3} \bar{u}$ exists, then so will $D_{x}^{3} u(x)$. To do this we differentiate (4.21) in $x$, obtaining

$$
\begin{aligned}
D_{x} D_{x}^{2} u(x) & =D_{x} F_{\sigma}\left(D_{\bar{x}}^{2} \bar{u}(\bar{x}(x))\right) \\
& =\frac{d F_{\sigma}}{d D_{\bar{x}}^{2} \bar{u}} \cdot \frac{d D_{\bar{x}}^{2} \bar{u}}{d \bar{x}} \cdot \frac{d \bar{x}}{d x} .
\end{aligned}
$$

Combining (4.20), the assumption that $D_{\bar{x}}^{3} \bar{u}$ exists, and the fact that all of these factors are well-defined and bounded, we conclude that $D_{x}^{3} u$ exists and is controlled in terms of $D_{\bar{x}}^{3} \bar{u}$.

Higher order estimates follow in the same way inductively.

## 5. Proof of Theorem 1.2

Proof. We are assuming that the function $\theta$ is a weak solution to a divergence type equation (1.2) on the set $\mathbb{B}_{1}(0) \backslash Q$. Because the conditions (1.6), (1.7) and (1.8) each guarantee uniform ellipticity of the Laplace equation, we may immediately apply Theorem 3.1 and conclude that $\theta$ is a weak solution over the whole ball $\mathbb{B}_{1}(0)$.

Recall that

$$
F\left(D^{2} u\right)=F\left(\lambda_{1}, \cdots, \lambda_{n}\right)=\sum_{i=1}^{n} \arctan \lambda_{i} .
$$

To begin, we claim that if either of the conditions (1.6) or (1.7) holds, then for $u$ satisfying

$$
F\left(D^{2} u\right)=\theta
$$

it follows that $u$ is a solution to a concave equation.
For the case $\theta \geq \delta+\frac{\pi}{2}(n-2)$, we recall that by [Yua06, Lemma 2.1] (see also [CNS85, section 8]) the level sets of $F$, at any level $c$ with $|c| \geq \frac{\pi}{2}(n-2)$, are convex. We have a uniform bound $\left|D^{2} u\right| \leq C_{0}$ wherever the Hessian exists, so we may find a compact set $\mathcal{K} \subset S^{n \times n}$, where $S^{n \times n}$ is the space of symmetric $n \times n$ real matrices, such that

$$
\begin{aligned}
D^{2} u\left(\mathbb{B}_{1}(0)\right) & \subset \mathcal{K} \\
F(M) & >\frac{\delta}{2}+\frac{\pi}{2}(n-2) \text { for all } M \in \mathcal{K} .
\end{aligned}
$$

We may smoothly modify $F$ on $\mathcal{K}$,

$$
\tilde{F}=f(F)
$$

so that $\tilde{F}$ is a uniformly concave function and has the same level sets as $F$ on $\mathcal{K}$. (For a recent detailed proof of this fact, see [CPW16, Lemma 2.2].) In this case

$$
\tilde{F}\left(D^{2} u\right)=\tilde{\theta}
$$

for some smoothly modified $\tilde{\theta}$, constructed from $f$ such that

$$
\|\tilde{\theta}\|_{C^{\alpha}} \leq C\|\theta\|_{C^{\alpha}}
$$

For the second case, (1.7), $u$ is uniformly convex, and the function $F$ is clearly concave in the eigenvalues. So by taking $\tilde{F}=F$ (see [CNS85, section 3]) we already have that

$$
\tilde{F}\left(D^{2} u\right)=\theta
$$

for some concave $\tilde{F}$. Again, because $\left|D^{2} u\right| \leq C_{0}$ where it exists, we can find a compact set $\mathcal{K}$ (still using the same notation as above for simplicity) such that $D^{2} u\left(\mathbb{B}_{1}(0)\right) \subset \mathcal{K}$ and $F$ is uniformly concave on $\mathcal{K}$.

