INTERIOR SCHAUDER ESTIMATES FOR THE FOURTH ORDER HAMILTONIAN STATIONARY EQUATION IN TWO DIMENSIONS

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ABSTRACT. We consider the Hamiltonian stationary equation for all phases in dimension two. We show that solutions that are $C^{1,1}$ will be smooth and we also derive a $C^{2,\alpha}$ estimate for it.

1. INTRODUCTION

In this paper, we study the regularity of the Lagrangian Hamiltonian stationary equation, which is a fourth order nonlinear PDE. Consider the function $u : B_1 \to \mathbb{R}$ where B_1 is the unit ball in $\mathbb{R}^2$. The gradient graph of u, given by $\{(x, Du(x))|x \in B_1\}$ is a Lagrangian submanifold of the complex Euclidean space. The function θ is called the Lagrangian phase for the gradient graph and is defined by

$$\theta = F(D^2u) = Im \log \det(I + iD^2u)$$

or equivalently,

$$\theta = \sum_i \arctan(\lambda_i) \tag{1.1}$$

where λ_i represents the eigenvalues of the Hessian.

The non-homogeneous special Lagrangian equation is given by the following second order nonlinear equation

$$F(D^2u) = f(x). \tag{1.2}$$

The Hamiltonian stationary equation is given by the following fourth order nonlinear PDE

$$\Delta_g \theta = 0 \tag{1.3}$$

where Δ_g is the Laplace-Beltrami operator, given by:

$$\Delta_g = \sum_{i,j=1}^{2} \frac{\partial_i(\sqrt{detg}g^{ij}\partial_j)}{\sqrt{detg}}$$

and g is the induced Riemannian metric from the Euclidean metric on $\mathbb{R}^4$, which can be written as

$$g = I + (D^2u)^2.$$

Recently, Chen and Warren [CW19] proved that in any dimension, a $C^{1,1}$ solution of the Hamiltonian stationary equation will be smooth with uniform estimates of all orders if the phase $\theta \geq \delta + (n-2)\pi/2$, or, if the bound on the Hessian is small. In the two dimensional case, using [CW19]'s result, we get uniform estimates for u when $|\theta| \geq \delta > 0$ (by symmetry). In this paper, we consider the Hamiltonian stationary equation for all phases in dimension two without imposing a smallness condition on the Hessian or on the range of θ, and we derive uniform estimates for u, in terms of the $C^{1,1}$ bound which we denote by Λ. We write $||u||_{C^{1,1}(B_1)} = ||Du||_{C^{0,1}(B_1)} = \Lambda$. Our main results are the following:

Theorem 1.1. *Suppose that $u \in C^{1,1}(B_1)$ and satisfies (1.3) on $B_1 \subset \mathbb{R}^2$ where $\theta \in W^{1,2}(B_1)$. Then u is a smooth function with interior Hölder estimates of all orders, based on the $C^{1,1}$ bound of u.*

Theorem 1.2. *Suppose that $u \in C^{1,1}(B_1)$ and satisfies (1.2) on $B_1 \subset \mathbb{R}^2$. If $f \in C^{\alpha}(B_1)$, then there exists $R = R(2, \Lambda, \alpha) < 1$ such that $u \in C^{2,\alpha}(B_R)$ and satisfies the following estimate*

$$|D^2u|_{C^{\alpha}(B_R)} \leq C_1(||u||_{L^{\infty}(B_1)}, \Lambda, |f|_{C^{\alpha}(B_1)}). \tag{1.4}$$

