

$C^{2,\alpha}$ ESTIMATES FOR SOLUTIONS TO ALMOST LINEAR ELLIPTIC EQUATIONS

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ABSTRACT. In this paper, we show explicit $C^{2,\alpha}$ interior estimates for viscosity solutions of fully non-linear, uniformly elliptic equations, which are close to linear equations and we also compute an explicit bound for the closeness.

1. INTRODUCTION

In this paper, we derive an a priori interior $C^{2,\alpha}$ estimate for viscosity solutions of the non-linear, uniformly elliptic equation

$$(1.1) \quad F(D^2u) = f(x),$$

under the assumption that $f(x) \in C^\alpha$ and F is C^1 -close to a linear operator.

For viscosity solutions of second order, fully non-linear equations of the form

$$(1.2) \quad F(D^2u) = 0$$

where F is concave and uniform elliptic, the landmark estimate is that of Krylov and Evans, who proved $C^{2,\alpha}$ estimates from C^2 estimates [N.V83], [Eva82]. For general F , the fundamental results on the regularity of solutions to fully non-linear uniformly elliptic equations of the form (1.2) include interior C^α estimates of [KS81] and interior $C^{1,\alpha}$ estimate of [CC95]. The structure of F plays a key role in deriving higher order estimates for fully non-linear elliptic equations of the forms (1.1) and (1.2). In [NV08], the authors produced counterexamples to Evans-Krylov type estimates for general fully non-linear equations. In fact, solutions need not even be $C^{1,1}$, [NV10].

Prior to Krylov and Evans, few fully non-linear equations were known to enjoy a $C^{1,1}$ to $C^{2,\alpha}$ regularity boost. The Monge-Ampère equation was shown to have this property (even stronger, $C^{1,1}$ to C^3) following Calabi's calculation [Cal58]. Other results, requiring stronger conditions on D^2F , are mentioned in [Eva82, pg 335.]. If the linearized operator for F satisfies a Cordes-Nirenberg condition, one can also obtain this boosting (see Section 5). Since the 1980s, it has been a

challenge to find equations with the regularity boosting property that are neither convex nor concave, see for example [CY00], [Yua01], [CC03], [Col16], [SW16], [Pin16]. Savin [Sav07] proved interior $C^{2,\alpha}$ (and higher) estimates for viscosity solutions of (1.2) that are sufficiently close to a quadratic polynomial, for F smooth. When full regularity is not available, partial regularity results can be found, see [ASS].

Here, we consider a space of uniformly elliptic, non-linear equations of the forms (1.2) and (1.1) where we assume that F is uniformly differentiable and DF lies in a set of diameter ε_0 . We formally define this property of F in definition 1.2. We show that given ellipticity constants and an $\alpha \in (0, 1)$ of your choice, there is a universal constant $\varepsilon_0(n, \lambda, \Lambda, \alpha)$ guaranteeing $C^{2,\alpha}$ regularity.

Differentiating (1.2) with respect to a direction i , one sees that u_i solves a linear equation with bounded measurable coefficients (now depending on x , not D^2u). One then hopes to achieve $C^{1,\alpha}$ estimates on u_i , yielding $C^{2,\alpha}$ estimates on u . In particular, it may be possible to apply estimates of Cordes and Nirenberg from the 1950s: Any solution v of a linear equation

$$(1.3) \quad a^{ij}(x)v_{ij}(x) = 0$$

with coefficients close to δ^{ij} will enjoy $C^{1,\alpha}$ regularity. Thus when a solution is already C^3 , universal interior $C^{2,\alpha}$ estimates should follow by the Cordes-Nirenberg theory. A delicate analysis of the Dirichlet boundary value problem, approximating u with C^3 mollifications on the boundary should also yield the estimates when u is not known to be C^3 , cf.[Eva82, Section 7]. The closeness constants of Cordes-Nirenberg are explicit and mildly restrictive, in fact much less restrictive than ours. As the historical literature is not widely known, we discuss the Cordes-Nirenberg results in more detail in Section 5.

Note that our result is stated for every $\alpha \in (0, 1)$. Also, note that for equation (1.1) one cannot hope to differentiate either side of (1.1) if the right hand side is merely C^α , so the regularity theory cannot be immediately reduced to the Cordes-Nirenberg theory. Our methods for proving 1.3 are much different in nature than the proof of Cordes and Nirenberg: we use the method of constructing approximating polynomials, instead of integral estimates. In Theorem 1.4, we prove interior $C^{2,\alpha}$ estimates for solutions of (1.1) using our $C^{2,\alpha}$ estimates for (1.2) together with estimates found in [CC95].

This paper is divided into the following sections. In the remainder of this section we state definitions and our main results. In section 2, we prove Theorem 1.3 and in section 3, we prove Theorem 1.4. In section 4

we explicitly state and prove an often used result involving Hölder estimates and in section 5 we further discuss the Cordes-Nirenberg regularity and some applications of Cordes-Nirenberg regularity to equations of the form (1.2).

1.1. Definitions and notations. We first define a few terms that we will be using to state the properties of the operator F .

Condition 1.1. *Throughout this paper we make the assumption*

$$(1.4) \quad F(\mathbf{0}) = 0.$$

Definition 1.2. *We define the uniformly elliptic, non-linear operator F to be **almost linear with constant ε** if*

$$(1.5) \quad \|DF(M) - DF(N)\| \leq \varepsilon$$

*for all $M, N \in S_n$ where S_n is the space of all real symmetric $n \times n$ matrices. We define ε to be the **closeness constant** of F .*

We say that F is λ, Λ elliptic if

$$F(M) + \lambda \|P\| \leq F(M + P) \leq F(M) + \Lambda \|P\|$$

for all positive matrices P . To be clear, for matrices and their dual (linear operators) we use $\|\cdot\|$ to denote the (L^2, L^2) norm, that is

$$\|M\| = \sup_{x \leq 1} \|Mx\|.$$

Theorem 1.3. *Given λ, Λ , and $0 < \bar{\alpha} < 1$ there exists a universal constant $\varepsilon_0(n, \lambda, \Lambda, \bar{\alpha}) > 0$ such that if F is almost linear with constant ε_0 and $u \in C(B_1)$ is a viscosity solution of (1.2) on B_1 , then $u \in C^{2, \bar{\alpha}}(B_{\frac{1}{4\Lambda}})$ and satisfies the following estimate*

$$(1.6) \quad \|D^2u\|_{C^{\bar{\alpha}}(B_{\frac{1}{4\Lambda}})} \leq C_1 \|u\|_{L^\infty(B_1)}$$

where

$$(1.7) \quad C_1 = \left(1 + n + 4n^2 + \frac{1}{\lambda} \varepsilon_0 \frac{25}{8} n^2\right) \left(1 + \frac{3}{1 - r_0^{\bar{\alpha}}}\right) \frac{2^{\bar{\alpha}}}{r_0^{1+\bar{\alpha}}} \Lambda^{2+\bar{\alpha}} (2 + 2^{2+\bar{\alpha}})^2.$$

The constant ε_0 is determined in (2.44), (2.27)

Theorem 1.4. *Given λ, Λ , and $0 < \alpha < \bar{\alpha} < 1$, suppose that F is almost linear with constant ε_0 for the same constant $\varepsilon_0(n, \lambda, \Lambda, \bar{\alpha})$ as in Theorem 1.3 and $u \in C(B_1)$ is viscosity solution of (1.1) on B_1 . If $f \in C^\alpha(B_1)$, then $u \in C^{2, \alpha}(B_{1/2})$ and the following estimate holds*

$$(1.8) \quad \|u\|_{C^{2, \alpha}(B_{1/2})} \leq C_2 (\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)})$$

where C_2 depends on $n, \lambda, \Lambda, C_1, \alpha, \bar{\alpha}$.

The methods involved in our proof include comparing equation (1.2) to the Laplace equation with boundary data equal to a mollification of u . We use the Krylov-Safanov Theorem [KS81] along with harmonic estimates to construct a quadratic polynomial that separates from u to order $r^{2+\alpha}$ on the ball of radius r . This is used in the construction of an iterative sequence of quadratic polynomials that leads to our desired estimate in the first theorem. The proof of Theorem 1.4 uses arguments from $W^{2,p}$ regularity found in [CC95, Chapter 7] .