In either case, (1.6) or (1.7), we may extend $\tilde{F}$ beyond $\mathcal{K}$ to a global function $\bar{F}$ on $S^{n \times n}$ to obtain a uniformly elliptic $\bar{F}$, satisfying $\bar{F}(M)=\tilde{F}(M)$ for $M \in \mathcal{K}, \bar{F}$ is uniformly elliptic, $\bar{F}$ is concave, and $\bar{F}$ is continuous on $S^{n \times n}$ and still smooth on the interior of $\mathcal{K}$. (For example, see [Col16, Lemma 2.2].)

Now we apply [CC95, Theorem 8.1 and Remarks 1 and 3 following, see also Remark 1 in 6.2], which is Schauder theory for uniformly elliptic concave equations. Note that [CC95, p. 54 ] only requires the function $\bar{F}$ to be concave and continuous. First note that by De Giorgi-Nash, when $u \in C^{1,1}$ the equation (1.2) is uniformly elliptic, so the function $\theta$ enjoys Hölder estimates. Thus we also have Hölder estimates on the modified $\tilde{\theta}$. Now our definition of weak solution is that $F\left(D^{2} u\right)=\theta$, almost everywhere, so also, $\bar{F}\left(D^{2} u\right)=\tilde{\theta}$ almost everywhere, and we may apply [Lio83, Corollary 3] to conclude that $u$ is also a viscosity solution to $\bar{F}\left(D^{2} u\right)=\tilde{\theta}$. Because the modification of $F$ was either smooth or away from a compact set containing the image of $D^{2} u$, we still have

$$
\left\|\bar{F}\left(D^{2} u\right)\right\|_{C^{\alpha}\left(\mathbb{B}_{4 / 5}(0)\right)} \leq C_{1}
$$

for some $C_{1}$ depending on the ellipticity constants obtained in our application of De GiorgiNash, noting that $\|\theta\|_{L^{\infty}} \leq n \pi / 2$. We conclude from [CC95] that

$$
\left\|D^{2} u\right\|_{C^{\alpha}\left(\mathbb{B}_{3 / 4}(0)\right)} \leq C_{2}
$$

for $C_{2}$ depending on the ellipticity constants, $C_{1}$, and the oscillation of $u$.
Now with interior $C^{2, \alpha}$ estimates in hand, we return to $\theta$, which is a solution to a divergence type equation with $C^{\alpha}$ coefficients, so we may apply [HL97, Theorem 3.13] to conclude that

$$
\|\theta\|_{C^{1, \alpha}\left(\mathbb{B}_{2 / 3}(0)\right)} \leq C_{3} .
$$

Now for $e_{k}$, consider the function

$$
\theta^{\left(h_{k}\right)}(x)=\frac{\theta\left(x+h e_{k}\right)-\theta(x)}{h}
$$

defined on some interior region, for small $h>0$. Because $\theta \in C^{1, \alpha}\left(\mathbb{B}_{2 / 3}(0)\right)$ we have

$$
\left\|\theta^{\left(h_{k}\right)}\right\|_{C^{\alpha}\left(\mathbb{B}_{2 / 3-h}(0)\right)} \leq C_{3}
$$

Now

$$
\begin{aligned}
\theta^{\left(h_{k}\right)}(x) & =\frac{1}{h} \int_{0}^{1} \frac{d}{d t} F\left(D^{2} u\left(x+h e_{k}\right) t+(1-t) D^{2} u(x)\right) d t \\
& =\frac{1}{h} \int_{0}^{1} g^{i j}\left(D^{2} u\left(x+h e_{k}\right) t+(1-t) D^{2} u(x)\right)\left(u\left(x+h e_{k}\right)_{i j}-u_{i j}(x)\right) d t \\
& =\int_{0}^{1} g^{i j}\left(D^{2} u\left(x+h e_{k}\right) t+(1-t) D^{2} u(x)\right)\left(\frac{u\left(x+h e_{k}\right)_{i j}-u_{i j}(x)}{h}\right) d t \\
& =G^{i j} u_{i j}^{\left(h_{k}\right)}(x) \\
& :=L u^{\left(h_{k}\right)}(x)
\end{aligned}
$$

