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Regularity Bootstrapping for 4th-order Nonlinear Elliptic Equations

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We consider nonlinear 4th-order elliptic equations of double divergence type. We show that for a certain class of equations where the nonlinearity is in the Hessian, solutions that are $C^{2,\alpha}$ enjoy interior estimates on all derivatives.

1 Introduction

In this paper, we develop Schauder and bootstrapping theory for solutions to 4th-order nonlinear elliptic equations of the following double divergence form:

$$\int_{\Omega} a^{ij,kl}(D^2u)u_{ij}\eta_{kl}\mathrm{d}x = 0, \forall \eta \in C_0^{\infty}(\Omega) \tag{1.1}$$

in $B_1 = B_1(0)$. For the Schauder theory, we require the standard Legendre–Hadamard ellipticity condition

$$a^{ij,kl}(D^2u(x))\xi_{ij}\xi_{kl} \geq \Lambda|\xi_{rs}|^2; \tag{1.2}$$

while in order to bootstrap, we will require the following condition:

$$b^{ij,kl}(D^2u(x)) = a^{ij,kl}(D^2u(x)) + \frac{\partial a^{pq,kl}}{\partial u_{ij}}(D^2u(x))u_{pq}(x) \tag{1.3}$$

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satisfies

$$b^{ij,kl}(D^2u(x))\xi_{ij}\xi_{kl} \geq \Lambda_1 \|\xi\|^2. \tag{1.4}$$

Our main result is the following: suppose that conditions (1.1) and (1.4) are met on some open set $U \subseteq S^{n\times n}$ (space of symmetric matrices). If u is a $C^{2,\alpha}$ solution with $D^2u(B_1) \subset U$, then u is smooth on the interior of the domain B_1.

One example of such an equation is the Hamiltonian stationary Lagrangian equation, which governs Lagrangian surfaces that minimize the area functional

$$\int_\Omega \sqrt{\det\left(I + (D^2u)^T D^2u\right)}\mathrm{d}x \tag{1.5}$$

among potential functions u. (cf. [9], [10, Proposition 2.2]). The minimizer satisfies a 4th-order equation, that, when smooth, can be factored into a Laplace type operator on a nonlinear quantity. Recently in [2], it is shown that a C^2 solution is smooth. The results in [2] are the combination of an initial regularity boost, followed by applications of the 2nd-order Schauder theory as in [1].

More generally, for a functional F on the space of matrices, one may consider a functional of the form

$$\int_M F(D^2u)\mathrm{d}x.$$

The Euler–Lagrange equation will generically be of the following double divergence type:

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial F}{\partial u_{ij}}(D^2u)\right) = 0. \tag{1.6}$$

Equation (1.6) need not factor into 2nd-order operators, so it may be genuinely a 4th-order double divergence elliptic-type equation. It should be noted that in general, (1.6) need not take the form of (1.1). It does when $F(D^2u)$ can be written as a function of $D^2u^TD^2u$ (as for example (1.5)). Our results in this paper apply to a class of Euler–Lagrange equations arising from such functionals. In particular, we will show that if F is a convex function of D^2u and a function of $D^2u^TD^2u$ (such as 1.5 when $|D^2u| \leq 1$) then $C^{2,\alpha}$ solutions will be smooth.

The Schauder theory for 2nd-order divergence and non-divergence type elliptic equations is by now well-developed, see [7], [6] and [1]. For higher-order non-divergence equations, Schauder theory is available, see [11]. However, for higher-order equations in divergence form, much less is known. One expects the results to be different; for 2nd-order equations, solutions to divergence type equations with C^{α} coefficients are known to be $C^{1,\alpha}$, [7, Theorem 3.13], whereas for non-divergence equations, solutions will be $C^{2,\alpha}$ [6, Chapter 6]. Recently, Dong and Zhang [4] have obtained general Schauder theory results for parabolic equations (of order $2m$) in divergence form, where the time coefficients are allowed to be merely measurable. Their proof (like ours) is in the spirit of Campanato techniques, but requires smooth initial conditions. Our result is aimed at showing that weak solutions are in fact smooth. Classical Schauder theory for general systems has been developed, [8, Chapters 5 and 6]. However, it is non-trivial to apply the general classical results to obtain the result we are after. Even so, it is useful to focus on a specific class of 4th-order double divergence operators, and offer random access to the nonlinear Schauder theory for these cases. Regularity for 4th-order equations remains an important developing area of geometric analysis.

Our proof goes as follows: we start with a $C^{2,\alpha}$ solution of (1.1) whose coefficient matrix is a smooth function of the Hessian of u. We first prove that $u \in W^{3,2}$ by taking a difference quotient of (1.1) and give a $W^{3,2}$ estimate of u in terms of its $C^{2,\alpha}$ norm. Again by taking a difference quotient and using the fact that now $u \in W^{3,2}$, we prove that $u \in C^{3,\alpha}$.

Next, we make a more general proposition where we prove a $W^{3,2}$ estimate for $u \in W^{2,2}$ satisfying a uniformly elliptic equation of the form

$$\int \left(c^{ij,kl}u_{ik} + h^{jl}\right)\eta_{jl}\mathrm{d}x = 0$$

in $B_1(0)$, where $c^{ij,kl}, h^{kl} \in W^{1,2}(B_1)$ and η is a test function in B_1. Using the fact that $u \in W^{3,2}$, we prove that $u \in C^{3,\alpha}$ and also derive a $C^{3,\alpha}$ estimate of u in terms of its $W^{3,2}$ norm. Finally, using difference quotients and dominated convergence, we achieve all higher orders of regularity.

Definition 1.1. We say an equation of the form (1.1) is **regular on** $U \subseteq S^{n\times n}$ when the coefficients of the equation satisfy the following conditions on U:

1. The coefficients $a^{ij,kl}$ depend smoothly on D^2u.
2. The coefficients $a^{ij,kl}$ satisfy (1.2).
3. Either b^{ijkl} or $-b^{ijkl}$ (given by (1.3)) satisfy (1.4).

The following is our main result.

Theorem 1.2. Suppose that $u \in C^{2,\alpha}(B_1)$ satisfies the following 4th-order equation

$$\int_{B_1(0)} a^{ij,kl}(D^2u(x))u_{ij}(x)\eta_{kl}(x)\mathrm{d}x = 0$$
$$\forall \eta \in C_0^\infty(B_1(0)).$$

If $a^{ij,kl}$ is regular on an open set containing $D^2u(B_1(0))$, then u is smooth on $B_r(0)$ for $r < 1$.

To prove this, we will need the following two Schauder type estimates.

Proposition 1.3. Suppose $u \in W^{2,\infty}(B_1)$ satisfies the following:

$$\int_{B_1(0)} \left[c^{ij,kl}(x)u_{ij}(x) + f^{kl}(x)\right]\eta_{kl}(x)\mathrm{d}x = 0 \tag{1.7}$$
$$\forall \eta \in C_0^\infty(B_1(0)),$$

where $c^{ij,kl}, f^{kl} \in W^{1,2}(B_1)$, and $c^{ij,kl}$ satisfies (1.2). Then $u \in W^{3,2}(B_{1/2})$ and

$$\|D^3u\|_{L^2(B_{1/2})} \le C\left(||u||_{W^{2,\infty}(B_1)}, \|f^{kl}\|_{W^{1,2}(B_1)}, \|c^{ij,kl}\|_{W^{1,2}}, \Lambda_1\right).$$

Proposition 1.4. Suppose $u \in C^{2,\alpha}(B_1)$ satisfies (1.7) in B_1 where $c^{ij,kl}, f^{kl} \in C^{1,\alpha}(B_1)$ and $c^{ij,kl}$ satisfies (1.2). Then we have $u \in C^{3,\alpha}(B_{1/2})$ with

$$||D^3u||_{C^{0,\alpha}(B_{1/4})} \le C\left(1 + ||D^3u||_{L^2(B_{3/4})}\right)$$

and $C = C(|c^{ij,kl}|_{C^\alpha(B_1)}, |f^{kl}|_{C^\alpha(B_1)}, \Lambda_1, \alpha)$ is a positive constant.