2. PROOF OF THEOREM 1.3

By calculus,

$$\begin{aligned} F(N) - F(\mathbf{0}) &= \int_0^1 DF(tN) \cdot N dt \\ &= \int_0^1 (DF(tN) - DF(\mathbf{0})) \cdot N dt + DF(\mathbf{0}) \cdot N. \end{aligned}$$

Thus, from (1.4) and (1.5)

$$(2.1) \quad |F(N) - DF(\mathbf{0}) \cdot N| \leq \varepsilon_0 \|N\|.$$

With this in mind we begin with the following Lemma.

Lemma 2.1. *Given $\bar{\alpha}, \lambda, \Lambda$ there exist universal constants $\tilde{\varepsilon}_0(n, \lambda, \Lambda, \bar{\alpha}) > 0$ and $r_0(n, \bar{\alpha}) > 0$, such that if the λ, Λ elliptic operator F satisfies*

$$(2.2) \quad |F(N) - \text{tr}(N)| < \tilde{\varepsilon}_0 \|N\|$$

for all $N \in S_n$, then for any viscosity solution $u \in C(B_1)$ of (1.2) in $B_1(0)$, we can find a polynomial P of degree 2 satisfying

$$\begin{aligned} F(D^2P) &= 0 \\ \sup_{B_{r_0}} |u - P| &\leq r_0^{2+\bar{\alpha}} \|u\|_{L^\infty(B_1)} \\ (2.3) \quad \|P\|_{L^\infty(B_1)} &\leq C_0 \|u\|_{L^\infty(B_1)}. \end{aligned}$$

We compute the explicit values of the universal constants to be

$$\begin{aligned} (i) \quad r_0 &= \left(\frac{3}{250n^3}\right)^{\frac{1}{1-\bar{\alpha}}} \\ (ii) \quad C_0 &= 1 + n + 4n^2 + \frac{1}{\lambda} \tilde{\varepsilon}_0 \frac{25}{8} n^{5/2} \\ (iii) \quad \tilde{\varepsilon}_0 &= \min \left\{ \lambda \frac{2}{25n^2} r_0^{\bar{\alpha}}, \left(\frac{1}{2}\right)^{1+6/\alpha_0} \frac{\lambda}{K_2} \frac{1}{K_1^{3/\alpha_0}} r_0^{(2+\bar{\alpha})(1+3/\alpha_0)} \right\} \text{ where } K_1, \\ &\quad \alpha_0, K_2 \text{ are defined in (2.7), (2.4), and (2.21) respectively.} \end{aligned}$$

The required constant α_0 is defined in the proof of the Lemma, and will require the Krylov-Safanov Theorem, so we state that here.

Theorem 2.2. [KS81, Theorem 1] *[Krylov-Safanov] Let $u \in C^0$ be a viscosity solution of $S(\frac{\lambda}{n}, \Lambda, 0) = 0$ in B_1 . Then u is Hölder continuous and*

$$(2.4) \quad \|u\|_{C^{\alpha_0}(B_{1/2})} \leq C\left(\frac{\lambda}{n}, \Lambda\right) \|u\|_{L^\infty(B_1)}$$

with (small) $\alpha_0 = \alpha_0(\frac{\lambda}{n}, \Lambda) > 0$.

We will apply the following result to the Laplace operator to determine the constant K_2 . We state a weaker version than in [CC95, Theorem 9.5].

Theorem 2.3. [CC95, Theorem 9.5] *Let g be a smooth function in \bar{B}_1 . If $u \in C^3(\bar{B}_1)$ is a solution of*

$$\begin{cases} \Delta u = 0 & \text{in } \bar{B}_1 \\ u = g & \text{on } \partial B_1 \end{cases}$$

then

$$(2.5) \quad \|u\|_{C^2(\bar{B}_1)} \leq C' \|g\|_{C^3(\partial B_1)}.$$

where C' is a universal constant.

Proof of Lemma 2.1. Let's denote $\|u\|_{L^\infty(B_1)} = M$. We consider a function h that satisfies the following boundary value problem:

$$\begin{cases} \Delta h = 0 & \text{in } \bar{B}_{4/5} \\ h = u^\gamma & \text{on } \partial B_{4/5} \end{cases}.$$

Here u^γ refers to a mollification of u for some $\gamma \in (0, 1/5)$, defined by

$$u^\gamma = \eta_\gamma * u$$

where

$$\eta_\gamma(x) = \frac{1}{\gamma^n} \eta\left(\frac{x}{\gamma}\right)$$

and $\eta \in C^\infty(\mathbb{R}^n)$ is given by

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

with the constant $C > 0$ being chosen such that $\int_{\mathbb{R}^n} \eta dx = 1$. Note that since u is defined on all of B_1 , the mollifier sequence u^γ is well defined on $B_{4/5}$ when $\gamma < 1/5$ and that

$$(2.6) \quad \|u^\gamma\|_{L^\infty(B_{4/5})} \leq M.$$

From the Krylov-Safanov theorem, we get the following estimate

$$(2.7) \quad \|u\|_{C^{\alpha_0}(B_{4/5})} \leq K_1 M.$$

This implies that u^γ converges to u uniformly on $B_{4/5}^-$ as $\gamma \rightarrow 0$ and satisfies the following estimate:

$$(2.8) \quad \|u^\gamma - u\|_{L^\infty(B_{4/5})} \leq K_1 \gamma^{\alpha_0} M.$$

Since h is harmonic and thus analytic there exists a polynomial $P_0(x)$ of degree two

$$P_0(x) = h(0) + x \cdot Dh(0) + x \cdot D^2 h(0)x$$

such that for all $|x| < 1/2$,

$$|h(x) - P_0(x)| \leq |R_3(x)|$$

where R_3 is the remainder term of order 3 in the Taylor series expansion of h . Estimates for harmonic functions (cf. [GT01, (2.31)]), considering (2.6) are of the form

$$\begin{aligned} \sup_{x \in B_{1/5}} |h_{ijk}(x)| &\leq \frac{n}{1/5} \sup_{x \in B_{2/5}} |h_{ij}(x)| \\ &\leq 5n \frac{n}{1/5} \sup_{x \in B_{3/5}} |h_i(x)| \\ &\leq 25n^2 \frac{n}{1/5} \sup_{x \in B_{4/5}} |h(x)| \\ &\leq 125n^3 M. \end{aligned}$$

Thus we have on $B_{1/5}$

$$|h(x) - P_0(x)| \leq \frac{125}{3!} n^3 M |x|^3.$$

Choosing

$$(2.9) \quad r_0 = \left(\frac{3}{250n^3} \right)^{\frac{1}{1-\bar{\alpha}}} \ll \frac{1}{5},$$

we have

$$(2.10) \quad \sup_{B_{r_0}} |h(x) - P_0(x)| \leq \frac{1}{4} M r_0^{2+\bar{\alpha}}.$$

Now from (2.2) and $\Delta P_0 = 0$, we see that

$$|F(D^2 P_0)| \leq \tilde{\varepsilon}_0 \|D^2 P_0\|.$$

So using λ -ellipticity, there is a $c \in [-\tilde{\varepsilon}_0, \tilde{\varepsilon}_0]$ such that the quadratic polynomial

$$(2.11) \quad P(x) = P_0(x) + \frac{|x|^2}{2\lambda} c \|D^2 h(0)\|$$

satisfies

$$F(D^2 P) = 0.$$

Using harmonic estimates again we see that

$$(2.12) \quad \|D^2 h(0)\| \leq \frac{25}{4} n^2 M.$$

Bringing in (2.11) we see

$$(2.13) \quad \sup_{B_{r_0}} |h - P| < \sup_{B_{r_0}} |h - P_0| + \frac{r_0^2}{2\lambda} \tilde{\varepsilon}_0 \frac{25}{4} n^2 M.$$

Insisting on a choice of $\tilde{\varepsilon}_0$ such that

$$(2.14) \quad \tilde{\varepsilon}_0 \leq \frac{\lambda}{2} \frac{r_0^{\bar{\alpha}}}{\frac{25}{4} n^2} = \lambda \frac{2}{25 n^2} \left(\frac{3}{250 n^3} \right)^{\frac{\bar{\alpha}}{1-\bar{\alpha}}}$$

we conclude from (2.13) and (2.10)

$$(2.15) \quad \sup_{B_{r_0}} |h - P| \leq \frac{1}{2} M r_0^{2+\bar{\alpha}}.$$

Again using harmonic estimates (2.12), we get the following estimate for P :

$$(2.16) \quad \|P\|_{L^\infty(B_1)} \leq C_0 M,$$

$$C_0 = 1 + n + \frac{25}{4} n^2 + \frac{1}{2\lambda} \tilde{\varepsilon}_0 \frac{25}{4} n^2$$

$$(2.17) \quad = 1 + n + \frac{25}{4} \left(1 + \frac{1}{2\lambda} \tilde{\varepsilon}_0 \right) n^2.$$

Next, by (2.2) for $x \in B_{4/5}$ we have

$$(2.18) \quad |F(D^2 h(x))| = |F(D^2 h) - \Delta h + \Delta h|$$

$$= |F(D^2 h) - \text{Tr}(D^2 h)|$$

$$(2.19) \quad \leq \tilde{\varepsilon}_0 \|D^2 h\|_{L^\infty(B_{4/5}^-)}.$$

Now recall (2.5):

$$\|D^2 h\|_{L^\infty(B_{4/5}^-)} \leq C' \|u_\gamma\|_{C^3(B_{4/5}^-)}.$$

We compute the value of $\|u_\gamma\|_{C^3(B_{4/5}^-)}$.