for some uniformly elliptic $L=G^{i j} \partial_{i} \partial_{j}$ which is an average of elliptic operators with $C^{\alpha}$ coefficients, where

$$
u^{\left(h_{k}\right)}(x)=\frac{u\left(x+h e_{k}\right)-u(x)}{h} .
$$

Thus, each $u^{\left(h_{k}\right)}$ satisfies an uniformly elliptic equation of non-divergence type, that is

$$
L u^{\left(h_{k}\right)}=\theta^{\left(h_{k}\right)} \in C^{\alpha}\left(\mathbb{B}_{2 / 3-h}(0)\right)
$$

with Hölder estimate uniform in $h$. Noting that each $u^{\left(h_{k}\right)} \in C^{2, \alpha}$ we may apply the nondivergence Schauder theory [GT01, Theorem 6.6] to conclude a uniform $C^{2, \alpha}$ estimate as $h \rightarrow 0$. Thus, for each $u_{k}=\lim _{h \rightarrow 0} u^{\left(h_{k}\right)}$, where $k \in 1, \ldots, n$, we have

$$
\left\|u_{k}\right\|_{C^{2, \alpha}\left(\mathbb{B}_{1 / 2}(0)\right)} \leq C_{4}
$$

that is

$$
\begin{aligned}
& u \in C^{3, \alpha}\left(\mathbb{B}_{1 / 2}(0)\right) \\
& g \in C^{1, \alpha}\left(\mathbb{B}_{1 / 2}(0)\right)
\end{aligned}
$$

with estimates.
Now from $\Delta_{g} \theta=0$ we get

$$
\sqrt{g} g^{i j} \theta_{i j}=-\partial_{i}\left(\sqrt{g} g^{i j}\right) \theta_{i} \in C^{\alpha}\left(\mathbb{B}_{1 / 2}(0)\right)
$$

thus $\theta$ satisfies a non-divergence equation with Hölder continuous right hand side $f$. By Schauder theory [GT01, Theorem 6.13], $\theta$ must be $C^{2, \alpha}$. (More precisely, $\theta$ is the unique viscosity solution to the Dirichlet problem $\sqrt{g} g^{i j} \varphi_{i j}=f$ on $\mathbb{B}_{1 / 2}(0)$ and $\varphi=\theta$ on $\partial \mathbb{B}_{1 / 2}(0)$.) Iterating the previous two steps, we may obtain all higher order estimates for any region further in the interior.

Next we assume that (1.8) holds. Suppose that a function $u$ satisfies (1.8). Let

$$
\kappa=\arctan (1-\delta)<\frac{\pi}{4}
$$

Condition (1.8) gives us that $u$ is $-\tan (\kappa)$-convex. Perform a downward rotation of the graph of $u$ with $\sigma=\frac{\pi}{4}$. Proposition 4.1 implies that the corresponding coordinate change $\bar{x}(x)$ defined by (4.7) is bi-Lipschitz. It will follow that any interior region of $\bar{\Omega}^{\varepsilon}$ (recall (4.13)) will be the homeomorphic image of an interior region $\Omega^{\prime}$ with

$$
\Omega^{\varepsilon_{2}} \subset \Omega^{\prime} \subset \Omega^{\varepsilon_{1}}
$$

with $\varepsilon_{1} / \varepsilon$ and $\varepsilon_{2} / \varepsilon$ bounded above and away from 0 . It follows that interior estimates for $\bar{u}$ on $\bar{\Omega}$ will correspond to interior estimates for $u$ on $\Omega$.