We note that the above estimates are appropriately scaling invariant; thus, we can use these to obtain interior estimates for a solution in the interior of any sized domain.

2 Preliminaries

We begin by considering a constant coefficient double divergence equation.

Theorem 2.1. Suppose $w \in H^2(B_r)$ satisfies the constant coefficient equation

$$\int c_0^{ik,jl} w_{ik} \eta_{jl} \mathrm{d}x = 0 \tag{2.1}$$

$$\forall \eta \in C_0^\infty(B_r(0)).$$

Then for any $0 < \rho \le r$ there holds

$$\int_{B_\rho} |D^2 w|^2 \le C_1 (\rho/r)^n ||D^2 w||^2_{L^2(B_r)}$$

$$\int_{B_\rho} |D^2 w - (D^2 w)_\rho|^2 \le C_2 (\rho/r)^{n+2} \int_{B_r} |D^2 w - (D^2 w)_r|^2.$$

Here $(D^2 w)_\rho$ is the average value of $D^2 w$ on a ball of radius ρ.

Proof. By dilation we may consider $r = 1$. We restrict our consideration to the range $\rho \in (0, a]$ noting that the statement is trivial for $\rho \in [a, 1]$ where a is some constant in $(0, 1/2)$.

First, we note that w is smooth [5, Theorem 33.10]. Recall [3, Lemma 2, Section 4, applied to elliptic case]; for an elliptic 4th-order L_0

$$L_0 u = 0 \text{ on } B_R$$

$$\implies \|Du\|_{L^\infty(B_{R/4})} \le C_3(\Lambda, n) \|u\|_{L^2(B_R)}.$$

We may apply this to the 2nd derivatives of w to conclude that

$$\left\|D^3 w\right\|^2_{L^\infty(B_a)} \le C_4(\Lambda, n) \int_{B_1} \left\|D^2 w\right\|^2. \tag{2.2}$$

For small enough $a < 1$. Now

$$\begin{aligned}
\int_{B_\rho} \left|D^2 w\right|^2 &\le C_5(n)\rho^n \left\|D^2 w\right\|^2_{L^\infty(B_a)} \\
&= C_5\rho^n \inf_{x\in B_a} \sup_{y\in B_a} \left|D^2 w(x) + D^2 w(y) - D^2 w(x)\right|^2 \\
&\le C_5\rho^n \inf_{x\in B_a} \left[D^2 w(x) + 2a \left\|D^3 w\right\|_{L^\infty(B_a)}\right]^2 \\
&\le 2C_5\rho^n \left[\inf_{x\in B_a} \left\|D^2 w(x)\right\|^2 + 4a^2 \left\|D^3 w\right\|_{L^\infty(B_a)}\right] \\
&\le 2C_5\rho^n \left[\frac{1}{|B_a|}||D^2 w||^2_{L^2(B_a)} + 4a^2 C_4 ||D^2 w||^2_{L^2(B_a)}\right] \\
&\le C_6(a,n)\rho^n ||D^2 w||^2_{L^2(B_1)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{B_\rho} \left|D^2 w - (D^2 w)_\rho\right|^2 &\le \int_{B_\rho} \left|D^2 w - D^2 w(0)\right|^2 \\
&\le \int_{S^{n-1}} \int_0^\rho r^2 \left|D^3 w\right|^2 r^{n-1} \mathrm{d}r\mathrm{d}\phi \\
&= C_7 \rho^{n+2} ||D^3 w||^2_{L^\infty(B_a)}. \qquad (2.3)
\end{aligned}$$

Next, observe that (2.1) is purely 4th order, so the equation still holds when a 2nd-order polynomial is added to the solution. In particular, we may choose

$$D^2 \bar{w} = D^2 w - \left(D^2 w\right)_1$$

for $\bar{w}$ also satisfying the equation. Then

$$D^3 \bar{w} = D^3 w;$$

so by the Poincaré inequality we have

$$\begin{aligned}
\left\|D^3 w\right\|^2_{L^\infty(B_a)} &= \left\|D^3 \bar{w}\right\|^2_{L^\infty(B_a)} \qquad (2.4) \\
&\le C_4 \int_{B_1} \left\|D^2 \bar{w}\right\|^2 = C_4 \int_{B_1} \left\|D^2 w - \left(D^2 w\right)_1\right\|^2.
\end{aligned}$$

We conclude from (2.4) and (2.3)

$$\int_{B_\rho} \left|D^2 w - (D^2 w)_\rho\right|^2 \leq C_7 \rho^{n+2} C_4 \int_{B_1} \left\|D^2 w - \left(D^2 w\right)_1\right\|^2.$$

■

Next, we have a corollary to the above theorem.

Corollary 2.2. Suppose w is as in the Theorem 2.1. Then for any $u \in H^2(B_r)$, and for any $0 < \rho \leq r$, there holds

$$\int_{B_\rho} \left|D^2 u\right|^2 \leq 4C_1(\rho/r)^n \left\|D^2 u\right\|^2_{L^2(B_r)} + \left(2 + 8C_1\right) \left\|D^2(u-w)\right\|^2_{L^2(B_r)} \tag{2.5}$$

and

$$\int_{B_\rho} \left|D^2 u - (D^2 u)_\rho\right|^2 \leq 4C_2(\rho/r)^{n+2} \int_{B_r} \left|D^2 u - (D^2 u)_r\right|^2 + \left(8 + 16C_2\right) \int_{B_r} \left|D^2(u-w)\right|^2. \tag{2.6}$$

Proof. Let $v = u - w$. Then (2.5) follows from direct computation:

$$\begin{aligned}
\int_{B\rho} |D^2 u|^2 &\leq 2\int_{B_\rho} |D^2 w|^2 + 2\int_{B_\rho} |D^2 v|^2. \\
&\leq 2C_1(\rho/r)^n ||D^2 w||^2_{L^2(B_r)} + 2\int_{B_r} |D^2 v|^2 \\
&\leq 4C_1(\rho/r)^n \left[||D^2 v||^2_{L^2(B_r)} + ||D^2 u||^2_{L^2(B_r)}\right] + 2\int_{B_r} |D^2 v|^2 \\
&= 4C_1(\rho/r)^n \left\|D^2 u\right\|^2_{L^2(B_r)} + 2[1 + 2C_1(\rho/r)^n] \left\|D^2 v\right\|^2_{L^2(B_r)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{B\rho}\left|D^2u-(D^2u)_\rho\right|^2 &\le 2\int_{B_\rho}\left|D^2w-(D^2w)_\rho\right|^2+2\int_{B_\rho}\left|D^2v-(D^2v)_\rho\right|^2\\ &\le 2\int_{B_\rho}\left|D^2w-(D^2w)_\rho\right|^2+8\int_{B_\rho}\left|D^2v\right|^2\\ &\le 2C_2(\rho/r)^{n+2}\int_{B_r}|D^2w-(D^2w)_r|^2+8\int_{B_\rho}\left|D^2v\right|^2\\ &\le 2C_2(\rho/r)^{n+2}\left\{\begin{array}{c}2\int_{B_r}\left|D^2u-(D^2u)_r\right|^2\\ +2\int_{B_r}\left|D^2v-(D^2v)_r\right|^2\end{array}\right\}+8\int_{B_r}\left|D^2v\right|^2\\ &\le 4C_2(\rho/r)^{n+2}\int_{B_r}\left|D^2u-(D^2u)_r\right|^2\\ &\quad+\left(8+16C_2(\rho/r)^{n+2}\right)\int_{B_r}\left|D^2v\right|^2.\end{aligned}$$

The statement follows, noting that $\rho/r\le 1$. ■

We will be using the following lemma frequently, so we state it here for the reader's convenience.

