# COMPACTIFICATION OF THE SPACE OF HAMILTONIAN STATIONARY LAGRANGIAN SUBMANIFOLDS OF BOUNDED TOTAL EXTRINSIC CURVATURE AND VOLUME 

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#### Abstract

For a sequence of closed Hamiltonian stationary Lagrangian submaniolds in $\mathbb{C}^{n}$ with uniform bounds on their volumes and the total extrinsic curvature, we prove that a subsequence converges either to a point or to a Hamiltonian stationary Lagrangian $n$-varifold locally uniformly in $C^{k}$ for any nonnegative integer $k$ away from a finite set of points. We also obtain a theorem on extending Hamiltonian stationary Lagrangian submanifolds across a compact set with certain volume control locally.


## 1. Introduction

Hamiltonian stationary Lagrangian submanifolds in $\mathbb{C}^{n}$ are critical points of the volume functional under Hamiltonian variations $X=J D f$ for any compactly supported smooth function $f$ on $\mathbb{C}^{n}$ [Oh93]. Any smooth Lagrangian submanifold can be locally defined by a graph over a region $\Omega$ in a Lagrangian tangent plane, in the form

$$
\Gamma_{u}=\{(x, D u(x)): x \in \Omega\}
$$

for some $u \in C^{\infty}(\Omega)$. If the Lagrangian phase

$$
\theta=\sum_{\lambda_{j} \text { eigenvalues of } D^{2} u} \arctan \lambda_{j}
$$

is constant, then the Lagrangian submanifold is volume minimizing among all submanifolds in the same homology class, as shown in [HL82]. If the phase $\theta$ is harmonic on $\Gamma_{u}$, that is,

$$
\begin{equation*}
\Delta_{g} \theta=0 \tag{1.1}
\end{equation*}
$$

where $\Delta_{g}$ is the Laplace-Beltrami operator on $\Gamma_{u}$ for the induced metric $g$, then $\Gamma_{u}$ is Hamiltonian stationary, and vice versa (cf. [Oh93], [SW03, Proposition 2.2]). Equation (1.1) is a fourth order nonlinear elliptic equation for the potential function $u$. An important feature of the fourth order operator is its decomposition into two second order elliptic operators, and this is the basis for our curvature estimate and smoothness estimates, as already used in our regularity theory on Hamiltonian stationary Lagrangian submanifolds [CW16].

In this paper, we prove a compactness result for closed immersed Hamiltonian stationary Lagrangian submanifolds of $\mathbb{C}^{n}$ with uniform bound on volume and total extrinsic curvature, namely, the $L^{n}$-norm of the second fundamental form. For any sequence of such submanifolds, we show that a subsequence converges, locally uniformly in every $C^{k}$-norm away from a finite set of points, to an integral varifold which is Hamiltonian

[^0]stationary in an appropriate sense. So we can compactify the space of these submanifolds by including Hamiltonian stationary integral $n$-varifolds with only point singularities (immersed elsewhere) and the number of the singular points is bounded by a constant only depends on the upper bound of the total extrinsic curvature. It is possible that the sequence converges to a point, such as shrinking circles in the plane. This can be excluded by scaling volume to one, while the total extrinsic curvature and being Hamiltonian stationary Lagrangian are both scaling invariant, although the Hamiltonian isotopy classes may change.

Theorem 1.1. Suppose that $\left\{L_{i}\right\}$ is a sequence of connected Lagrangian Hamiltonian stationary closed (compact without boundary) immersed manifolds into $\mathbb{C}^{n}, n \geq 1$, with $0 \in L_{i}$ and

$$
\operatorname{Volume}\left(L_{i}\right)<C_{1} \text { and } \int_{L_{i}}|A|^{n} d \mu<C_{2} .
$$

Then there is a bounded set $B_{R_{0}}(0) \subset \mathbb{C}^{n}$ where $R_{0}\left(n, C_{1}, C_{2}\right)$ such that $L_{i} \subset B_{R_{0}}(0)$. Moreover, either there exists a subsequence of $\left\{L_{i}\right\}$ converging to a point, or there exists a finite set $S \subset B_{R_{0}}(0)$ and a subsequence of $\left\{L_{i}\right\}$ that converges in the $C^{k}$ topology on any compact subset of $B_{R_{0}}(0) \backslash S$ to a Lagrangian varifold $L \subset \mathbb{C}^{n}$ which is locally Hamiltonian stationary, the closure $\bar{L}$ is Hamiltonian stationary in the sense that a generalized mean curvature $\mathcal{H}$ of the varifold $\bar{L}$ exists and satisfies