To be clear, for any given function u we denote

$$\theta(x) = F(D^2u(x)) \tag{1.5}$$

so that for solutions of (1.2) we always have

$$\theta(x) \equiv f(x). \tag{1.6}$$

Our proof of Theorem 1.1 goes as follows: We start by applying the De Giorgi-Nash theorem to the uniformly elliptic Hamiltonian stationary equation (1.3) on B_1 to prove that $\theta \in C^{\alpha}(B_{1/2})$. Next we consider the non-homogeneous special Lagrangian equation (1.2) where $\theta \in C^{\alpha}(B_{1/2})$. Using a rotation of Yuan [Yua02] we rotate the gradient graph so that the new phase $\bar{\theta}$ of the rotated gradient graph satisfies $\left|\bar{\theta}\right| \geq \delta > 0$. Now we apply [CC03] to the new potential $\bar{u}$ of the rotated graph to obtain a $C^{2,\alpha}$ interior estimate for it. On rotating back the rotated gradient graph to our original gradient graph, we see that our potential u turns out to be $C^{2,\alpha}$ as well. A computation involving change of co-ordinates gives us the corresponding $C^{2,\alpha}$ estimate, shown in (1.4). Once we have a $C^{2,\alpha}$ solution of (1.3), smoothness follows by [CW19, Corollary 5.1].

In two dimensions, solutions to the second order special Lagrangian equation

$$F(D^2u) = C$$

enjoy full regularity estimates in terms of the potential u [WY09]. For higher dimensions, such estimates fail [WY13] for $\theta = C$ with $|C| < (n-2)\pi/2$.

2. Proof of theorems:

We first prove Theorem 1.2, followed by the proof of Theorem 1.1. We prove Theorem 1.2 using the following lemma. Recalling (1.5, 1.6) we state the following lemma:

Lemma 2.1. *Suppose that $u \in C^{1,1}(B_1)$ satisfies (1.2) on $B_1 \subset \mathbb{R}^2$. Suppose*

$$0 \leq \theta(0) < (\pi/2 - \arctan\Lambda)/4. \tag{2.1}$$

If $\theta \in C^{\bar{\alpha}}(B_1)$, then there exists $0 < \alpha < \bar{\alpha}$ and C_0 such that

$$|D^2u(x) - D^2u(0)| \leq C_0(||u||_{L^\infty(B_1)}, \Lambda, |\theta|_{C^\alpha(B_1)}) * |x|^\alpha.$$

Proof. Consider the gradient graph $\{(x, Du(x))|x \in B_1\}$ where u has the following Hessian bound

$$-\Lambda I_n \leq D^2u \leq \Lambda I_n$$

a.e. where it exists.

Define δ as

$$\delta = (\pi/2 - \arctan\Lambda)/2 > 0. \tag{2.2}$$

Since by (2.1) we have $0 \leq \theta(0) < \delta/2$, there exists $R'(\delta, |\theta|_{C^{\bar{\alpha}}}) > 0$ such that

$$|\theta(x) - \theta(0)| < \delta/2$$

for all $x \in B_{R'} \subseteq B_1$. This implies for every x in $B_{R'}$ for which D^2u exists, we have

$$\delta > \theta > \theta(0) - \delta/2.$$

So now we rotate the gradient graph $\{(x, Du(x))|x \in B_{R'}\}$ downward by an angle of δ.

Let the new rotated co-ordinate system be denoted by $(\bar{x}, \bar{y})$ where

$$\bar{x} = \cos(\delta)x + \sin(\delta)Du(x) \tag{2.3}$$

$$\bar{y} = -\sin(\delta)x + \cos(\delta)Du(x). \tag{2.4}$$

On differentiating $\bar{x}$ (2.3) with respect to x we see that

$$\frac{d\bar{x}}{dx} = \cos(\delta)I_n + \sin(\delta)D^2u(x) \leq \cos(\delta)I_n + \Lambda\sin(\delta)I_n$$

Thus

$$\cos(\delta)I_n - \Lambda\sin(\delta)I_n \leq \frac{d\bar{x}}{dx} \leq \cos(\delta)I_n + \Lambda\sin(\delta)I_n.$$

To obtain Lipschitz constants so that

$$\frac{1}{L_2}I_n \leq \frac{d\bar{x}}{dx} \leq L_1 I_n \tag{2.5}$$

let

$$\begin{aligned} L_1 &= \cos(\delta) + \Lambda\sin(\delta) \\ L_2 &= \max\{\left|\frac{1}{\cos(\sigma)I_n + D^2u(x)\sin(\sigma)}\right| |x \in B_{R'}\}. \end{aligned}$$