Lemma 2.3. [7, Lemma 3.4]. Let ϕ be a nonnegative and nondecreasing function on $[0,R]$. Suppose that

$$\phi(\rho)\le A\left[\left(\frac{\rho}{r}\right)^\alpha+\varepsilon\right]\phi(r)+Br^\beta$$

for any $0<\rho\le r\le R$, with A,B,α,β nonnegative constants and $\beta<\alpha$. Then for any $\gamma\in(\beta,\alpha)$, there exists a constant $\varepsilon_0=\varepsilon_0(A,\alpha,\beta,\gamma)$ such that if $\varepsilon<\varepsilon_0$ we have for all $0<\rho\le r\le R$

$$\phi(\rho)\le c\left[\left(\frac{\rho}{r}\right)^\gamma\phi(r)+Br^\beta\right],$$

where c is a positive constant depending on A,α,β,γ. In particular, we have for any $0<r\le R$

$$\phi(r)\le c\left[\frac{\phi(R)}{R^\gamma}r^\gamma+Br^\beta\right].$$

3 Proofs of the Propositions

We begin by proving Proposition 1.3.

Proof. By approximation, (1.7) holds for $\eta \in W_0^{2,2}$. We are assuming that $u \in W^{2,\infty}$, so (1.7) must hold for the test function

$$\eta = -\left[\tau^4 u^{h_p}\right]^{-h_p},$$

where $\tau \in C_c^\infty$ is a cutoff function in B_1 that is 1 on $B_{1/2}$, and the superscript h_p refers to taking difference quotient in the e_p direction. We choose h small enough after having fixed τ, so that η is well defined. We have

$$\int_{B_1} (c^{ij,kl} u_{ij} + f^{kl}) \left[\tau^4 u^{h_p}\right]_{kl}^{-h_p} \mathrm{d}x = 0$$

For h small we can integrate by parts with respect to the difference quotient to get

$$\int_{B_1} (c^{ij,kl} u_{ij} + f^{kl})^{h_p} \left[\tau^4 u^{h_p}\right]_{kl} \mathrm{d}x = 0.$$

Using the product rule for difference quotients we get

$$\int_{B_1} \left[\left(c^{ij,kl}(x)\right)^{h_p} u_{ij}(x) + c^{ij,kl}\left(x + he_p\right) u_{ij}^{h_p} + (f^{kl})^{h_p}\right] \left[\tau^4 u^{h_p}\right]_{kl} \mathrm{d}x = 0$$

Letting $v = u^{h_p}$, differentiating the 2nd factor gives

$$\int_{B_1} \left[\left(c^{ij,kl}(x)\right)^{h_p} u_{ij}(x) + c^{ij,kl}\left(x + he_p\right) v_{ij}(x) + (f^{kl})^{h_p}(x)\right] \times \begin{bmatrix} \tau^4 v_{kl} + 4\tau^3 \tau_k v_l + 4\tau^3 \tau_l v_k \\ +4v\left(\tau^3 \tau_{kl} + 3\tau^2 \tau_k \tau_l\right) \end{bmatrix} (x) \mathrm{d}x = 0$$

from which

$$\begin{aligned}
\int_{B_1} \tau^4 c^{ij,kl}(x + he_p) v_{ij} v_{kl} \mathrm{d}x = \\
& - \int_{B_1} \left[\left(c^{ij,kl}(x)\right)^{h_p} u_{ij}(x) + c^{ij,kl}(x + he_p) v_{ij}(x) + (f^{kl})^{h_p}(x)\right] \\
& \times \begin{bmatrix} 4\tau^3 \tau_k v_l + 4\tau^3 \tau_l v_k \\ +4v\left(\tau^3 \tau_{kl} + 3\tau^2 \tau_k \tau_l\right) \end{bmatrix} \mathrm{d}x \qquad (3.1) \\
& - \int_{B_1} \left[\left(c^{ij,kl}(x)\right)^{h_p} u_{ij}(x) + (f^{kl})^{h_p}(x)\right] \tau^4 v_{kl} \mathrm{d}x.
\end{aligned}$$

First we bound the terms on the right side of (3.1). Starting at the top:

$$\int_{B_1}\left[(c^{ij,kl}(x))^{h_p}u_{ij}(x)+(f^{kl})^{h_p}(x)\right]\times\left[\begin{array}{c}4\tau^3\tau_k v_l+4\tau^3\tau_l v_k\\ +4v\left(\tau^3\tau_{kl}+3\tau^2\tau_k\tau_l\right)\end{array}\right]\mathrm{d}x$$

$$\leq\left[\|u\|^2_{W^{2,\infty}(B_1)}+1\right]\int_{B_1}\left(\left|(c^{ij,kl}(x))^{h_p}\right|^2+\left|(f^{kl})^{h_p}(x)\right|^2\right)\mathrm{d}x \tag{3.2}$$

$$+C_8(\tau,D\tau,D^2\tau)\int_{B_1}\left(|Dv|^2+|v|^2\right)\mathrm{d}x.$$

Next, by Young's inequality we have

$$\int_{B_1}c^{ij,kl}(x+he_p)v_{ij}(x)\times \tag{3.3}$$

$$\left[4\tau^3\tau_j v_l+4\tau^3\tau_l v_j+4v\left(\tau^3\tau_{jl}+3\tau^2\tau_j\tau_l\right)\right]\mathrm{d}x$$

$$\leq\frac{C_9(\tau,D\tau,D^2\tau,c^{ij,kl})}{\varepsilon}\int_{B_1}\left(|Dv|^2+v^2\right)\mathrm{d}x+\varepsilon\int_{B_1}\tau^4\left|D^2v\right|^2\mathrm{d}x \tag{3.3}$$

and also

$$\int_{B_1}\left[(c^{ij,kl}(x))^{h_p}u_{ij}(x)+(f^{kl})^{h_p}(x)\right]\tau^4v_{kl}\mathrm{d}x$$

$$\leq\varepsilon\int_{B_1}\tau^4\left\|D^2v\right\|^2\mathrm{d}x$$

$$+\frac{C_{10}}{\varepsilon}\left(||u||^2_{W^{2,\infty}(B_1)},|\tau|_{L^\infty(B_1)}\right)\int_{B_1}\left[|(c^{ijkl})^{h_p}|^2+|(h^{jl})^{h_p}|^2\right]\mathrm{d}x. \tag{3.4}$$