$$
\begin{equation*}
\int_{\bar{L}}\langle J \nabla f, \mathcal{H}\rangle d \mu_{L}=0 \tag{1.2}
\end{equation*}
$$

for any $f \in C_{0}^{\infty}\left(\mathbb{R}^{2 n}\right)$. Also, $\bar{L}$ is connected.
We also obtain an extendibility result in Theorem 4.1 which asserts that a properly immersed Hamiltonian stationary Lagrangian submanifold $L$ in $\mathbb{C}^{n} \backslash N$ (i.e. for Hamiltonian vector fields supported away from $N$ ) is Hamiltonian stationary in $\mathbb{C}^{n}$ (i.e. for all compactly supported Hamiltonian vector fields), provided $N$ is a compact set with finite $k$-dimensional Hausdorff measure which is locally $k$-noncollapsing, $k \leq n-2$, and the volume of $L \cap B_{r}(x)$ for $x \in N$ is dominated by a power of $r$ involving $n, k$. Local control on volume is important for extension problems, our consideration is inspired by those for extending minimal varieties (general dimension and codimension) across small closed sets in [HL70, Theorem 5.1, 5.2], also see [CL17]. A special case of Theorem 4.1, namely, when $N$ is a finite set of points, is used in concluding the limiting varifold in Theorem1.1 is Hamiltonian stationary in $\mathbb{C}^{n}$. A removable singularity theorem for Hamiltonian stationary Lagrangian graphs was proven in [CW16] under a weaker assumption.

We now outline the structure of the paper.
In section 2, we set up basic framework for dealing with properly immersed Hamiltonian stationary Lagrangian submanifolds. In particular, for a proper Lagrangian immersion $\iota: M \rightarrow \mathbb{C}^{n}$, we show equivalence of $L=\iota(M)$ being Hamiltonian stationary (seemingly weaker due to non-embedded points) and $M$ being Hamiltonian stationary. This leads to the definition of Hamiltonian stationary varifolds which fits naturally in convergence of a sequence of immersed ones.

In section 3, we derive curvature and smoothness estimates. Simons's identity ([S68]) plays an important role for minimal submanifolds in deriving higher order estimates in terms of the second fundamental form $A$ and in proving the $\varepsilon$-regularity (cf. [CS85], [And86]). However, such useful technique is not available for the Hamiltonian stationary
case for $\nabla^{2} H$ involved in $\Delta_{g}|A|^{2}$ is not reduced to lower order terms of $A$. Instead, we use a priori estimates for the potential function $u$ by viewing (1.1) as a second order elliptic operator $\Delta_{g}$ acting on the fully nonlinear second order elliptic operator $\theta$ as in [CW16].

In section 4, we show in Theorem 4.1 that a Hamiltonian stationary Lagrangian submanifold away from a small set extends across the set as a Hamiltonian stationary varifold provided its mean curvature $H$ is in $L^{n}$ and a volume condition near the small set is satisfied. This volume condition follows directly from the monotonicity formula if $H=0$, and it also valid if the set is of isolated points and $n \geq 2$, see Proposition 4.3.

In section 5, we prove Theorem 1.1. The structure of the proof is similar to that in [CS85] and [And86]. For precise definitions of convergence, see Definition 5.1 and Definition 5.2.

## 2. Hamiltonian stationary immersions

In this section we set up the basic framework for dealing with compact smooth Lagrangian Hamiltonian stationary immersions.

We will need to deal with immersed submanifolds that may be non-embedded, so we define the following.

Definition 2.1. Let $L$ be defined by a smooth immersion $\iota: M^{n} \rightarrow \mathbb{R}^{n+l}$. Given any connected open set $U \subset \mathbb{R}^{n+l}$, decompose the inverse image into connected components as

$$
\iota^{-1}(U)=\bigsqcup_{\alpha} O_{\alpha}(U) .
$$

If $\iota$ restricted to each $O_{\alpha}(U)$ is a smooth embedding into $\mathbb{R}^{n+l}$, then we say that $\iota\left(O_{\alpha}(U)\right)$ is an embedded connected component of $U \cap L$ and that $L$ splits into embedded components on $U$.

Proposition 2.2. Let $\iota: M \rightarrow \mathbb{R}^{n+l}$ be a proper immersion fo a smooth manifold $M$, and set $L=\iota(M)$. For any point $y \in L$, there is an open ball $B_{r}^{n+l}(y)$ such that $\iota(M)$ splits into embedded components on $B_{r}^{n+l}(y)$, and each component contains $y$.