Let p be a multi-index such that $|p| = 3$. For any $x \in B_{4/5}^-$ we observe the following:

$$\begin{aligned} |D^p(u_\gamma(x))| &= |D^p(\eta_\gamma) * u(x)| = \left| \int_{B_1} D^p \eta_\gamma(x-y) u(y) dy \right| \\ &\leq \sup_{y \in B_1} |u(y)| \int_{B_1} |D^p \eta_\gamma(x-y)| dy \\ &\leq M \int_{B_1} \left| \frac{1}{\gamma^{n+3}} D^p \eta\left(\frac{x-y}{\gamma}\right) \right| dy. \end{aligned}$$

We do a change of variable $z = \frac{x-y}{\gamma}$ to reduce the above expression to

$$\leq M \frac{1}{\gamma^3} \int_{B_1} \left| \frac{1}{\gamma^n} D^p \eta(z) \gamma^n \right| dz = M \frac{1}{\gamma^3} \int_{B_1} |D^p \eta(z)| dz.$$

This shows that

$$(2.20) \quad \|D^2 h\|_{L^\infty(B_{4/5}^-)} \leq C' M \frac{1}{\gamma^3} \sup_{|p|=3} \int_{\mathbb{R}^n} |D^p \eta(z)| dz.$$

Let's define

$$(2.21) \quad K_2 = C' \sup_{|p|=3} \int_{\mathbb{R}^n} |D^p \eta(z)| dz$$

so that

$$(2.22) \quad \|D^2 h\|_{L^\infty(B_{4/5}^-)} \leq K_2 M \frac{1}{\gamma^3}.$$

Using uniform ellipticity, (2.19), and (2.22) we see that the following inequalities hold on $B_{4/5}$:

$$\begin{aligned} F(D^2 h + D^2(\frac{\tilde{\varepsilon}_0}{2\lambda} K_2 M \frac{1}{\gamma^3} (1 - |x|^2))) &\leq 0. \\ F(D^2 h - D^2(\frac{\tilde{\varepsilon}_0}{2\lambda} K_2 M \frac{1}{\gamma^3} (1 - |x|^2))) &\geq 0. \end{aligned}$$

Using comparison principles [GT01, Theorem 17.1] and (2.8) we see that for all $x \in B_{4/5}$ we have:

$$(2.23) \quad |u(x) - h(x)| \leq K_1 M \gamma^{\alpha_0} + \frac{\tilde{\varepsilon}_0}{2\lambda} K_2 M \frac{1}{\gamma^3}.$$

On combining (2.23), (2.15) we see that

$$\begin{aligned} \sup_{B_{r_0}} |u - P| &< \sup_{B_{r_0}} |u - h| + \sup_{B_{r_0}} |h - P| \\ (2.24) \quad &< K_1 M \gamma^{\alpha_0} + \frac{\tilde{\varepsilon}_0}{2\lambda} K_2 M \frac{1}{\gamma^3} + \frac{1}{2} M r_0^{2+\bar{\alpha}}. \end{aligned}$$

The right hand side of (2.24) will be no greater than $Mr_0^{2+\bar{\alpha}}$ provided

$$K_1\gamma^{\alpha_0} + \frac{\tilde{\varepsilon}_0}{2\lambda}K_2\frac{1}{\gamma^3} \leq \frac{1}{2}r_0^{2+\bar{\alpha}}$$

for some choice of γ and $\tilde{\varepsilon}_0$. While this could be optimized with some messy calculus, we scare up constants as follows. Choose

$$(2.25) \quad \gamma = \left(\frac{\frac{1}{4}r_0^{2+\bar{\alpha}}}{K_1} \right)^{1/\alpha_0}$$

so that

$$K_1\gamma^{\alpha_0} = \frac{1}{4}r_0^{2+\bar{\alpha}}$$

and then we want

$$\frac{\tilde{\varepsilon}_0}{2\lambda}K_2\frac{1}{\left(\frac{\frac{1}{4}r_0^{2+\bar{\alpha}}}{K_1}\right)^{3/\alpha_0}} \leq \frac{1}{4}r_0^{2+\bar{\alpha}}$$

so we choose

$$(2.26) \quad \begin{aligned} \tilde{\varepsilon}_0 &\leq \frac{1}{4}r_0^{2+\bar{\alpha}} \frac{2\lambda}{K_2} \left(\frac{\frac{1}{4}r_0^{2+\bar{\alpha}}}{K_1} \right)^{3/\alpha_0} \\ &= \left(\frac{1}{2} \right)^{1+6/\alpha_0} r_0^{(2+\bar{\alpha})(1+3/\alpha_0)} \frac{\lambda}{K_2} \frac{1}{K_1^{3/\alpha_0}} \end{aligned}$$

where K_1 , α_0 and K_2 are defined in (2.7) and (2.22) respectively and r_0 defined by (2.9), From (2.14) and (2.26) we see that

$$(2.27) \quad \tilde{\varepsilon}_0 = \min \left\{ \begin{aligned} &\lambda \frac{2}{25n^2} \left(\frac{3}{250n^3} \right)^{\frac{\bar{\alpha}}{1-\bar{\alpha}}}, \\ &\left(\frac{1}{2} \right)^{1+6/\alpha_0} \frac{\lambda}{K_2} \frac{1}{K_1^{3/\alpha_0}} \left(\frac{3}{250n^3} \right)^{\frac{(2+\bar{\alpha})(1+3/\alpha_0)}{1-\bar{\alpha}}} \end{aligned} \right\}.$$

□

We now make a proposition similar to the statement of Theorem 1.2, but with the operator close to the Laplacian. Throughout this proof the constants C_0 and r_0 will refer to the constants obtained in (2.16) and (2.9) respectively.

Proposition 2.4. *Given $\bar{\alpha}, \lambda, \Lambda$, if the λ, Λ elliptic operator F satisfies*

$$(2.28) \quad |F(N) - \text{tr}(N)| < \tilde{\varepsilon}_0 \|N\|$$

for all $N \in S_n$, then any viscosity solution $u \in C(B_1)$ of (1.2) will be in $C^{2,\bar{\alpha}}(B_{1/4})$ and satisfy the following estimate

$$\|u\|_{C^{2,\bar{\alpha}}(B_{1/4})} \leq \tilde{C}_1 \|u\|_{L^\infty(B_1)}$$

for

$$(2.29) \quad \tilde{C}_1 = C_0 \left(1 + \frac{3}{1 - r_0^{\bar{\alpha}}}\right) \frac{2^{\bar{\alpha}}}{r_0^{1+\bar{\alpha}}} (2 + 2^{2+\bar{\alpha}})^2$$

where C_0, r_0 and $\tilde{\varepsilon}_0$ are as stated in Lemma 2.1.

Proof. We first prove that the $C^{2,\bar{\alpha}}$ estimate holds at the origin. As before, we denote $\|u\|_{L^\infty(B_1)} = M$.

We prove that there exists a polynomial P of degree 2 such that

$$(2.30) \quad |u(x) - P(x)| \leq MC'_0 |x|^{2+\bar{\alpha}} \quad \forall x \in B_1$$

$$F(D^2 P) = 0$$

$$\|P\|_{L^\infty(B_1)} \leq MC'_0$$

where $C'_0 = C_0 \left(1 + \frac{3}{1 - r_0^{\bar{\alpha}}}\right) \frac{1}{r_0^{1+\bar{\alpha}}}$. In order to prove the existence of such a polynomial P , we need the following claim.