Now by uniform ellipticity (1.2), the left-hand side of (3.1) is bounded below by

$$\Lambda\int_{B_1}\tau^4\left\|D^2v\right\|^2\mathrm{d}x\leq\int_{B_1}\tau^4c^{ij,kl}(x+he_p)v_{ik}(x)v_{kl}(x)\mathrm{d}x. \tag{3.5}$$

Combining all (3.1), (3.2),(3.4), (3.3) and (3.5) and choosing ε appropriately, we get

$$\frac{\Lambda}{2}\int_{B_1}\tau^4\left\|D^2v\right\|^2\mathrm{d}x$$

$$\leq C_{11}\left(||\tau||_{W^{2,\infty}(B_1)},||u||^2_{W^{2,\infty}(B_1)}\right)\left(\int_{B_1}|(f^{kl})^{h_p}|^2+|c^{ij,kl}|^2+|(c^{ij,kl})^{h_p}|^2\right)$$

$$\leq C_{12}\left(||\tau||_{W^{2,\infty}(B_1)},||u||^2_{W^{2,\infty}(B_1)},||f^{kl}||^2_{W^{1,2}(B_1)},\left\|c^{ij,kl}\right\|^2_{W^{1,2}(B_1)},\Lambda\right).$$

Now this estimate is uniform in h and direction e_p so we conclude that the difference quotients of u are uniformly bounded in $W^{2,2}(B_{1/2})$. Hence, $u \in W^{3,2}(B_{1/2})$ and

$$\begin{aligned}&||D^3 f||_{L^2(B_{1/2})}\\&\quad\leq \frac{2C_{12}}{\Lambda}\left(||\tau||_{W^{2,\infty}(B_1)}, ||u||^2_{W^{2,\infty}(B_1)}, ||f^{kl}||^2_{W^{1,2}(B_1)}, \left\|c^{ij,kl}\right\|^2_{W^{1,2}(B_1)}, \Lambda\right).\end{aligned}$$

■

We now prove Proposition 1.4

Proof. We begin by taking a difference quotient of the equation

$$\int (c^{ij,kl}u_{ij} + f^{kl})\eta_{kl}\mathrm{d}x = 0$$

along the direction h_m. This gives

$$\int \left[(c^{ij,kl}(x))^{h_m}u_{ij}(x) + c^{ij,kl}(x+he_m)u_{ij}^{h_m}(x) + (f^{kl})^{h_m}\right]\eta_{kl}(x)\mathrm{d}x = 0,$$

which gives us the following partial differential equation in $u_{ij}^{h_m}$:

$$\int c^{ij,kl}(x+he_m)u_{ij}^{h_m}(x)\eta_{kl}(x)\mathrm{d}x = \int q(x)\eta_{kl}(x)\mathrm{d}x,$$

where

$$q(x) = -(f^{kl})^{h_m}(x) - (c^{ij,kl}(x))^{h_m}u_{ij}(x).$$

Note that $q \in C^\alpha(B_1)$ and $c^{ij,kl}(x+he_m)$ is still an elliptic term for all x in $B_{1.}$ For compactness of notation we denote

$$g = u^{h_m} \tag{3.6}$$

and replace $c^{ij,kl}(x+he_m)$ with $c^{ij,kl}$, as the difference is immaterial. Our equation reduces to

$$\int c^{ij,kl}g_{ij}\eta_{kl}\mathrm{d}x = \int q\eta_{kl}\mathrm{d}x \tag{3.7}$$

Using integration by parts, we have

$$\begin{aligned}\int c^{ij,kl} g_{ij}\eta_{kl}\mathrm{d}x &= -\int q_l \eta_k \mathrm{d}x \\ &= -\int (q-q(0))_l \eta_k \mathrm{d}x \\ &= \int (q-q(0))\eta_{kl}\mathrm{d}x.\end{aligned}$$

Now for each fixed $r < 1$ we write $g = v + w$ where w satisfies the following constant coefficient partial differential equation on $B_r \subseteq B_1$:

$$\int_{B_1(0)} c^{ij,kl}(0) w_{ij}\eta_{kl}\mathrm{d}x = 0 \tag{3.8}$$

$$\forall \eta \in C_0^\infty(B_r(0))$$

$$w = g \text{ on } \partial B_r$$

$$\nabla w = \nabla g \text{ on } \partial B_r.$$

By the Lax–Milgram theorem the above partial differential equation with the given boundary condition has a unique solution in the space H_0^2. By combining (3.7) and (3.8) we conclude

$$\int_{B_r} c^{ij,kl}(0) v_{ij}\eta_{kl}\mathrm{d}x = \int_{B_r} (c^{ij,kl}(0) - c^{ij,kl}(x)) g_{ij}\eta_{kl}\mathrm{d}x + \int_{B_r} q\eta_{kl}\mathrm{d}x. \tag{3.9}$$

Now w is smooth (again see [5, Theorem 33.10]), and $g = u^{h_m}$ is $C^{2,\alpha}$, so $v = g - w$ is $C^{2,\alpha}$ and can be well approximated by smooth test functions in $H_0^2(B_r)$. It follows that v can be used as a test function in (3.9); on the left-hand side we have by (1.2)

$$\left[\int_{B_r} c^{ij,kl}(0) v_{ij} v_{kl}\mathrm{d}x\right]^2 \geq \left[\Lambda \int_{B_r} |D^2 v|^2 \mathrm{d}x\right]^2.$$

Defining

$$\zeta(r) = \sup\left\{| \, c^{ij,kl}(x) - c^{ij,kl}(y)| : x, y \in B_r\right\} \tag{3.10}$$

and using the Cauchy–Schwarz inequality we get

$$\left[\int_{B_r} (c^{ij,kl}(0) - c^{ij,kl}(x)) g_{ij} v_{kl}\mathrm{d}x\right]^2 \leq \zeta^2(r) \int_{B_r} |D^2 g|^2 \mathrm{d}x \int_{B_r} |D^2 v|^2 \mathrm{d}x.$$

Using Holder's inequality

$$\left[\int_{B_r} \left|(q(x)-q(0))v_{kl}(x)\right| \mathrm{d}x\right]^2 \leq \int_{B_r} |q(x)-q(0)|^2 \mathrm{d}x \int_{B_r} |D^2 v|^2 \mathrm{d}x.$$

This gives us

$$\Lambda^2\left[\int_{B_r} |D^2 v|^2 \mathrm{d}x\right]^2 \leq \zeta^2(r)\int_{B_r} |D^2 g|^2 \mathrm{d}x \int_{B_r} |D^2 v|^2 \mathrm{d}x + \int_{B_r} |q(x)-q(0)|^2 \mathrm{d}x \int_{B_r} |D^2 v|^2 \mathrm{d}x,$$

which implies

$$\Lambda^2 \int_{B_r} |D^2 v|^2 \mathrm{d}x \leq \zeta^2(r)\int_{B_r} |D^2 g|^2 \mathrm{d}x + \int_{B_r} |q(x)-q(0)|^2 \mathrm{d}x. \tag{3.11}$$

Using Corollary 2.2 for any $0<\rho\leq r$, we get

$$\int_{B_\rho} \left|D^2 g\right|^2 \mathrm{d}x \leq 4C_1(\rho/r)^n \left\|D^2 g\right\|^2_{L^2(B_r)} + \left(2+8C_1\right)\left\|D^2 v\right\|^2_{L^2(B_r)}. \tag{3.12}$$