Proof. For any fixed $y \in \iota(M)$, since $\iota$ is a proper immersion, the pre-image of $y$ is a finite set $\iota^{-1}(\{y\})=\left\{x_{1}, \ldots, x_{m}\right\}$. Let $B\left(x_{1}\right), \ldots, B\left(x_{m}\right)$ be disjoint coordinate balls centered at $x_{i}$ and $\iota$ is injective on each $B\left(x_{i}\right)$. Take a decreasing sequence $r_{k} \rightarrow 0$. Let

$$
S_{k}=\iota^{-1}\left(B_{r_{k}}^{2 n}(y)\right) \bigcap\left(M \backslash \bigcup_{i=1}^{m} B\left(x_{i}\right)\right)
$$

Clearly, $S_{k+1} \subset S_{k}$. If there exists $x$ in all $S_{k}$ then $\iota(x)=y$. So $x \in\left\{x_{1}, \ldots, x_{m}\right\}$, but this violates the definition of $S_{k}$. Thus there is $k_{0}$ such that $S_{k_{0}}=\emptyset$. Then

$$
\iota^{-1}\left(B_{r_{k_{0}}}(y)\right) \subseteq \bigcup_{i=1}^{m} B\left(x_{i}\right)
$$

and this implies

$$
\iota(M) \cap B_{r_{k_{0}}}^{2 n}(y) \subset \bigcup_{i=1}^{m} \iota\left(B\left(x_{i}\right)\right)
$$

and then

$$
\iota(M) \cap B_{r_{k_{0}}}^{2 n}(y)=B_{r_{k_{0}}}^{2 n}(y) \bigcap \bigcup_{i=1}^{m} \iota\left(B\left(x_{i}\right)\right)=\bigcup_{i=1}^{m} \iota\left(B\left(x_{i}\right) \cap B_{r_{k_{0}}}^{2 n}(y) .\right.
$$

We shall finish the proof by showing that $\iota\left(B\left(x_{i}\right)\right) \cap B_{r}^{2 n}(y)$ is connected for all $r \geq r_{i}$ for some positive $r_{i}$ and then taking the smallest $r_{i}, i=1, \ldots, m$. Represent $\iota\left(B\left(x_{i}\right)\right.$ locally as a graph of a vector valued function $F: B_{\rho}^{n}(0) \subset \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, where we identify $T_{y} \iota\left(B\left(x_{i}\right)\right)$ with $\mathbb{R}^{n}$ and $y$ with 0 ; we further assume $F(0)=0, \mathrm{D} F(0)=0,|\mathrm{DF}| \leq C(\rho)$ on $B_{\rho}^{n}(0)$. Then any point $x$ with $(x, F(x)) \in \partial B_{\rho}^{2 n}(y)$ satisfies

$$
\rho^{2}=|x|^{2}+|F(x)|^{2} \leq\left(1+C_{\rho}^{2}\right)|x|^{2}
$$

therefore

$$
|x| \geq \frac{\rho}{\sqrt{1+C_{\rho}^{2}}}
$$

If $\iota\left(B\left(x_{i}\right) \cap B_{\rho}^{2 n}(y)\right.$ is disconnected, there must be a point $p \in \iota\left(B\left(x_{i}\right)\right) \cap \partial B_{\rho}^{2 n}(y)$ that is not on the connected component containing $y$. On the ray $\sigma(t)=t x_{p} /\left|x_{p}\right|$ from 0 to $x_{p}$ in $B_{\rho}^{n}(0)$ where $p=\left(x_{p}, F\left(x_{p}\right)\right)$, there must be two distinct points $\sigma\left(t_{1}\right), \sigma\left(t_{2}\right)$ with $\rho / \sqrt{1+C_{\rho}^{2}} \leq t_{1}, t_{2} \leq \rho$ such that $t_{1}$ is the last departing time for $\iota\left(B\left(x_{i}\right)\right.$ to leave $B_{\rho}^{2 n}(y)$ and $s_{2}$ is the first returning time. Thus, for the smooth function

$$
f(t)=|x(t)|^{2}+|F(x(t))|^{2}
$$

we have $f^{\prime}\left(t_{1}\right) \geq 0$ and $f^{\prime}\left(t_{2}\right) \leq 0$. So there is $t_{0} \in\left[t_{1}, t_{2}\right]$ with $f^{\prime}\left(t_{0}\right)=0$, i.e.