To find the value of L_2, we see that in $B_{R'}$ we have the following: let $\min\{\theta_1, \theta_2\} \geq -A$ where $A = \arctan\Lambda$.

$$\begin{aligned} \cos(\delta)I_n + \sin(\delta)D^2u(x) &\geq \cos(\delta) - \sin(\delta)\tan(A) \\ &= \cos(\delta)(1 - \tan(\delta)\tan(A)) \\ &= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\delta + A)} \\ &= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\frac{\pi/2 - A}{2} + A)} \\ &= \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - \delta)}. \end{aligned}$$

This shows that

$$\frac{1}{L_2} = \cos(\delta)\frac{\tan(\delta) + \tan(A)}{\tan(\pi/2 - \delta)}.$$

Clearly $1/L_2$ is positive.

Now, by [CW19, Prop 4.1] we see that there exists a function $\bar{u}$ such that

$$\bar{y} = D_{\bar{x}}\bar{u}(\bar{x})$$

where

$$\bar{u}(x) = u(x) + \sin\delta\cos\delta\frac{|Du(x)|^2 - |x|^2}{2} - \sin^2(\delta)Du(x)\cdot x \tag{2.6}$$

defines $\bar{u}$ implicitly in terms of $\bar{x}$ (since $\bar{x}$ is invertible). Here $\bar{x}$ refers to the rotation map (2.3).

Note that

$$\bar{\theta}(\bar{x}) - \bar{\theta}(\bar{y}) = \theta(x) - \theta(y)$$

which implies that $\bar{\theta}$ is also a $C^{\bar{\alpha}}$ function

$$\frac{|\bar{\theta}(\bar{x}_1)-\bar{\theta}(\bar{x}_2)|}{|\bar{x}_1-\bar{x}_2|^{\alpha}} = \frac{|\theta(x_1)-\theta(x_2)|}{|x_1-x_2|^{\bar{\alpha}}} * \frac{|x_1-x_2|^{\bar{\alpha}}}{|\bar{x}_1-\bar{x}_2|^{\bar{\alpha}}}$$

thus,

$$|\bar{\theta}|_{C^{\bar{\alpha}}(B_{r_0})} \leq L_2^{\bar{\alpha}}|\theta|_{C^{\bar{\alpha}}(B_{R'})}.$$

Let $\Omega = \bar{x}(B_{R'})$. Note that $B_{r_0} \subset \Omega$ where $r_0 = R'/2L_2$. So our new gradient graph is $\{(\bar{x}, D_{\bar{x}}\bar{u}(\bar{x}))|\bar{x} \in \Omega\}$. The function $\bar{u}$ satisfies the equation

$$F(D_{\bar{x}}^2\bar{u}) = \bar{\theta}(\bar{x})$$

in B_{r_0} where $\bar{\theta} \in C^{\bar{\alpha}}(B_{r_0})$. Observe that on B_{r_0} we have

$$\bar{\theta} = \theta - 2\delta < \delta - 2\delta = -\delta < 0$$

as $\theta < \delta$ on $B_{R'}$.

Claim 2.2. *: If $|\bar{\theta}| > \delta$, then $F(D^2\bar{u}) = \bar{\theta}$ is a solution to a uniformly elliptic concave equation.*

Proof. The proof follows from [CPW17, lemma 2.2] and also from [CW19, pg 24]. □

Now using [CC03, Corollary 1.3] we get interior Schauder estimates for $\bar{u}$:

$$|D^2\bar{u}(\bar{x}) - D^2\bar{u}(0)| \leq C(||\bar{u}||_{L^\infty(B_{r_0/2})} + |\bar{\theta}|_{C^\alpha(B_{r_0/2})}) \tag{2.7}$$

for all $\bar{x}$ in $B_{r_0/2}$ where $C = C(\Lambda, \alpha)$. This is our $C^{2,\alpha}$ estimate for $\bar{u}$.