Now combing (3.12) and (3.11) we get

$$\begin{aligned}\int_{B_\rho} \left|D^2 g\right|^2 \mathrm{d}x &\leq 4C_1(\rho/r)^n \left\|D^2 g\right\|^2_{L^2(B_r)} \\ &\quad + \frac{\left(2+8C_1\right)}{\Lambda^2}\left[\zeta^2(r)\int_{B_r} |D^2 g|^2 \mathrm{d}x + \int_{B_r} |q(x)-q(0)|^2 \mathrm{d}x\right] \\ &= \left[\frac{\left(2+8C_1\right)\zeta^2(r)}{\Lambda^2} + 4C_1(\rho/r)^n\right]\int_{B_r} |D^2 g|^2 \mathrm{d}x \\ &\quad + \frac{\left(2+8C_1\right)}{\Lambda^2}\int_{B_r} |q(x)-q(0)|^2 \mathrm{d}x. \end{aligned} \tag{3.13}$$

Also from Corollary 2.2

$$\begin{aligned}\int_{B_\rho} \left|D^2 g - (D^2 g)_\rho\right|^2 \mathrm{d}x &\leq 4C_2(\rho/r)^{n+2}\int_{B_r} \left|D^2 g - (D^2 g)_r\right|^2 \mathrm{d}x \\ &\quad + \left(8+16C_2\right)\int_{B_r} \left|D^2 v\right|^2 \mathrm{d}x \\ &\leq 4C_2(\rho/r)^{n+2}\int_{B_r} \left|D^2 g - (D^2 g)_\rho\right|^2 \mathrm{d}x \\ &\quad + \frac{\left(8+16C_2\right)}{\Lambda^2}\left[\zeta^2(r)\int_{B_r} |D^2 g|^2 \mathrm{d}x + \int_{B_r} |q(x)-q(0)|^2 \mathrm{d}x\right].\end{aligned}$$

Because $c^{ij,kl} \in C^{1,\alpha}$ we have from (3.10) that

$$\zeta(r)^2 \le C_{13} r^{2\alpha}. \tag{3.14}$$

Again q is a C^α function that implies

$$|q(x) - q(0)| \le \|q\|_{C^\alpha(B_1)} |x - 0|^\alpha$$

and

$$\int_{B_r} |q - q(0)|^2 \mathrm{d}x \le C_{14} \|q\|_{C^\alpha(B_1)} r^{n+2\alpha}.$$

So we have

$$\begin{aligned}
&\int_{B_\rho} |D^2 g - (D^2 g)_\rho|^2 \\
&\le 4C_2(\rho/r)^{n+2} \int_{B_r} \left|D^2 g - (D^2 g)_\rho\right|^2 \\
&+ \frac{(8 + 16C_2)}{\Lambda^2} C_{13} r^{2\alpha} \int_{B_r} |D^2 g|^2 \\
&+ \frac{(8 + 16C_2)}{\Lambda^2} C_{14} \|q\|_{C^\alpha(B_1)} r^{n+2\alpha}.
\end{aligned} \tag{3.15}$$

For $r < r_0 < 1/4$ to be determined, we have (3.13)

$$\int_{B_\rho} \left|D^2 g\right|^2 \le C_{15} \left\{ [(\rho/r)^n + r^{2\alpha}] \int_{B_r} \left|D^2 g\right|^2 + r_0^{2\alpha+2\delta} r^{n-2\delta} \right\},$$

where δ is some positive number. Now we apply [7, Lemma 3.4]. In particular, take

$$\phi(\rho) = \int_{B_\rho} \left|D^2 g\right|^2$$

$$A = C_{15}$$

$$B = r_0^{2\alpha+2\delta}$$

$$\alpha = n$$

$$\beta = n - 2\delta$$

$$\gamma = n - \delta.$$

There exists $\varepsilon_0(A, \alpha, \beta, \gamma)$ such that if

$$r_0^{2\alpha} \leq \varepsilon_0, \tag{3.16}$$

we have

$$\phi(\rho) \leq C_{15} \left\{ \left[(\rho/r)^n + \varepsilon_0 \right] \phi(r) + r_0^{2\alpha+2\delta} r^{n-2\delta} \right\},$$

and the conclusion of [7, Lemma 3.4] says that for $\rho < r_0$

$$\begin{aligned}
\phi(\rho) &\leq C_{16} \left\{ [(\rho/r)^{\gamma}] \phi(r) + r_0^{2\alpha+2\delta} \rho^{n-2\delta} \right\} \\
&\leq C_{16} \frac{1}{r_0^{n-\delta}} \rho^{n-\delta} \left\| D^2 g \right\|_{L^2(B_{r_0})} + r_0^{2\alpha+2\delta} \rho^{n-2\delta} \\
&\leq C_{17} \rho^{n-\delta}.
\end{aligned}$$

This C_{17} depends on r_0 that is chosen by (3.16) and $\left\| D^2 g \right\|_{L^2(B_{3/4})}$. So there is a positive uniform radius upon which this holds for points well in the interior. In particular, we choose $r_0 \in (0, 1/4)$ so that the estimate can be applied uniformly at points centered in $B_{1/2}(0)$ whose balls remain in $B_{3/4}(0)$. Turning back to (3.15), we now have

$$\begin{aligned}
\int_{B_\rho} |D^2 g - (D^2 g)_\rho|^2 &\leq 4C_2 (\rho/r)^{n+2} \int_{B_r} \left| D^2 g - (D^2 g)_\rho \right|^2 + C_{18} r^{2\alpha} \rho^{n-\delta} \\
&\quad + C_{19} \|q\|_{C^\alpha(B_1)} r^{n+2\alpha} \\
&\leq 4C_2 (\rho/r)^{n+2} \int_{B_r} \left| D^2 g - (D^2 g)_\rho \right|^2 + C_{20} r^{n+2\alpha-\delta}.
\end{aligned}$$

Again we apply [7, Lemma 3.4]; this time, take

$$\phi(\rho) = \int_{B_\rho} |D^2 g - (D^2 g)_\rho|^2$$

$$A = 4C_2$$

$$B = C_{20}$$

$$\alpha = n + 2$$

$$\beta = n + 2\alpha - \delta$$

$$\gamma = n + 2\alpha$$

and conclude that for any $r < r_0$

$$\int_{B_r} |D^2 g - (D^2 g)_\rho|^2 \le C_{21} \left\{ \frac{1}{r_0^{n+2\alpha}} \int_{B_{r_0}} |D^2 g - (D^2 g)_{r_0}|^2 r^{n+2\alpha} + C_{20} r^{n+2\alpha-\delta} \right\}$$
$$\le C_{22} r^{n+2\alpha-\delta}$$

with C_{22} depending on r_0, $\|D^2 g\|_{L^2(B_{3/4})}$, $\|q\|_{C^\alpha(B_1)}$ etc. It follows by [7, Theorem 3.1] that $D^2 g \in C^{(2\alpha-\delta)/2}(B_{1/4})$, in particular, must be bounded locally:

$$\left\|D^2 g\right\|_{L^\infty(B_{1/4})} \le C_{23} \left\{1 + \left\|D^2 g\right\|_{L^2(B_{1/2})}\right\}. \tag{3.17}$$

This allows us to bound

$$\int_{B_r} |D^2 g|^2 \le C_{24} r^n,$$

which we can plug back in to (3.15):

$$\int_{B_\rho} |D^2 g - (D^2 g)_\rho|^2 \le 4C_2 (\rho/r)^{n+2} \int_{B_r} \left|D^2 g - (D^2 g)_\rho\right|^2 + C_{25} r^{2\alpha} C_{24} r^n$$
$$+ C_{19} \|q\|_{C^\alpha(B_1)} r^{n+2\alpha}$$
$$\le C_{26} r^{n+2\alpha}.$$

This is precisely the hypothesis in [7, Theorem 3.1]. We conclude that

$$\left\|D^2 g\right\|_{C^\alpha(B_{1/4})} \le C_{27} \left\{\sqrt{C_{26}} + \left\|D^2 g\right\|_{L^2(B_{1/2})}\right\}.$$

Recalling (3.6) we see that u must enjoy uniform $C^{3,\alpha}$ estimates on the interior, and the result follows. ∎

4 Proof of the Theorem

The propositions in the previous section allow us to prove the following corollary, from which the main theorem will follow.