$$
x\left(t_{0}\right) \cdot \sigma^{\prime}\left(t_{0}\right)+F\left(x\left(t_{0}\right)\right) \cdot \mathrm{D} F_{x\left(t_{0}\right)}\left(\sigma^{\prime}\left(t_{0}\right)\right)=0
$$

Since $\sigma^{\prime}\left(t_{0}\right)$ is a unit vector, we have

$$
\frac{\rho}{\sqrt{1+C_{\rho}^{2}}} \leq\left|x\left(t_{0}\right)\right|=\left|x\left(t_{0}\right) \cdot \sigma^{\prime}\left(t_{0}\right)\right| \leq C_{\rho}\left|F\left(x\left(t_{0}\right)\right)\right| \leq C_{\rho} \rho .
$$

But this becomes impossible for small $\rho$ since $C_{\rho} \rightarrow 0$.
To describe convergence of immersions to a possibly nonsmooth limit, we will treat the immersed submanifolds as varifolds. Recall that an $n$-varifold on $U \subset \mathbb{R}^{n+l}$ is a Radon measure on the space $U \times G(n, n+l)$, where $G(n, n+l)$ is the Grassmannian of $n$-planes in $\mathbb{R}^{n+l}$. Given a smooth immersion $\iota: M \rightarrow \mathbb{R}^{n+l}$, one can define a volume measure $\mathrm{d} \mu_{M}$ on $M$ via

$$
\mathrm{d} \mu_{g}=\iota^{*}\left(d H^{n}\right)
$$

where $d H^{n}$ is the $n$-dimensional Hausdorff measure on $\mathbb{R}^{n+l}$. In turn, this defines an $n$-varifold: If we define $\Gamma(x) \in G(n, n+l)$ to be the tangent plane at the point $\iota(x)$ for the immersion $\iota$, then we can define a varifold associated to the immersion via

$$
V=(\iota \times \Gamma)_{\#} \mathrm{~d} \mu_{M}
$$

where

$$
\iota \times \Gamma: M \rightarrow U \times G(n, n+l)
$$

is defined by

$$
\iota \times \Gamma(x)=(\iota(x), \Gamma(x))
$$

and $(\iota \times \Gamma)_{\#}$ is the measure pushforward. There is a natural measure supported on the image set $L=\iota(M)$ via

$$
\mu_{L}=\left(\pi_{\mathbb{R}^{n+l}}\right)_{\#} V=\iota_{\#} \mathrm{~d} \mu_{M}
$$

where $\pi_{\mathbb{R}^{n+l}}$ is the projection from $U \times G(n, n+l)$ onto $U$. This measure agrees with $d H^{n}$ near points where $\iota$ is an embedding.

From now on, we assume that the immersion $\iota$ is proper. For any $y \in L$, let $\beta(y)$ be the cardinality of the preimage set $\iota^{-1}(\{y\})=\{x \in M: \iota(x)=y\}$, and $\beta(y)$ is finite as $\iota$ is a proper immersion. For any $x \in M$ let $y=\iota(x)$, then $x$ belongs to a unique $O_{\alpha}(U)$, where $U$ is the open ball given in Proposition 2.2, let $H_{\iota}(x)$ be the mean curvature vector of the embedded submanifold $\iota\left(O_{\alpha}(U)\right)$ at $y$. The vector field $H_{\iota}$ is globally defined as a smooth section of the pullback bundle $\iota^{*}\left(T \mathbb{R}^{n+l}\right)$ over $M$. For the immersed submanifold $L=\iota(M) \subset \mathbb{R}^{n+l}$, its (weak) mean curvature is defined as

$$
\begin{equation*}
\bar{H}(y)=\frac{1}{\beta(y)} \sum_{x \in \iota^{-1}(y)} H_{\iota}(x) \tag{2.1}
\end{equation*}
$$