Claim 2.5. *There exists a sequence of polynomials $\{P_k\}_{k=1}^\infty$ of degree 2 such that*

$$(2.31) \quad F(D^2 P_k) = 0$$

$$(2.32) \quad \|u - P_k\|_{L^\infty(B_{r_0^k})} \leq M r_0^{k(2+\bar{\alpha})}$$

where F and u are as defined in Proposition 2.4.

We first prove the claim.

Proof. : Let $P_0 = 0$. Then (2.32) holds good for the $k = 0$ case. We assume that (2.32) holds for $k \leq i$ and we prove it for $k = i + 1$.

Consider

$$v_i(x) = \frac{u(r_0^i x) - P_i(r_0^i x)}{r_0^{2i}}$$

for all $x \in B_1$. Define

$$F_i(N) = F(N + D^2 P_i)$$

for all $N \in S_n$. Since $F(D^2 P_i) = 0$ we see that $F_i(D^2 v_i) = 0$. Since

$$\|u - P_i\|_{L^\infty(B_{r_0^i})} \leq M r_0^{i(2+\bar{\alpha})},$$

we observe that

$$\|v_i\|_{L^\infty(B_1)} \leq \frac{M r_0^{i(2+\bar{\alpha})}}{r_0^{2i}} = M r_0^{i\bar{\alpha}}.$$

Note that the operator F_i satisfies the same properties as the operator F :

$$|DF_i(M) - DF_i(N)| = |DF(M + D^2 P_i) - DF(N + D^2 P_i)| \leq \tilde{\varepsilon}_0$$

and F_i also has the same ellipticity constants as F . We apply Lemma 2.1 to the equation $F_i(D^2 v_i) = 0$. This gives us the existence of a quadratic polynomial

$$(2.33) \quad \bar{P}_i = a_i + \vec{b}_i \cdot x + x^T \mathbf{c}_i \cdot x$$

such that

$$(2.34) \quad F_i(D^2 \bar{P}_i) = 0$$

$$(2.35) \quad \|v_i - \bar{P}_i\|_{L^\infty(B_{r_0})} \leq M r_0^{i\bar{\alpha}} r_0^{(2+\bar{\alpha})}$$

$$(2.36) \quad \|\bar{P}_i\|_{L^\infty(B_1)} \leq C_0 M r_0^{i\bar{\alpha}}.$$

We conclude immediately from (2.36) that

$$(2.37) \quad \begin{aligned} |a_i| &\leq C_0 M r_0^{i\bar{\alpha}} \\ \|b_i\| &\leq C_0 M r_0^{i\bar{\alpha}} \\ \|c_i\| &\leq C_0 M r_0^{i\bar{\alpha}}. \end{aligned}$$

Next, we define

$$(2.38) \quad P_{i+1} = P_i + r_0^{2i} \bar{P}_i(r_0^{-i} x).$$

From (2.34) we see that

$$F(D^2 P_{i+1}) = F_i(D^2 \bar{P}_i) = 0$$

and on substituting the expression for v_i into (2.35) we see that

$$\left\| \frac{u(r_0^i x) - P_i(r_0^i x)}{r_0^{2i}} - \bar{P}_i \right\|_{L^\infty(B_{r_0})} \leq M r_0^{i\bar{\alpha}} r_0^{(2+\bar{\alpha})}$$

which reduces to

$$\|u - P_{i+1}\|_{L^\infty(B_{r_0^{i+1}})} \leq M r_0^{(i+1)(2+\bar{\alpha})}.$$

This completes the inductive construction of the quadratic polynomial sequence. Hence the claim 2.5. \square

Using the above claim, we return to proving Proposition 2.4.

We show that this sequence $\{P_k\}_{k=1}^\infty$ is convergent and $\lim_{k \rightarrow \infty} P_k = P$ is the required polynomial in (2.30).

From (2.38), (2.33) we see that

$$(2.39) \quad P_{i+1} - P_i = r_0^{2i} a_i + r_0^{2i} \vec{b}_i \cdot x + x^T \mathbf{c}_i \cdot x.$$

Inequality (2.36) guarantees that the series $\sum_{i=1}^\infty (P_{i+1} - P_i)$ is bounded by a convergent geometric series

$$|P_{i+1} - P_i| \leq M C_0 r_0^{i\bar{\alpha}}.$$

Hence the telescopic series $\sum_{i=1}^{\infty} (P_{i+1} - P_i)$ converges uniformly on the unit ball and we define

$$P = \lim_{i \rightarrow \infty} P_i = \sum_{i=1}^{\infty} (P_{i+1} - P_i).$$

Note that $F(D^2P) = 0$ as $F(D^2P_i) = 0$ for all i . The limit P will be a quadratic polynomial as well.

For $x \in B_{r_0^i}$ we have, using (2.39), (2.37)

$$\begin{aligned} |P(x) - P_i(x)| &\leq \\ \sum_{j=i}^{\infty} |P_{j+1} - P_j| &\leq C_0 M \sum_{j=i}^{\infty} (r_0^{2j} r_0^{j\bar{\alpha}} + r_0^j r_0^{j\bar{\alpha}} r_0^i + r_0^i r_0^{j\bar{\alpha}} r_0^i) \\ &= C_0 M \left(\frac{r_0^{(2+\bar{\alpha})i}}{1 - r_0^{2+\bar{\alpha}}} + \frac{r_0^{(1+\bar{\alpha})i}}{1 - r_0^{1+\bar{\alpha}}} r_0^i + \frac{r_0^{i\bar{\alpha}}}{1 - r_0^{\bar{\alpha}}} r_0^{2i} \right) \\ &\leq 3C_0 M \frac{1}{1 - r_0^{\bar{\alpha}}} r_0^{(2+\bar{\alpha})i}. \end{aligned}$$

If we fix $x \in B_1$, we can choose an integer i such that

$$r_0^{i+1} < \|x\| \leq r_0^i.$$

Then we have the estimate

$$\begin{aligned} |u(x) - P(x)| &\leq |u(x) - P_i(x)| + |P_i(x) - P(x)| \\ &\leq MC_0 r_0^{i(2+\bar{\alpha})} + \frac{3MC_0}{1 - r_0^{\bar{\alpha}}} r_0^{i(2+\bar{\alpha})} \\ (2.40) \quad &\leq MC'_0 \|x\|^{2+\bar{\alpha}} \end{aligned}$$

where

$$(2.41) \quad C'_0 = C_0 \left(1 + \frac{3}{1 - r_0^{\bar{\alpha}}} \right) \frac{1}{r_0^{1+\bar{\alpha}}}.$$

This completes the proof of (2.30).

Next, consider any point x_0 in $B_{1/2}$. Let $v(x') = 4u(x'/2 + x_0)$ where $x \in B_1$. Note that $B_{1/2}(x_0) \subset B_1$ and hence $F(D^2v) = 0$ makes sense on B_1 . Applying estimate (2.40) to v for $x' = 2(x - x_0)$ yields a polynomial $P_{x_0}(x)$ such that

$$|u(x) - P_{x_0}(x)| \leq MC'_0 2^{\bar{\alpha}} \|x - x_0\|^{2+\bar{\alpha}}$$

holds on $B_{1/2}(x_0)$.

The following Lemma has been used in passing in the literature [CC95, Remark 3, page 74]. We state it here for precision in the estimate. For the proof see Corollary 4.2 in Appendix 1.

Lemma 2.6. *Suppose for all $x_0 \in B_{1/2}$ there a second order polynomial P_{x_0} such that*

$$|u(x) - P_{x_0}(x)| \leq K \|x - x_0\|^{2+\bar{\alpha}}$$

and

$$|P_{x_0}| \leq K$$

on B_1 . Then $\|D^2u\|_{C^{\bar{\alpha}}(\bar{B}_{1/4})} \leq (2 + 2^{2+\bar{\alpha}})^2 K$.