Corollary 4.1. Suppose $u \in C^{N,\alpha}(B_1)$, $N \ge 2$, and u satisfies the following regular (recall (1.3)) 4th-order equation

$$\int_\Omega a^{ij,kl}(D^2 u) u_{ij} \eta_{kl} \mathrm{d}x = 0, \forall \eta \in C_0^\infty(\Omega).$$

Then

$$\|u\|_{C^{N+1,\alpha}(B_r)} \le C(n, b, \|u\|_{W^{N,\infty}(B_1)}).$$

In particular

$$u \in C^{N,\alpha}(B_1) \implies u \in C^{N+1,\alpha}(B_r).$$

Case 1 $N = 2$. The function $u \in C^{2,\alpha}(B_1)$ and hence also in $W^{2,\infty}(B_1)$. By approximation (1.1) holds for $\eta \in W_0^{2,\infty}$, in particular, for

$$\eta = -\left[\tau^4 u^{h_m}\right]^{-h_m},$$

where $\tau \in C_c^\infty(B_1)$ is a cutoff function in B_1 that is 1 on $B_{1/2}$, and superscript h_m refers to the difference quotient. As before, we have chosen h small enough (depending on τ) so that η is well defined. We have

$$\int_\Omega a^{ij,kl}(D^2u)u_{ij}\left[\tau^4 u^{h_m}\right]_{kl}\mathrm{d}x = 0.$$

Integrating by parts as before with respect to the difference quotient, we get

$$\int_{B_1} \left[a^{ij,kl}(D^2u)u_{ij}\right]^{h_m}\left[\tau^4 u^{h_m}\right]_{kl}\mathrm{d}x = 0.$$

Let $v = u^{h_m}$. Observe that the 1st difference quotient can be expressed as

$$\begin{aligned}\left[a^{ij,kl}(D^2u)u_{ij}\right]^{h_m}(x) &= a^{ij,kl}\left(D^2u(x+he_m)\right)\frac{u_{ij}(x+he_m)-u_{ij}(x)}{h} \\ &\quad + \frac{1}{h}\left[a^{ij,kl}\left(D^2u(x+he_m)\right) - a^{ij,kl}(D^2u(x))\right]u_{ij}(x) \\ &= a^{ij,kl}\left(D^2u(x+he_m)\right)v_{ij}(x) \\ &\quad + \left[\int_0^1 \frac{\partial a^{ij,kl}}{\partial u_{pq}}\left(tD^2u(x+he_m)+(1-t)D^2u(x)\right)\mathrm{d}t\right]v_{pq}(x)u_{ij}(x).\end{aligned} \tag{4.1}$$

We get

$$\int_{B_1} \tilde{b}^{ij,kl}v_{ij}[\tau^4 v]_{kl}\mathrm{d}x = 0, \tag{4.2}$$

where

$$\tilde{b}^{ij,kl}(x) = a^{ij,kl}\big(D^2u(x+he_m)\big) + \left[\int_0^1 \frac{\partial a^{pq,kl}}{\partial u_{ij}}\big(tD^2u(x+he_m) + (1-t)D^2u(x)\big)\mathrm{d}t\right] u_{pq}(x). \tag{4.3}$$

Expanding derivatives of the 2nd factor in (4.2) and collecting terms gives us

$$\int_{B_1} \tilde{b}^{ij,kl} v_{ij}\tau^4 v_{kl}\mathrm{d}x \le \int_{B_1} \left|\tilde{b}^{ij,kl}\right| \left|v_{ij}\right| \tau^2 C_{28}(\tau, D\tau, D^2\tau)\,(1+|v|+|Dv|)\,\mathrm{d}x.$$

Now for h small, $\tilde{b}^{ij,kl}$ very closely approximates $b^{ij,kl}$, so we may assume h is small. Applying (1.4) and Young's inequality,

$$\int_{B_1} \tau^4\Lambda_1|D^2v|^2 \le C_{28}\sup\tilde{b}^{ij,kl}\int_{B_1}\left(\varepsilon\tau^4|D^2v|^2 + C_{32}\frac{1}{\varepsilon}(1+|v|+|Dv|)^2\right)\mathrm{d}x.$$

That is,

$$\int_{B_{1/2}} |D^2v|^2 \le C_{29}\int_{B_1}(1+|v|+|Dv|)^2\mathrm{d}x.$$

Now this estimate is uniform in h (for h small enough) and direction $e_{m,}$ so we conclude that the derivatives are in $W^{2,2}(B_{1/2})$. This also shows that

$$||D^3u||_{L^2(B_{1/2})} \le C_{30}\left(||Du||_{L^2(B_1)}, \left\|D^2u\right\|_{L^2(B_1)}\right).$$

Remark: We only used uniform continuity of D^2u to allow us to take the limit, but we did require the precise modulus of continuity.

For the next step, we are not quite able to use Proposition 1.4 because the coefficients $a^{ij,kl}$ are only known to be $W^{1,2}$. So we proceed by hand.

We begin by taking a single difference quotient

$$\int_{B_1}\left[a^{ij,kl}(D^2u)u_{ij}\right]^{h_m}\eta_{kl}\mathrm{d}x = 0$$

and arriving at the equation in the same fashion as to (4.2) above (this time letting $g = u^{h_m}$); we have

$$\int_{B_1}\tilde{b}^{ij,kl}g_{ij}(x)\eta_{kl}\mathrm{d}x = 0.$$

Inspecting (4.3) we see that $\tilde{b}^{ij,kl}$ is C^α :

$$\left\|\tilde{b}^{ij,kl}(x) - \tilde{b}^{ij,kl}(y)\right\| \le C_{31}\,|x-y|^\alpha\,,$$

where C_{31} depends on $\left\|D^2u\right\|_{C^\alpha}$ and on bounds of $Da^{ij,kl}$ and $D^2a^{ij,kl}$. As in the proof of Proposition 1.4, for a fixed $r<1$ we let w solve the boundary value problem

$$\int_{B_r}\tilde{b}^{ij,kl}(0)w_{ij}\eta_{kl}\mathrm{d}x = 0, \forall\eta\in C_0^\infty(B_r)$$

$$w = g \text{ on } \partial B_r$$

$$\nabla w = \nabla g \text{ on } \partial B_r.$$

Let $v = g - w$. Note that

$$\int_{B_r}\tilde{b}^{ij,kl}(0)v_{ij}\eta_{kl}\mathrm{d}x = \int_{B_r}\left(\tilde{b}^{ij,kl}(0)-\tilde{b}^{ij,kl}(x)\right)g_{ij}\eta_{kl}\mathrm{d}x.$$