Note that $\bar{H}$ is the ordinary mean curvature $H_{\iota}$ at any point $y$ where $L$ is embedded (i.e. $\beta(y)=1)$. For any smooth compactly supported vector field $X$ on $\mathbb{R}^{2 n}$, the first variation formula (for rectifiable $n$-varifolds) asserts

$$
\int_{L} \operatorname{div}_{L}(X) \mathrm{d} \mu_{L}=-\int_{L}\langle X, \mathcal{H}\rangle \mathrm{d} \mu_{L}
$$

where $\mathcal{H}$ is the generalized mean curvature of $L$ as a d $\mu_{L}$-integrable $\mathbb{R}^{2 n}$-valued function, provided such an $\mathcal{H}$ exists. For smooth immersions, $\mathcal{H}$ is just $\bar{H}$ since they agree $H^{n}$-a.e. On the other hand, using the general area formula (see 8.5 in [Sim83])

$$
\begin{align*}
\int_{L}\langle X(y), \mathcal{H}(y)\rangle \mathrm{d} \mu_{L}(y) & =\int_{L} \frac{1}{\beta(y)} \sum_{x \in \iota^{-1}(y)}\left\langle X(y), H_{\iota}(x)\right\rangle \mathrm{d} \mu_{L}(y) \\
& =\int_{\mathbb{R}^{2 n}} \frac{1}{\beta(y)} \sum_{x \in \iota^{-1}(y)}\left\langle X(y), H_{\iota}(x)\right\rangle \beta(y) \mathrm{d} H^{n}(y) \\
& =\int_{\mathbb{R}^{2 n}} \sum_{x \in \iota^{-1}(y)}\left\langle X(y), H_{\iota}(x)\right\rangle \mathrm{d} H^{n}(y)  \tag{2.2}\\
& =\int_{\mathbb{R}^{2 n}} \int_{\iota^{-1}(y)}\left\langle X \circ \iota, H_{\iota}\right\rangle d H^{0}(x) \mathrm{d} H^{n}(y) \\
& =\int_{M}\left\langle X \circ \iota, H_{\iota}\right\rangle \mathrm{d} \mu_{M} .
\end{align*}
$$

Any vector field $X$ along $L$ yields a vector field $X \circ \iota$ along $M$, so in light of (2.2), "stationary" $M$ gives "stationary" $L$. We show the other direction is also true when $X$ is restricted to Hamiltonian vector fields. In particular, we can use (2.2) to get information about the ordinary mean curvature of a proper immersion if its image is stationary as variolds.

Definition 2.3. Let $V$ be an integral rectifiable $k$-varifold on an open subset $U$ of $\mathbb{C}^{n}$ with generalized mean curvature $\mathcal{H}$. We say $V$ is Hamiltonian stationary if

$$
\int_{U}\langle J D f, \mathcal{H}\rangle \mathrm{d} \mu_{V}=0
$$

for any $f \in C_{0}^{\infty}(U)$. If $k=n$ and every approximate tangent space $T_{x} V$ is a Lagrangian $n$-plane in $\mathbb{C}^{n}$, we say $V$ is a Lagrangian varifold. If a Lagrangian varifold is Hamiltonian stationary, it is a Hamiltonian stationary Lagrangian n-varifold.

We say a proper smooth immersion $\iota: M \rightarrow U$ is Hamiltonian stationary on $U$, if

$$
\begin{equation*}
\int_{M}\left\langle J \mathrm{D} f(\iota(x)), H_{\iota}(x)\right\rangle \mathrm{d} \mu_{M}(x)=0 \tag{2.3}
\end{equation*}
$$

for all $f \in C_{c}^{\infty}(U)$. We remark that this definition precludes the possibility of the manifold having manifold boundary portions inside the neighborhood $U$.
Proposition 2.4. Let $\iota: M \rightarrow \mathbb{C}^{n}$ be a proper smooth immersion of a connected smooth manifold $M$ and suppose that $L=\iota(M)$ is Hamiltonian stationary and Lagrangian. Then the Lagrangian angle function $\theta$ of each embedded connected component of $L \cap U$ satisfies (1.1) for any open subset $U$ of $\mathbb{C}^{n}$.

Proof. Any embedded connected component $\tilde{L}$ of $L \cap U$ is Lagrangian, and its Lagrangian angle $\theta_{\iota}$ is defined (up to orientation and modulo $2 \pi$ ) by

$$
\left.d z^{1} \wedge \cdots \wedge d z^{n}\right|_{\tilde{L}}=e^{\sqrt{-1} \theta_{\iota}} \mathrm{d} \mu_{\tilde{L}}
$$

and satisfies $H_{\iota}=J \nabla \theta_{\iota}$ on $\tilde{L}$ where $J$ is the complex structure on $\mathbb{C}^{n}$ (cf. [HL82]).
We divide the point-set $L$ into two pieces. We say a point $y \in L$ is an embedded point if there is an open set $W$ in $\mathbb{R}^{2 n}$ containing $y$ so that the point set $L \cap W$ is an embedded submanifold in $\mathbb{R}^{2 n}$ and let $E$ be the set of all embedded points of $L$. We show first that (1.1) holds on $E$, and then argue that $\bar{E}=L$.