It follows from Lemma 2.6 that $u \in C^{2,\bar{\alpha}}(\bar{B}_{1/4}(0))$ with bounds given by

$$(2.42) \quad \|D^2u\|_{C^{\bar{\alpha}}(\bar{B}_{1/4}(0))} \leq C'_0 2^{\bar{\alpha}} (2 + 2^{2+\bar{\alpha}})^2 \|u\|_{L^\infty(B_1)}.$$

Combining (2.41) with (2.42) proves the estimate (2.29). \square

Proof of Theorem 1.3. :

We are assuming that F is an operator on the space of symmetric matrices, and thus we can take a DF that is symmetric. Let

$$W = DF(\mathbf{0})$$

which will be a positive symmetric matrix, by ellipticity. In particular

$$\lambda I \leq W \leq \Lambda I.$$

We can find a positive square root of the inverse, namely

$$(2.43) \quad AA^T = W^{-1}.$$

Now define

$$\tilde{F}(N) = F(ANA^T).$$

Observe

$$\begin{aligned} \frac{\partial \tilde{F}}{\partial n_{ij}}|_{N=\mathbf{0}} &= \frac{\partial F}{\partial a_{pq}}|_{\mathbf{0}} \frac{\partial (ANA^T)_{pq}}{\partial n_{ij}} \\ &= \sum_{p,q} \frac{\partial F}{\partial a_{pq}}|_{\mathbf{0}} A_{pi} A_{jq}^T \\ &= W_{pq} A_{pi} A_{jq}^T = (A^T W A)_{ij}. \end{aligned}$$

But by (2.43),

$$A^T W A = I.$$

It follows that $D\tilde{F}(\mathbf{0}) = I$. Note that \tilde{F} has ellipticity constants in $[\frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}]$.

Finally, note that if F satisfies a ε_0 closeness condition then

$$\begin{aligned} \left\| D\tilde{F}(M) - D\tilde{F}(N) \right\| &= \left\| A^T DF(AMA^T)A - A^T DF(ANA^T)A \right\| \\ &= \left\| A^T \left(DF(AMA^T) - DF(ANA^T) \right) A \right\| \\ &\leq \varepsilon_0 \left\| A^T A \right\| \leq \varepsilon_0 \Lambda. \end{aligned}$$

Therefore, \tilde{F} is almost linear with constant $\varepsilon_0 \Lambda$.

Now we let

$$(2.44) \quad \varepsilon_0(n, \lambda, \Lambda, \bar{\alpha}) := \frac{1}{\Lambda} \tilde{\varepsilon}_0\left(n, \frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}, \bar{\alpha}\right)$$

for $\tilde{\varepsilon}_0$ defined in Lemma 2.1. It follows that \tilde{F} satisfies the $\tilde{\varepsilon}_0$ criterion of 2.1 when F satisfies the ε_0 closeness condition. Now let

$$v(x) = u\left((A^{-1})^T x\right).$$

Notice that

$$\begin{aligned} v_{ij}(x) &= u_{kl} A_{jk}^{-1} A_{il}^{-1} \\ D^2 v &= A^{-1} D^2 u \left((A^{-1})^T x \right) (A^{-1})^T \end{aligned}$$

so

$$\begin{aligned} \tilde{F}(N) &= F(AA^{-1} D^2 u \left((A^{-1})^T x \right) (A^{-1})^T A^T) \\ &= F\left(D^2 u \left((A^{-1})^T x \right)\right) = 0. \end{aligned}$$

Now if u is defined on B_1 , the new function v will be defined on $B_{\frac{1}{\sqrt{\Lambda}}}$.

Rescaling

$$\tilde{v} = \Lambda v \left(\frac{x}{\sqrt{\Lambda}} \right)$$

we have a function defined on B_1 and can apply Proposition 2.4 to \tilde{v} :

$$\|D^2 \tilde{v}\|_{C^{\bar{\alpha}}(B_{1/4})} \leq \tilde{C}_1 \|\tilde{v}\|_{L^\infty(B_1)} \leq \Lambda \tilde{C}_1 \|u\|_{L^\infty(B_1)}.$$

Meanwhile, provided that

$$\begin{aligned} \sqrt{\Lambda} A^T x &\in B_{1/4} \\ \sqrt{\Lambda} A^T y &\in B_{1/4}, \end{aligned}$$

we have

$$\begin{aligned}
\frac{\|D^2u(x) - D^2u(y)\|}{|x - y|^{\bar{\alpha}}} &= \frac{\|A(D^2v(A^Tx) - D^2v(A^Ty))A^T\|}{|x - y|^{\bar{\alpha}}} \\
&= \frac{\|A(D^2\tilde{v}(\sqrt{\Lambda}A^Tx) - D^2\tilde{v}(\sqrt{\Lambda}A^Ty))A^T\|}{|x - y|^{\bar{\alpha}}} \\
&\leq \frac{\Lambda}{|x - y|^{\bar{\alpha}}} \|D^2\tilde{v}\|_{C^{\bar{\alpha}}(B_{1/4})} \left| \sqrt{\Lambda}A^Tx - \sqrt{\Lambda}A^Ty \right|^{\bar{\alpha}} \\
&\leq \Lambda^{1+\bar{\alpha}} \|D^2\tilde{v}\|_{C^{\bar{\alpha}}(B_{1/4})} \\
&\leq \Lambda^{2+\bar{\alpha}} \tilde{C}_1 \|u\|_{L^\infty(B_1)}.
\end{aligned}$$

We conclude that for $x \in B_{1/4\Lambda}$ the estimate holds. \square

3. PROOF OF THEOREM 1.4

To begin proving Theorem 1.4 we require the following version of [CC95, Lemma 7.9]:

Lemma 3.1. *Let u be a viscosity solution of (1.1) in $B_{4/7}$ such that $\|u\|_{L^\infty(B_{4/7})} \leq 1$ and $f \in L^n(B_{4/7})$. Assume that $F(D^2w) = 0$ has $C^{1,1}$ interior estimates (with constant C_1). Then there exists a function $h \in C^2(\bar{B}_{3/7})$ such that h satisfies $\|h\|_{C^{1,1}(\bar{B}_{3/7})} \leq c(n)C_1$ (for a constant $c(n)$ depending only on n) and*

$$\begin{aligned}
(3.1) \quad &\|u - h\|_{L^\infty(B_{3/7})} \leq C_3 \|f\|_{L^n(B_{4/7})} \\
&F(D^2h) = 0 \quad \text{in } B_{1/2} \\
&h = u \quad \text{on } \partial B_{1/2}.
\end{aligned}$$

Here C_3 is a positive constant depending on n, λ, Λ, C_1 .

Note: We say that $F(D^2w) = 0$ has $C^{1,1}$ interior estimates (with constant C_1) if for any $w_0 \in C(\partial B)$ there exists a solution $w \in C^2(B_1) \cap C(\bar{B}_1)$ of

$$\begin{aligned}
F(D^2w) &= 0 \quad \text{in } B_1 \\
w &= w_0 \quad \text{on } \partial B_1
\end{aligned}$$

such that $\|w\|_{C^{1,1}(\bar{B}_{1/2})} \leq C_1 \|w_0\|_{L^\infty(\partial B_1)}$.

Proof. The statement in [CC95, lemma 7.9] is given for elliptic operators $F(D^2w, x)$ that may depend also on x . The obvious approximation argument when there is no dependence on x gives the proof of Lemma 3.1. \square

Lemma 3.2. *There exists $\delta > 0$ depending on n, λ, Λ , and $\alpha < \bar{\alpha}$ such that if u is a viscosity solution of (1.1) in B_1 with F almost linear with constant $\varepsilon_0(n, \lambda, \Lambda, \bar{\alpha})$ with*

$$\|u\|_{L^\infty(B_1)} \leq 1$$

and

$$(3.2) \quad \left(\frac{1}{|B_r|} \int_{B_r} |f|^n \right)^{1/n} \leq \delta r^\alpha \quad \forall r \leq 1$$

then there exists a polynomial P of degree 2 such that

$$(3.3) \quad \begin{aligned} \|u - P\|_{L^\infty(B_r)} &\leq C_4 r^{2+\alpha} \quad \forall r \leq 1, \\ |DP(0)| + \|D^2P\| &\leq C_4 \end{aligned}$$

for some constant $C_4 > 0$ depending only on $n, \lambda, \Lambda, \alpha$.

Proof. The proof follows from the following claim.

Claim 3.3. *Given λ, Λ , and $\alpha < \bar{\alpha}$, suppose that u is a viscosity solution of (1.1) in B_1 for F almost linear with constant $\varepsilon_0(n, \lambda, \Lambda, \bar{\alpha})$, with f satisfying (3.2) and u satisfying*

$$(3.4) \quad \|u\|_{L^\infty(B_1)} \leq 1.$$

Then there exists $\delta > 0$, $0 < \mu < 1$ and a sequence

$$P_k(x) = a_k + b_k \cdot x + \frac{1}{2} x^t c_k \cdot x$$

satisfying

$$(3.5) \quad F(D^2P_k) = 0$$

$$(3.6) \quad \|u - P_k\|_{L^\infty(B_{\mu^k})} \leq \mu^{k(2+\alpha)}$$

$$(3.7) \quad |a_k - a_{k-1}| + \mu^{k-1} |b_k - b_{k-1}| + \mu^{2(k-1)} |c_k - c_{k-1}| \leq C_1 \mu^{(k-1)(2+\alpha)}.$$

We first prove the claim.