Now v vanishes to 2nd order on the boundary, and we may use v as a test function. We get

$$\int_{B_r}\tilde{b}^{ij,kl}(0)v_{ij}v_{kl}\mathrm{d}x = \int_{B_r}\left(\tilde{b}^{ij,kl}(0)-\tilde{b}^{ij,kl}(x)\right)g_{ij}v_{kl}\mathrm{d}x.$$

As before,

$$\left(\Lambda\int_{B_r}\left|D^2v\right|^2\mathrm{d}x\right)^2 \leq \left[\sup_{x\in B_r}\left|\tilde{b}^{ij,kl}(0)-\tilde{b}^{ij,kl}(x)\right|\right]^2\int_{B_r}\left|D^2g\right|^2\mathrm{d}x\int_{B_r}\left|D^2v\right|^2\mathrm{d}x.$$

Defining

$$\begin{aligned}\zeta(r) &= \sup\left\{\left|\tilde{b}^{ij,kl}(x)-\tilde{b}^{ij,kl}(y)\right| x,y\in B_r\right\} \\ &\leq 4^\alpha C_{31}r^{2\alpha}\end{aligned} \tag{4.4}$$

then

$$\int_{B_r}(\tilde{b}^{ij,kl}(0)-\tilde{b}^{ij,kl}(x))g_{ij}v_{kl}\mathrm{d}x)^2 \leq \zeta^2(r)\int_{B_r}\left|D^2g\right|^2\int_{B_r}\left|D^2v\right|^2.$$

So now we have

$$\int_{B_r}\left|D^2v\right|^2 \leq \frac{\zeta^2(r)}{\Lambda^2}\int_{B_r}\left|D^2g\right|^2.$$

Using Corollary 2.2, for any $0 < \rho \le r$ we get

$$\begin{aligned}\int_{B_\rho} \left|D^2 g - (D^2 g)_\rho\right|^2 &\le 4C_2(\rho/r)^{n+2} \int_{B_r} \left|D^2 g - (D^2 g)_r\right|^2 \\ &\quad + \left(8 + 16C_2\right) \int_{B_r} \left|D^2 v\right|^2 \\ &\le 4C_2(\rho/r)^{n+2} \int_{B_r} \left|D^2 g - (D^2 g)_r\right|^2 + \frac{\left(8+16C_2\right)\zeta^2(r)}{\Lambda^2} \left\|D^2 g\right\|^2_{L^2(B_r)}.\end{aligned} \tag{4.5}$$

Also by Corollary 2.2

$$\begin{aligned}\int_{B_\rho} \left|D^2 g\right|^2 &\le 4C_1(\rho/r)^n \left\|D^2 g\right\|^2_{L^2(B_r)} + \left(2 + 8C_1\right) \left\|D^2 v\right\|^2_{L^2(B_r)} \\ &\le 4C_1(\rho/r)^n \left\|D^2 g\right\|^2_{L^2(B_r)} + \left(2 + 8C_1\right) \frac{\zeta^2(r)}{\Lambda^2} \left\|D^2 g\right\|^2_{L^2(B_r)}.\end{aligned}$$

This implies

$$\int_{B_\rho} \left|D^2 g\right|^2 \le \left(4C_1(\rho/r)^n + \left(2 + 8C_1\right) 4^{2\alpha} C_{31}^2 r^{2\alpha}\right) \left\|D^2 g\right\|^2_{L^2(B_r)}.$$

Now we can apply [7, Lemma 3.4] again, this time with

$$\begin{aligned}\phi(\rho) &= \int_{B_\rho} \left|D^2 g\right|^2 \\ A &= 4C_1 \\ \alpha &= n \\ B, \beta &= 0 \\ \gamma &= n - 2\delta \\ \varepsilon &= \left(2 + 8C_1\right) 4^{2\alpha} C_{31}^2 r^{2\alpha}.\end{aligned}$$

There exists a constant $\varepsilon_0(A, \alpha, \gamma)$ such that by chosing

$$r_0^{2\alpha} \le \frac{\varepsilon_0}{\left(2 + 8C_1\right) 4^{2\alpha} C_{31}^2} < \frac{1}{4}$$

we may conclude that for $0 < r \le r_0$

$$\int_{B_r} \left|D^2 g\right|^2 \le C_{32} r^{n-2\delta} \frac{\int_{Br_0} \left|D^2 g\right|^2}{r_0^{n-2\delta}}. \tag{4.6}$$

Next, for small $\rho < r < r_0$ we have, combining (4.5), (4.4) and (4.6),

$$\begin{aligned} \int_{B\rho} \left|D^2 g - (D^2 g)_\rho\right|^2 &\le 4C_2 (\rho/r)^{n+2} \int_{B_r} \left|D^2 g - (D^2 g)_r\right|^2 \\ &+ \frac{(8+16C_2)\, 4^\alpha}{\Lambda^2} \frac{\int_{Br_0} \left|D^2 g\right|^2}{r_0^{n-2\delta}} C_{31} C_{32} r^{n-2\delta} r^{2\alpha} \\ &\le C_{33} r^{n+2\alpha-\delta} \end{aligned} \tag{4.7}$$

with C_{33} depending on $\left\|D^2 g\right\|_{L^2(B_{3/4})}, r_0, \varepsilon_0$. Again, we apply [7, Theorem 3.1] to $D^2 g \in C^{(2\alpha-\delta)/2}(B_{1/4})$. From here, the argument is identical to the argument following (3.17). We conclude that

$$\left\|D^2 g\right\|_{C^\alpha(B_{1/4})} \le C_{34} \left\{1 + \left\|D^2 g\right\|_{L^2(B_{3/4})}\right\}.$$

Substituting $g = u^{h_m}$ we see that u must enjoy uniform $C^{3,\alpha}$ estimates on the interior, and the result follows.

Case 2 $N = 3$. We may take a difference quotient of (1.1) directly.

$$\int_\Omega \left[a^{ij,kl}(D^2 u) u_{ij}\right]^{h_m} \eta_{kl} \mathrm{d}x = 0, \forall \eta \in C_0^\infty(\Omega).$$

(To be more clear we are using a slightly offset test function $\eta(x + he_m)$ and then using a change of variables, subtracting and dividing by h.)