Next, in order to show the same Schauder type inequality as (2.7) for u in place of $\bar{u}$, we establish relations between the following pairs:

(i) oscillations of the Hessian of D^2u and $D^2\bar{u}$
(ii) oscillations of θ and $\bar{\theta}$
(iii) the supremum norms of u and $\bar{u}$.

We rotate back to our original gradient graph by rotating up by an angle of δ and consider again the domain $B_{R'}(0)$. This gives us the following relations:

$$\begin{aligned} x &= \cos(\delta)\bar{x} - \sin(\delta)D_{\bar{x}}\bar{u}(\bar{x}) \\ y &= \sin(\delta)\bar{x} + \cos(\delta)D_{\bar{x}}\bar{u}(\bar{x}). \end{aligned} \tag{2.8}$$

This gives us:

$$\begin{aligned} \frac{dx}{d\bar{x}} &= \cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2\bar{u}(\bar{x}) \\ D_{\bar{x}}y &= \sin(\delta)I_n + \cos(\delta)D_{\bar{x}}^2\bar{u}(\bar{x}). \end{aligned}$$

So we have

$$D_x^2 u(x) = D_{\bar{x}} y \frac{d\bar{x}}{dx} = [\sin(\delta)I_n + \cos(\delta)D_{\bar{x}}^2 \bar{u}(\bar{x})][\cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2 \bar{u}(\bar{x})]^{-1}.$$

The above expression is well defined everywhere because $D_{\bar{x}}^2 \bar{u}(\bar{x}) < \cot(\delta)I_n$ for all $\bar{x} \in B_{r_0}$.

Note that we have $\cos(\delta)I_n - D_{\bar{x}}^2 \bar{u}(\bar{x})\sin(\delta) \geq \frac{1}{L_1}$, since

$$\frac{dx}{d\bar{x}} = \cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2 \bar{u}(\bar{x}) = \left(\frac{d\bar{x}}{dx}\right)^{-1} \geq \frac{1}{L_1} I_n$$

by (2.5).

Next,

$$\begin{aligned} D_x^2 u(x) - D_x^2 u(0) &= [\sin(\delta)I_n + \cos(\delta)D_{\bar{x}}^2 \bar{u}(\bar{x})][\cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2 \bar{u}(\bar{x})]^{-1} \\ &\quad - [\sin(\delta)I_n + \cos(\delta)D_{\bar{x}}^2 \bar{u}(0)][\cos(\delta)I_n - \sin(\delta)D_{\bar{x}}^2 \bar{u}(0)]^{-1}. \end{aligned} \tag{2.9}$$

For simplification of notation we write

$$\begin{aligned} D_{\bar{x}}^2 \bar{u}(\bar{x}) &= A \\ D_{\bar{x}}^2 \bar{u}(0) &= B \\ \cos(\delta) &= c, \sin(\delta) = s. \end{aligned}$$

Noting that $[sI_n + cA]$ and $[cI_n - sA]^{-1}$ commute with each other we can write (2.9) as the following equation

$$\begin{aligned} D_x^2 u(x) - D_x^2 u(0) = \\ &[cI_n - sB]^{-1}[cI_n - sB][sI_n + cA][cI_n - sA]^{-1} - \\ &[cI_n - sB]^{-1}[sI_n + cB][cI_n - sA][cI_n - sA]^{-1}. \end{aligned}$$