Proof. Let $P_0 = 0$. Then for $k = 0$, we see that (3.6) holds trivially for any $\mu > 0$ from (3.4). For μ determined by (3.8), we will show that whenever (3.6) holds for $k = i$, then there exist P_{i+1} so that (3.6) holds for $k = i + 1$.

We choose μ small enough such that

$$(3.8) \quad 2C_1 \mu^{\bar{\alpha}} \leq \mu^\alpha$$

and

$$(3.9) \quad \mu^\alpha \leq 3/7.$$

We define

$$(3.10) \quad \begin{aligned} v_i(x) &= \frac{(u - P_i)(\mu^i x)}{\mu^{i(2+\alpha)}}, \\ F_i(N) &= \frac{F(\mu^{i\alpha} N + c_i)}{\mu^{i\alpha}}, \\ f_i(x) &= \frac{f(\mu^i x)}{\mu^{i\alpha}}, \end{aligned}$$

where $P_i(x) = a_i + \vec{b}_i \cdot x + \frac{1}{2}x^T \cdot \mathbf{c}_i x$. Thus

$$(3.11) \quad F_i(D^2 v_i(x)) = f_i(x).$$

Note that

$$(3.12) \quad \|v_i\|_{L^\infty(B_1)} \leq 1$$

by (3.6). Now we choose δ small enough such that

$$(3.13) \quad \omega_n^{1/n} C_3 \delta \leq C_1 \mu^{2+\bar{\alpha}}$$

where ω_n is the volume of a unit ball in n dimensions and C_3 is the constant appearing in the first inequality of (3.1) in Lemma 3.1.

We consider the equation (3.11). Observe that (3.2) implies

$$(3.14) \quad \|f_i\|_{L^n(B_1)} = \mu^{-i\alpha} \mu^{-i} \|f\|_{L^n(B_{\mu^i})}$$

$$(3.15) \quad \leq \mu^{-i\alpha} \mu^{-i} |B_{\mu^i}|^{1/n} \delta \mu^{i\alpha} = (\omega_n)^{1/n} \delta.$$

Note that F_i satisfies

$$F_i(0) = \frac{F(c_i)}{\mu^{i\alpha}} = \frac{F(D^2 P_i)}{\mu^{i\alpha}} = 0$$

and

$$DF_i(N) = DF(\mu^{i\alpha} N + c_i)$$

so F_i also satisfies the $\varepsilon_0(n, \lambda, \Lambda, \bar{\alpha})$ closeness condition (1.5) when F does. Since $\|v_i\|_{L^\infty(B_1)} \leq 1$, by applying Lemma 3.1 to (3.11) considering (3.14) we see that there exists $h \in C^2(\bar{B}_{3/7})$ such that

$$(3.16) \quad \|v_i - h\|_{L^\infty(B_{3/7})} \leq \omega_n^{1/n} C_3 \delta$$

and h solves the following boundary value problem:

$$(3.17) \quad \begin{aligned} F_i(D^2 h) &= 0 \text{ in } B_{1/2} \\ h &= v_i \text{ on } \partial B_{1/2}. \end{aligned}$$

Then from the definition of F_i above, it follows that

$$(3.18) \quad F(\mu^{i\alpha} D^2 h + c_i) = 0 \text{ in } B_{1/2}.$$

Now apply Theorem 1.3 to h so see that

$$(3.19) \quad \|h\|_{C^{2,\bar{\alpha}}(B_{1/4})} \leq C_1 \|h\|_{L^\infty(\partial B_{1/2})}$$

$$(3.20) \quad \leq C_1 \|v_i\|_{L^\infty(\partial B_{1/2})}$$

$$(3.21) \quad \leq C_1$$

from (3.17) and the maximum principle (cf. [CC95, Proposition 2.13]), and the last inequality follows from (3.12). Since h is $C^{2,\bar{\alpha}}$, there exists a polynomial \bar{P} given by

$$\bar{P}(x) = h(0) + Dh(0) \cdot x + \frac{1}{2} x^t D^2 h(0) \cdot x$$

such that

$$(3.22) \quad \|h - \bar{P}\|_{L^\infty(B_\mu)} \leq C_1 \mu^{2+\bar{\alpha}}.$$

From (3.16), (3.9) and (3.22) we have

$$\begin{aligned} \|v_i - \bar{P}\|_{L^\infty(B_\mu)} &\leq \|v_i - h\|_{L^\infty(B_\mu)} + \|h - \bar{P}\|_{L^\infty(B_\mu)} \\ &\leq \omega_n^{1/n} C_3 \delta + C_1 \mu^{2+\bar{\alpha}} \\ &\leq 2C_1 \mu^{2+\bar{\alpha}} \\ (3.23) \quad &\leq \mu^{2+\alpha} \end{aligned}$$

where the last two inequalities follow from (3.13) and (3.8).

Rescaling the bound (3.23) back via (3.10) we see that

$$(3.24) \quad |u(x) - P_i(x) - \mu^{i(2+\alpha)} \bar{P}(\mu^{-i}x)| \leq \mu^{(2+\alpha)(i+1)}$$

for all $x \in B_{\mu^{i+1}}$.

We define

$$(3.25) \quad P_{i+1}(x) = P_i(x) + \mu^{i(2+\alpha)} \bar{P}(\mu^{-i}x)$$

and we have

$$(3.26) \quad \mathbf{c}_{i+1} = \mathbf{c}_i + \mu^{i\alpha} D^2 h(0).$$

From (3.24) we see that

$$\|u - P_{i+1}\|_{L^\infty(B_{\mu^{i+1}})} \leq \mu^{(i+1)(2+\alpha)}$$

which proves (3.6) for $k = i + 1$. Now from (3.18) and (3.26) we get

$$F(\mathbf{c}_{i+1}) = 0$$

proving (3.5). Now evaluating (3.25) and its first and second derivatives at $x = 0$ yields

$$\begin{aligned} a_{i+1} - a_i &= \mu^{i(2+\alpha)} \bar{P}(0) \\ \vec{b}_{i+1} - \vec{b}_i &= \mu^{i(1+\alpha)} D\bar{P}(0) \\ \mathbf{c}_{i+1} - \mathbf{c}_i &= \mu^{i\alpha} D^2\bar{P}(0). \end{aligned}$$

Thus

$$\begin{aligned} &|a_{i+1} - a_i| + \mu^i \left\| \vec{b}_{i+1} - \vec{b}_i \right\| + \mu^{2i} \|\mathbf{c}_{i+1} - \mathbf{c}_i\| \\ &= \mu^{i(2+\alpha)} (|h(0)| + \|Dh(0)\| + \|D^2h(0)\|) \\ &\leq \mu^{i(2+\alpha)} C_1 \end{aligned}$$

by (3.21), proving (3.7). This proves claim 3.3. \square

Now we return to proving Lemma 3.2, which will follow by arguments similar to those used in the proof of Theorem 1.3 following (2.39). In particular, define

$$P = \lim_{i \rightarrow \infty} P_i = \sum_{i=0}^{\infty} (P_{i+1} - P_i)$$

which will have coefficients

$$\begin{aligned} a &= \sum_{i=0}^{\infty} (a_{i+1} - a_i) \\ \vec{b} &= \sum_{i=0}^{\infty} (\vec{b}_{i+1} - \vec{b}_i) \\ \mathbf{c} &= \sum_{i=0}^{\infty} (\mathbf{c}_{i+1} - \mathbf{c}_i). \end{aligned}$$

Note that by (3.7)

$$\begin{aligned} |a_{i+1} - a_i| &\leq C_1 \mu^{i(2+\alpha)} \\ \left\| \vec{b}_{i+1} - \vec{b}_i \right\| &\leq C_1 \mu^{i(1+\alpha)} \\ \|\mathbf{c}_{i+1} - \mathbf{c}_i\| &\leq C_1 \mu^{i\alpha}. \end{aligned}$$