We get

$$\int_{B_1} \left[a^{ij,kl}(D^2 u(x + he_m)) u_{ij}^{h_m}(x) + \frac{\partial a^{ij,kl}}{\partial u_{pq}}(M^*(x)) u_{pq}^{h_m}(x) u_{ij}(x)\right] \eta_{kl} = 0,$$

where $M^*(x) = t^* D^2 u(x + h_m) + (1 - t^*) D^2 u(x)$ and $t^* \in [0,1]$. Now we are assuming that $u \in C^{3,\alpha}(B_1)$, so the 1st and 2nd derivatives of the difference quotient will converge to the 2nd and 3rd derivatives, uniformly. We can then apply dominated convergence;

passing the limit as $h \to 0$ inside the integral and recalling $u_m = v$ as before, we obtain

$$\int_{B_1} \left[[\ a^{ij,kl}(D^2u(x))v_{ij}(x) + \frac{\partial a^{pq,kl}}{\partial u_{ij}} \left(D^2u(x)\right) v_{ij}(x)u_{pq}(x) \right] \eta_{kl} = 0$$

that is

$$\int_{B_1} b^{ij,kl}(D^2u(x))v_{ij}(x)\eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega). \tag{4.8}$$

It follows that $v \in C^{2,\alpha}$satisfies a -order double divergence equation, with coefficients in $C^{1,\alpha}$. First, we apply Proposition 1.3:

$$\left\| D^3 v \right\|_{L^2(B_{1/2})} \leq C_{35} \left(||v||_{W^{2,\infty}(B_1)}\right) (1 + ||b^{ij,kl}||_{W^{1,2}(B_1)}).$$

In particular, $u \in W^{4,2}(B_{1/2})$. Next, we apply 1.4

$$\begin{aligned} ||D^3v||_{C^{0,\alpha}(B_{1/4})} &\leq C(1 + ||D^3v||_{L^2(B_{1/2})}) \leq C(||u||_{W^{2,\infty}(B_1)}, |b^{ij,kl}||_{W^{1,2}(B_1)}) \\ &\leq C_{36}(n, b, \|u\|_{C^{3,\alpha}(B_1)}). \end{aligned}$$

We conclude that $u \in C^{4,\alpha}(B_r)$ for any $r < 1$.

Case 3 $N \geq 4$. Let $v = D^\alpha u$ for some multi-index α with $|\alpha| = N - 2$. Observe that taking the 1st difference quotient and then taking a limit yields (4.8), when $u \in C^{3,\alpha}$. Now if $u \in C^{4,\alpha}$ we may take a difference quotient and limit of (4.8) to obtain

$$\int_{B_1} \left[b^{ij,kl}(D^2u(x))u_{ijm_1m_2}(x) + \frac{\partial b^{ij,kl}}{\partial u_{pq}}(D^2u(x))u_{pqm_2}u_{ij} \right] \eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega),$$

and if $u \in C^{N,\alpha}$, then $v \in C^{2,\alpha}$, so we may take $N - 2$ difference quotients to obtain

$$\int_{B_1} \left[b^{ij,kl}(D^2u(x))v_{ij}(x) + f^{kl}(x) \right] \eta_{kl}(x) = 0, \quad \forall \eta \in C_0^\infty(\Omega), \tag{4.9}$$

where

$$f^{kl} = D^\alpha \left(b^{ij,kl}(D^2u(x))u_{ij} \right) - b^{ij,kl}(D^2u(x))D^\alpha u_{ij}.$$

One can check by applying the chain rule repeatedly that f^{kl} is $C^{1,\alpha}$. So we may apply Proposition 1.3 to (4.9) and obtain that

$$\left\| D^3 v \right\|_{L^2(B_{1/2})} \leq C_{37}(\|v\|_{W^{2,\infty}(B_1)})(1 + ||b^{ij,kl}||_{W^{1,2}(B_1)})$$

that is

$$\|u\|_{W^{N+1,2}(B_r)} \leq C_{38}(n, b, \|u\|_{W^{N,\infty}(B_1)}).$$

Now apply Proposition 1.4:

$$||D^3 v||_{C^{0,\alpha}(B_{1/4})} \leq C_{39}(1 + ||D^3 v||_{L^2(B_{3/4})}),$$

that is,

$$\|u\|_{C^{N+1,\alpha}(B_r)} \leq C_{40}(n, b, \|u\|_{W^{N,\infty}(B_1)}).$$

The main theorem follows.

5 Critical Points of Convex Functions of the Hessian

Suppose that $F(D^2u)$ is either a convex or a concave function of D^2u, and we have found a critical point of

$$\int_\Omega F(D^2u)\mathrm{d}x \tag{5.1}$$

for some $\Omega \subset \mathbb{R}^n$, where we are restricting to compactly supported variations, so that the Euler–Lagrange equation is (1.6). If we suppose that F also has the additional structure condition,

$$\frac{\partial F(D^2u)}{\partial u_{ij}} = a^{pq,ij}(D^2u)u_{pq} \tag{5.2}$$

for a some $a^{ij,kl}$ satisfying (1.2), then we can derive smoothness from $C^{2,\alpha}$ as follows.

Corollary 5.1. Suppose $u \in C^{2,\alpha}(B_1)$ is the critical point of (5.1), where F is a smooth function satisfying (5.2) with $a^{ij,kl}$ satisfying (1.2) and F is uniformly convex or uniformly concave on $U \subseteq S^{n\times n}$ where U is the range of $D^2u(B_1)$ in the Hessian space.

Then $u \in C^\infty(B_r)$, for all $r < 1$.

Proof. If u is a critical point of (5.1), then it satisfies the weak equation (1.1), for $a^{ij,kl}$ in (5.2). To apply the main theorem, all we need to show is that

$$b^{ij,kl}(D^2u(x)) = a^{ij,kl}(D^2u(x)) + \frac{\partial a^{pq,kl}}{\partial u_{ij}}(D^2u(x))u_{pq}(x)$$

satisfies (1.2). From (5.2):

$$\frac{\partial}{\partial u_{kl}}\left(\frac{\partial F(D^2u)}{\partial u_{ij}}\right) = a^{kl,ij}(D^2u) + \frac{\partial a^{pq,ij}(D^2u)}{\partial u_{kl}}u_{pq}. \tag{5.3}$$

So

$$b^{ij,kl}(D^2u(x))\xi_{ij}\xi_{kl} = \frac{\partial}{\partial u_{kl}}\left(\frac{\partial F(D^2u)}{\partial u_{ij}}\right)\xi_{ij}\xi_{kl} \geq \Lambda\,|\xi|^2$$

for some $\Lambda > 0$, because F is convex. If F is concave, u is still a critical point of $-F$ and the same argument holds. ■

We mention one special case.

Lemma 5.2. Suppose $F(D^2u) = f(w)$ where $w = (D^2u)^T(D^2u)$. Then

$$\frac{\partial F(D^2u)}{\partial u_{ij}} = a^{ij,kl}(D^2u)u_{kl}. \tag{5.4}$$

Proof. Let

$$w_{kl} = u_{ka}\delta^{ab}u_{bl}.$$

Then

$$\begin{aligned}
\frac{\partial F(D^2u)}{\partial u_{ij}} &= \frac{\partial f(w)}{\partial w_{kl}}\frac{\partial w_{kl}}{\partial u_{ij}} \\
&= \frac{\partial f(w)}{\partial w_{kl}}\left(\delta_{ka,ij}\delta^{ab}u_{bl} + u_{ka}\delta^{ab}\delta_{bl,ij}\right) \\
&= \frac{\partial f(w)}{\partial w_{kl}}\left(\delta_{ki}u_{jl} + u_{ki}\delta_{lj}\right) \\
&= \frac{\partial f(w)}{\partial w_{il}}\delta_{jm}u_{ml} + \frac{\partial f(w)}{\partial w_{kj}}u_{km}\delta_{im} \\
&= \frac{\partial f(w)}{\partial w_{il}}\delta_{jk}u_{kl} + \frac{\partial f(w)}{\partial w_{kj}}u_{kl}\delta_{il}.
\end{aligned}$$

This shows (5.4) for

$$a^{ij,kl} = \frac{\partial f(w)}{\partial w_{il}}\delta_{jk} + \frac{\partial f(w)}{\partial w_{kj}}\delta_{il}.$$

■

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