Now by Proposition 4.4, $\bar{u}$ is $\beta_{0}$-convex for

$$
\beta_{0}=\tan \left(\arctan (\delta-1)-\frac{\pi}{4}\right)=\frac{\delta-2}{\delta} .
$$

Now letting $v=-u$, we may also rotate upward by $\sigma=\frac{\pi}{4}$, to obtain a function $\bar{v}$ that is $\beta_{1}$ convex for

$$
\beta_{1}=\tan \left(\arctan (\delta-1)+\frac{\pi}{4}\right)=\frac{\delta}{2-\delta}
$$

by Proposition 4.4. From the discussion in the proof of Corollary 4.2, we have that $\bar{v}=-\bar{u}$. In particular, $-\bar{u}$ is $C^{1,1}$, uniformly convex, and clearly is also a weak solution of (1.2), as the quantity $\theta$ is odd in $D^{2} u$. We are then back to the case (1.7), and may conclude interior estimates on the derivatives of $-\bar{u}$ for any order, and hence also for derivatives of $\bar{u}$. Now certainly (4.20) holds for $\epsilon=1$, so we may apply Proposition 4.7 and get interior derivative estimates on $u$.

### 5.1. Proof of Theorem 1.3.

Proof. Let $u$ be a $W^{2, n}(\Omega)$ solution to (1.1). Let $\Gamma_{u}=\{(x, D u(x)): x \in \Omega\}$. First note that the Grassmannian geometry (in particular, the distance function) is invariant under unitary actions on $\mathbb{C}^{n}$. Observe also that for small enough $c_{0}(n)$, all Lagrangian planes within distance $c_{0}(n)$ from each other must be graphical over each other. Thus at any point $p$ where $D^{2} u$ exists, the tangent space to $\Gamma$ is well-defined, and we can locally take $\Gamma$ to be a graph over $T_{p} L$. By taking a unitary map sending $T_{p} \Gamma$ to $\mathbb{R}^{n} \times\{0\}$, we may express the isometric image $\bar{\Gamma}$ locally as a gradient graph of some function $\bar{u}$ over a region $\bar{\Omega} \subset \mathbb{R}^{n}$, with $D^{2} \bar{u}(p)=0$. For Lagrangian tangent planes near $\mathbb{R}^{n} \times\{0\}$, the topology on the Lagrangian Grassmannian is equivalent to the topology on Hessian space, so by choosing $c_{0}(n)$ small we have also guaranteed that

$$
\|u\|_{C^{1,1}(\Omega)} \leq c(n)<1
$$

where $c(n)$ is from Theorem 1.1. Applying Theorem 1.1, we may conclude that $u$ is a weak solution to (1.2). By Theorem 1.2, $\bar{u}$ is smooth inside $\bar{\Omega}$. So $\bar{\Gamma}$ is the gradient graph of a smooth function over $\bar{\Omega}$, hence it is a smooth submanifold of $\mathbb{R}^{2 n}$.

Our result allows for the Hessian of the potential function $u$ to be just continuous or even have mild discontinuities provided that $\|u\|_{C^{1,1}} \leq c(n)$. The following result is obtained by Schoen and Wolfson [SW01, Proposition 4.6], for Lagrangian stationary surfaces (when the potential functions are locally in $C^{2, \alpha}$ ) in general Kählerian ambient manifolds.
Corollary 5.1. Suppose that $u \in C^{2}$ is a weak solution to (1.1). Then $u$ is smooth.
Proof. Let $\Gamma=\{(x, D u(x)): x \in \Omega\}$. Near any point $x_{0} \in \Gamma$, we may write $\Gamma$ locally as as gradient graph of a function $v$ over its tangent plane $T_{x_{0}} \Gamma$. Necessarily, this choice gives us $D^{2} v(0)=0$. Now $v$ is also stationary for compactly supported variations near $x_{0}$, so $v$ must satisfy (1.1) as well. Because $D^{2} u \in C^{0}$, the tangent planes change continuously. It follows that also $D^{2} v \in C^{0}$, and because we have chosen $D^{2} v(0)=0$, we may find a small neighborhood for which

$$
\left\|D^{2} v\right\|_{C^{0}} \leq c(n) .
$$

Applying Theorem 1.3, $v$ is smooth near $x$. It follows that $\Gamma$ is smooth near $x$. Now because $D^{2} u$ was bounded, we may project the smooth object $\Gamma$ back to the original coordinates $\Omega$, and the Jacobian does not vanish. Thus we conclude that $u$ is a smooth function on $\Omega$.

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