We conclude that the tails of the constant, linear, and quadratic terms of the polynomial series converge uniformly with upper bounds given

by

$$\begin{aligned} \left| \sum_{j=i}^{\infty} (a_{j+1} - a_j) \right| &\leq C_1 \mu^{i(2+\alpha)} \frac{1}{1 - \mu^{(2+\alpha)}} \\ \left| \sum_{j=i}^{\infty} (\vec{b}_{j+1} - \vec{b}_j) \right| &\leq C_1 \mu^{i(1+\alpha)} \frac{1}{1 - \mu^{(1+\alpha)}} \\ \left| \sum_{j=i}^{\infty} (\mathbf{c}_{j+1} - \mathbf{c}_j) \right| &\leq C_1 \mu^{i\alpha} \frac{1}{1 - \mu^\alpha} \end{aligned}$$

respectively. Thus P is well-defined. Next,

$$\begin{aligned} \|u - P\|_{L^\infty(B_{\mu^i})} &\leq \|u - P_i\|_{L^\infty(B_{\mu^i})} + \sum_{j=i}^{\infty} \|P_{j+1} - P_j\|_{L^\infty(B_{\mu^i})} \\ &\leq \mu^{i(2+\alpha)} + \sum_{j=i}^{\infty} [|a_{j+1} - a_j| + \mu^i \|\vec{b}_{j+1} - \vec{b}_j\| + \frac{1}{2} \mu^{2i} \|\mathbf{c}_{j+1} - \mathbf{c}_j\|] \\ &\leq \mu^{i(2+\alpha)} + C_1 \left\{ \begin{array}{l} \mu^{i(2+\alpha)} \frac{1}{1 - \mu^{(2+\alpha)}} \\ + \mu^i \mu^{i(1+\alpha)} \frac{1}{1 - \mu^{(1+\alpha)}} \\ + \mu^{2i} \mu^{i\alpha} \frac{1}{1 - \mu^\alpha} \end{array} \right\} \\ &\leq C_4 \mu^{i(2+\alpha)} \end{aligned}$$

where

$$C_4 = 1 + C_1 \left(\frac{1}{1 - \mu^\alpha} + \frac{1}{1 - \mu^\alpha} + \frac{1}{1 - \mu^\alpha} \right).$$

Clearly we have

$$|DP(0)| + \|D^2P\| \leq C_1 \frac{1}{1 - \mu^{1+\alpha}} + C_1 \frac{1}{1 - \mu^\alpha}.$$

We see that (3.3) holds good for C_4 . This proves the lemma. \square

Proof of Theorem 1.4. Fix $\alpha < \bar{\alpha}$. We will first prove that the estimate (1.8) holds at the origin, in particular, we show that there exists a polynomial of degree 2 such that

$$\begin{aligned} \|u - P\|_{L^\infty(B_r)} &\leq C'_2 r^{2+\alpha}, \forall r \leq 1 \\ (3.27) \quad |DP(0)| + \|D^2P\| &\leq C'_2 \end{aligned}$$

where $C'_2 = C'_2(\|u\|_{L^\infty(B_1)}, |f|_{C^\alpha(B_1)}, n, \lambda, \Lambda, \bar{\alpha}, \alpha, C_1)$, $0 < \alpha < \bar{\alpha}$ and $\bar{\alpha}$ is the Hölder exponent appearing in (1.6):

$$\|u\|_{C^{2,\bar{\alpha}}(B_{1/2})} \leq C_1 \|u\|_{L^\infty(B_1)}.$$

Let

$$\tilde{f}(x) = f(x) - f(0)$$

so that the C^α function $\tilde{f}(x)$ satisfies the following

$$\left(\frac{1}{|B_1|} \int_{B_1} |\tilde{f}|^n\right)^{1/n} \leq \|\tilde{f}\|_{C^\alpha(B_1)}.$$

The proof now follows directly from Lemma 3.2, if we do the following rescaling for all $x \in B_1$: Consider the following function

$$\tilde{u}(x) = \frac{u(x)}{\delta^{-1}|f|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}} = \frac{u(x)}{T}.$$

with $\delta(n, \lambda, \Lambda, \alpha, \bar{\alpha})$ as defined in (3.13). Note that

$$\begin{aligned} \delta T &= \|\tilde{f}\|_{C^\alpha(B_1)} + \delta\|u\|_{L^\infty(B_1)} \\ (3.28) \quad &> \|\tilde{f}\|_{C^\alpha(B_1)} \end{aligned}$$

and that

$$\|\tilde{u}\|_{L^\infty(B_1)} \leq 1.$$

Now we consider the operator

$$F_T(N) = \frac{1}{T}F(TN)$$

defined for all $N \in S_n$.

Note that F_T satisfies the following properties:

- (i) F_T has the same ellipticity constants λ and Λ as F .
- (ii) DF_T satisfies condition (1.5) with the same constant $\varepsilon_0(n, \lambda, \Lambda, \bar{\alpha})$ if DF does.

We see that \tilde{u} satisfies the equation

$$F_T(D^2\tilde{u}(x)) = \frac{1}{T}F(TD^2\tilde{u}(x)) = \frac{1}{T}F(D^2u(x)) = \frac{\tilde{f}(x)}{T} = f_T(x),$$

where for $r \leq 1$ we compute

$$\begin{aligned} \left(\frac{1}{|B_r|} \int_{B_r} |f_T|^n\right)^{1/n} &\leq \frac{\|\tilde{f}\|_{C^\alpha(B_r)}}{T} \left(\frac{1}{1+\alpha}\right)^{1/n} r^\alpha \\ &< \delta r^\alpha \end{aligned}$$

recalling (3.28) in the last inequality.

Therefore, the equation

$$F_T(D^2\tilde{u}(x)) = f_T(x)$$

satisfies all the conditions of Lemma 3.2 and hence the function \tilde{u} satisfies the estimates (3.3). In particular, there exists \tilde{P} such that

$$(3.29) \quad \|\tilde{u} - \tilde{P}\|_{L^\infty(B_r)} \leq C_4 r^{2+\alpha}, \quad \forall r \leq 1,$$

$$(3.30) \quad |D\tilde{P}(0)| + \|D^2\tilde{P}\| \leq C_4(n, \lambda, \Lambda, \alpha)$$

that is, letting

$$P = (\delta^{-1}|f|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}) \tilde{P}$$

we have

$$(3.31) \quad \begin{aligned} \|u - P\|_{L^\infty(B_r)} &\leq C_4 r^{2+\alpha} \quad \forall r \leq 1, \\ |DP(0)| + \|D^2P\| &\leq (\delta^{-1}|f|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)}) C_4(n, \lambda, \Lambda, \alpha). \end{aligned}$$

Next, consider any point x_0 in $B_{1/2}$. The remainder of the proof follows verbatim from the argument following (2.41). \square

4. APPENDIX 1: POINTWISE HÖLDER IMPLIES HÖLDER

Lemma 4.1. *Suppose that*

$$U : B_R(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

satisfies the following condition for some fixed $p > 0$. For every y there exists a linear function L_y such that

$$(4.1) \quad |U(x) - L_y(x)| \leq C_1 |x - y|^{1+p}.$$

Then for all $x \in B_{R/2}(0)$ we have

$$|DU(x) - DU(0)| \leq C_1 (2 + 2^{1+p}) |x|^p.$$

Proof. We will assume by adding a linear function that

$$(4.2) \quad \begin{aligned} U(0) &= 0 \\ DU(0) &= 0. \end{aligned}$$

First note that (4.1) implies that the derivative exists at any x_0 and

$$L_{x_0} = DU(x_0) \cdot (x - x_0) + U(x_0)$$

thus

$$|U(x) - DU(x_0) \cdot (x - x_0) - U(x_0)| \leq C_1 |x - x_0|^{1+p}.$$

That is

$$(4.3) \quad |DU(x_0) \cdot (x - x_0)| \leq C_1 |x - x_0|^{1+p} + |U(x)| + |U(x_0)|.$$

Now consider any point $x_0 \neq 0$ with $x_0 \in B_{R/2}$. Let

$$(4.4) \quad A = DU(x_0)$$

and let

$$e = \frac{A}{\|A\|}.$$

Now consider the point

$$x_1 = \left(x_0 + |x_0| \frac{A}{\|A\|} \right)$$

which satisfies

$$|x_1| \leq 2|x_0|.$$

So $x_1 \in B_R$. Letting $y = 0$ in (4.1) and using (4.2) we conclude

$$(4.5) \quad |U(x_1)| \leq C_1 2^{1+p} |x_0|^{1+p}.$$

Plugging x_1 into (4.3) and using (4.5)

$$(4.6) \quad |DU(x_0) \cdot (x_1 - x_0)| \leq C_1 |x_1 - x_0|^{1+p} + |U(x_1)| + |U(x_0)|$$

$$(4.7) \quad \leq C_1 |x_1 - x_0|^{1+p} + C_1 2^{1+p} |x_0|^{1+p} + C_1 |x_0|^{1+p}.$$