Again we see that

$$[cI_n - sB][sI_n + cA] - [sI_n + cB][cI_n - sA] = A - B.$$

This means

$$D_x^2 u(x) - D_x^2 u(0) = [cI_n - sB]^{-1}[A - B][cI_n - sA]^{-1}.$$

We have already shown that

$$|cI_n - sA| \geq \frac{1}{L_1}$$

which implies

$$|cI_n - sA|^{-1} \leq L_1.$$

Thus we get

$$\begin{aligned}
|D_x^2u(x) - D_x^2u(0)| &\le L_1^2|D_{\bar{x}}^2\bar{u}(\bar{x}) - D_{\bar{x}}^2\bar{u}(0)|. \\
&\le CL_1^2(||\bar{u}||_{L^\infty(B_{r_0/2})} + |\bar{\theta}|_{C^\alpha(B_{r_0/2})})|\bar{x}|^\alpha \\
&\le CL_1^{2+\alpha}(||\bar{u}||_{L^\infty(B_{r_0/2})} + |\bar{\theta}|_{C^\alpha(B_{r_0/2})}|x|^\alpha \qquad (2.10)
\end{aligned}$$

where L_1 is the Lipschitz constant of the co-ordinate change map. This implies

$$\frac{1}{L_1^{\alpha+2}}|D_x^2u(x)|_{C^\alpha(B_R)} \le |D_{\bar{x}}^2u(\bar{x})|_{C^\alpha(B_{r_0/2})}. \qquad (2.11)$$

Recall from (2.6) that

$$\bar{u}(x) = u(x) + v(x).$$

This shows

$$\begin{aligned}
||\bar{u}(\bar{x})||_{L^\infty(B_{r_0/2})} = ||\bar{u}(x)||_{L^\infty(\bar{x}^{-1}(B_{r_0/2}))} &\le ||\bar{u}(x)||_{L^\infty(B_{R'})} \\
&\le ||u(x)||_{L^\infty(B_{R'})} + ||v||_{L^\infty(B_{R'})}. \qquad (2.12)
\end{aligned}$$

Note that

$$||v||_{L^\infty(B_R)} \le R||Du||_{L^\infty(B_R)} + \frac{1}{2}[R^2 + ||Du||^2_{L^\infty(B_R)}] \qquad (2.13)$$

and combining (2.11), (2.12), (2.13) with (2.10) we get

$$\begin{aligned}
&|D_x^2u(x) - D_x^2u(0)| \\
&\le CL_1^{\alpha+2}\left\{ \begin{array}{c} ||u||_{L^\infty(B_{R'})} + R||Du||_{L^\infty(B_R)} + \\ \frac{1}{2}[R^2 + ||Du||^2_{L^\infty(B_R)}] + L_2^\alpha r_0|\theta|_{C^\alpha(B_{R'})} \end{array} \right\} |x|^\alpha .
\end{aligned}$$

This proves the Lemma. □

Proof of Theorem 1.2. First note that the lemma provides a bound for the Hölder norm of the Hessian on any interior ball, so by a rescaling of the form

$$u_\rho(x) = \frac{u(\rho x)}{\rho^2}$$

for values of $\rho > 0$ and translation of any point to the origin. Consider the gradient graph $\{(x, Du(x))|x \in B_1\}$ where u satisfies

$$F(D^2u) = \theta$$

on B_1 and $\theta \in C^{\bar{\alpha}}(B_1)$. Then there exists a ball of radius r inside B_1 on which $osc\theta < \delta/4$ where δ is as defined in (2.2).
Now this means that either we have $\theta(x) < \delta/2$ in which case, by the above lemma we see that $u \in C^{2,\alpha}(B_r)$ satisfying the given estimates; or we have $\theta(x) > \delta/4$ in which case $u \in C^{2,\alpha}(B_r)$ with uniform estimates, by claim (2.2) and [CC03, Corollary 1.3]. □

Proof of Theorem 1.1. Since $u \in C^{1,1}(B_1)$ and $\theta \in W^{1,2}(B_1)$ satisfies the uniformly elliptic equation

$$\Delta_g \theta = 0,$$

by the De Giorgi-Nash Theorem we have that $\theta \in C^{\alpha}(B_{1/2})$. This means that u satisfies

$$F(D^2 u) = \theta.$$

By Theorem 1.2 we see that $u \in C^{2,\alpha}(B_r)$ where $r < 1/2$. Smoothness follows by [CW19, Corollary 5.1]. $\square$

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