But

$$x_1 - x_0 = x_0 + |x_0| \frac{A}{\|A\|} - x_0 = |x_0| \frac{A}{\|A\|} = |x_0| \frac{DU(x_0)}{\|DU(x_0)\|}$$

and

$$|x_1 - x_0| = \left| |x_0| \frac{DU(x_0)}{\|DU(x_0)\|} \right| = |x_0|.$$

So we have shown that

$$(4.8) \quad |x_0| \|DU(x_0)\| \leq C_1 |x_0|^{1+p} + C_1 2^{1+p} |x_0|^{1+p} + C_1 |x_0|^{1+p}$$

that is

$$\|DU(x_0)\| \leq C_1 (2 + 2^{1+p}) |x_1 - x_0|^p.$$

□

Corollary 4.2. *Suppose that*

$$u : B_1(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

satisfies the following condition for some fixed $p > 0$. For every $y \in B_{1/2}$ there exists a quadratic function Q_y such that

$$(4.9) \quad |U(x) - Q_y(x)| \leq C_1 |x - y|^{2+\alpha}.$$

Then for $x \in B_{1/4}(0)$

$$\sup_{i,j} |u_{ij}(x) - u_{ij}(0)| \leq (2 + 2^{2+\alpha})^2 C_1 |x|^\alpha.$$

Proof. As before subtract off a quadratic function so that u vanishes at second order at 0. Apply the previous Lemma with $p = 1 + \alpha$ and conclude that for all $x \in B_{1/2}$

$$|DU(x) - DU(0)| \leq C_1 (2 + 2^{2+\alpha}) |x|^{1+\alpha}$$

that is

$$|u_i(x)| \leq (2 + 2^{2+\alpha}) C_1 |x|^{1+\alpha}.$$

So we apply the previous Lemma, with $R = 1/2$ and conclude that

$$|Du_i(x)| \leq (2 + 2^{2+\alpha})^2 C_1 |x|^\alpha.$$

□

5. APPENDIX 2: CORDES-NIRENBERG

In [Nir53, Lemma 3], Nirenberg proved the following result (slightly reworded).

Lemma 5.1. *Let $U = (u_1, u_2)$ be an \mathbb{R}^2 -valued continuous function defined in a domain $B_1 \subset \mathbb{R}^2$ having continuous first derivatives satisfying*

$$(5.1) \quad u_{1,1}^2 + u_{1,2}^2 + u_{2,1}^2 + u_{2,2}^2 \leq k (u_{1,2}u_{2,1} - u_{1,1}u_{2,2}) + k_1$$

and let $d < 1$.

Then there exists M, α depending on k, k_1 , and d such that

$$\int \int_{B_d(0)} r^{-\alpha} (u_{1,1}^2 + u_{1,2}^2 + u_{2,1}^2 + u_{2,2}^2) dx dy \leq M.$$

With this integral estimate in hand, a universal Hölder estimate on the functions u_1 and u_2 follows.

Now suppose that

$$(5.2) \quad a^{ij} u_{ij} = f.$$

Note

$$\Delta u = (\delta^{ij} - a^{ij}) u_{ij} + f$$

which implies

$$|\Delta u| \leq \|\delta^{ij} - a^{ij}\|_{HS} \|u_{ij}\|_{HS} + f.$$

In two dimensions, we have

$$\|D^2 u\|^2 = (\Delta u)^2 - 2 \det(D^2 u)$$

so

$$\|D^2 u\|^2 \leq (1+\varepsilon) \|\delta^{ij} - a^{ij}\|_{HS}^2 \|D^2 u\|_{HS}^2 + (1+\frac{1}{\varepsilon}) f^2 + 2(u_{1,2}u_{2,1} - u_{1,1}u_{2,2}).$$

In particular, (5.1) holds with constants

$$k = \frac{2}{1 - (1 + \varepsilon) \|\delta^{ij} - a^{ij}\|_{HS}^2}$$

$$k_1 = \left(1 + \frac{1}{\varepsilon}\right) \|f\|_{L^\infty}.$$

Thus in two dimensions a $C^{1,\alpha}$ estimate is available provided

$$\|\delta^{ij} - a^{ij}\|^2 < 1.$$

For higher dimensions, in [Nir54, Lemma 3], Nirenberg stated the following generalization

Theorem 5.2. *Let $U = (u_1, u_2, \dots, u_n)$ be an \mathbb{R}^n -valued continuous function defined in a domain $B_1 \subset \mathbb{R}^n$ having continuous first derivatives satisfying*

$$(5.3) \quad \sum_{i,j} u_{i,j}^2 \leq k \sum_{i,j} (u_{i,j} u_{j,i} - u_{i,i} u_{j,j}) + k_1$$

and in addition

$$k < \frac{n-1}{n-2}.$$

Then the functions u_i are Hölder continuous on the interior domain.

The proof of Theorem 5.2 would follow from an integral estimate of the form Lemma 5.1. However a proof is not given, although it is stated [Nir54, Section 3] that the proof of Theorem 5.2 is “similar” to the proof of Lemma 5.1.

In any case, the result of Cordes in 1956 [Cor56, page 292] provides better constants: Cordes defines the K'_ε -condition for a symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ as:

$$(n-1) \left(1 + \frac{n(n-2)}{(n+1)(n-1)}\right) \sum_{i < k} (\lambda_i - \lambda_k)^2 \leq (1 - \varepsilon) \left(\sum_i \lambda_i\right)^2.$$

Cordes proves the following [Cor56, Satz 8, page 303] :

Theorem 5.3. *Suppose the coefficients a^{ij} satisfy a K'_ε -condition. There exists an α depending on ε such that the solutions to (5.2) satisfy an estimate of the form*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} < c \left(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)} \right).$$

The proof involves pages of integrals. In 1961, [Cor61, Theorem 2], Cordes offered an outline for a refined argument, and summarized the results (in English).

The “Cordes condition” in the literature is often phrased as the following:

$$(5.4) \quad \|A\|_{HS}^2 < \frac{1}{n-1+\delta} |Tr(A)|^2.$$

Note that this is equivalent (for ε not equal to but depending on δ) to the K_ε -condition defined by Cordes in [Cor56, page 292]:

$$(n-1) \sum_{i < k} (\lambda_i - \lambda_k)^2 \leq (1-\varepsilon) \left(\sum_i \lambda_i \right)^2.$$

Cordes showed solutions to (5.2) will be C^α for f bounded. Talenti [Tal65] applied this condition to show that solutions to (5.2) exist in $W^{2,2}$ when $f \in L^2$.

It is interesting to look at the linearized operator for nonlinear equations of the form (1.2), in particular when equation (1.2) is neither convex nor concave. If the linearized operator satisfies a K'_ε -condition, then C^3 solutions will be $C^{2,\alpha}$ with uniform estimates based on the C^1 norm.

In general, a regularity boosting with estimates for equations of the form (1.2) can follow by applying Cordes-Nirenberg type results, locally, to smooth solutions, even when the operator does not globally satisfy such a condition. For a given nonlinear equation one may differentiate (1.2). When the oscillation of the linearized operator F^{ij} depends continuously on the oscillation of D^2u , there will be a δ_0 such that if the oscillation of the Hessian is smaller than δ_0 the oscillation of F^{ij} will be less than ε_0 , thus $C^{2,\alpha}$ estimates apply. In particular, any modulus of continuity on the Hessian can be used to derive Hölder continuity: Essentially, the results in [CLW11] can be “quantized”. (Keep in mind that we may always use a transformation like the one following (2.43), locally, so that the equation satisfies a K'_ε -condition nearby). Bootstrapping, using Schauder theory on difference quotients, one can derive estimates of all orders. In particular, the full suite of estimates can be derived by knowing the Hessian is nearly continuous.

We record the following corollary which follows immediately from this discussion.

Corollary 5.4. *Suppose that u is an entire quadratic solution to $F(D^2u) = 0$, for $F \in C^{1,\beta}$. Then there is an $\varepsilon_0(\|F\|_{C^{1,\beta}}, n) > 0$ such that any solution u' with*

$$\|D^2u - D^2u'\| < \varepsilon_0$$

must also be quadratic.

Thus quadratic solutions are rigid with respect to the global C^2 norm.

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