For any $y \in E \cap \tilde{L}, L$ is embedded in a neighborhood around $y$, by Proposition 2.2, there exists a sufficiently small ball $B_{r_{0}}^{2 n}(y)$ in $\mathbb{R}^{2 n}$, we may assume the ball is in $U$ as well, such that $\iota^{-1}\left(B_{r_{0}}^{2 n}(y)\right)$ is a finite disjoint union of $O_{1}, \ldots, O_{m(y)}$, and $\left.\iota\right|_{O_{i}}$ is an embedding with $L_{i}:=\iota\left(O_{i}\right)=L \cap B_{r_{0}}^{2 n}(y)$ for each $i$, and $m(y)$ is constant on $L \cap B_{r_{0}}^{2 n}(y)$. Pulling back the Euclidean metric on $\mathbb{R}^{2 n}$ and the $n$-form $d z^{1} \wedge \cdots \wedge d z^{n}$ by $\iota$, we see that $O_{i}, O_{j}$ are isometric in their induced metrics, and the Lagrangian angles $\theta_{\iota}$ are the same, since $\left.\left.\iota\right|_{O_{j}} ^{-1} \circ \iota\right|_{O_{i}}: O_{i} \rightarrow O_{j}$ is diffeomorphic.

Now for any $\phi \in C_{c}^{\infty}(M)$ with support in $O_{1}, \varphi(y)=\phi\left(\iota^{-1}(\{y\})\right)$ is a well defined function on $L$ which is smooth on $\iota\left(O_{1}\right)$ and equals zero outside $\iota\left(O_{1}\right)$. We can extend $\varphi$ to function $f \in C_{c}^{\infty}\left(\mathbb{R}^{2 n}\right)$. Since $L$ is Hamiltonian stationary, by (2.2), we have

$$
\begin{aligned}
0 & =\int_{L}\langle J \mathrm{D} f(y), \mathcal{H}(y)\rangle \mathrm{d} \mu_{L}(y) \\
& =\int_{M}\left\langle J \mathrm{D} f(\iota(x)), H_{\iota}(x)\right\rangle \mathrm{d} \mu_{M} \\
& =\int_{M}\left\langle J \mathrm{D} f(\iota(x)), J \nabla \theta_{\iota}(x)\right\rangle \mathrm{d} \mu_{M} \\
& =\int_{M} f(\iota(x)) \Delta \theta_{\iota}(x) \mathrm{d} \mu_{M} \\
& =\int_{O_{1} \cup \cdots \cup O_{m(y)}} f(\iota(x)) \Delta \theta_{\iota}(x) \mathrm{d} \mu_{M} \\
& =m(y) \int_{M} \phi \Delta \theta_{\iota} \mathrm{d} \mu_{M}
\end{aligned}
$$

This shows that $\theta_{\iota}$ is harmonic around any $y$ where $L$ is embedded, regardless whether the $m(y)$-sheeted cover of $L$ near $y$ given by $\iota$ is single sheeted or not.

Next, we show that $E \cap L_{i}$ is dense in $L_{i}$. First, we consider two embedded connected components $L_{i}$ and $L_{j}$ (if there there are more than one, otherwise we are done), and let $E_{i j}$ be the embedded points of $L_{i}$ relative to the set $L_{i} \cup L_{j}$. The set $E_{i j}$ is open in $L_{i}$. The closed set $E_{i j}^{c} \subseteq L_{i} \cap L_{j}$ has no interior points in $L_{i}$ for any such point would imply $L_{i} \cap L_{j}$ contains some open neighborhood of the point but then the neighborhood would be in $E_{i j}$. So $E_{i j}^{c}$ is closed and nowhere dense in $L_{i}$, in turn, $E_{i j}$ is dense and open in $L_{i}$. Now $E \cap L_{i} \subseteq L_{i} \backslash \cup_{j} E_{i j}^{c}$ is open and dense in $L_{i}$, since the range of $j$ is finite.

It then follows that for any $y \in L_{i}$ there exists $y_{k} \in E \cap L_{i}$. Now by smoothness of $\Delta_{g} \theta_{\iota}$ when restricted to $\tilde{L}$ and that $\Delta_{g} \theta_{\iota}\left(y_{k}\right)=0$, we conclude that $\Delta_{g} \theta_{\iota}(y)=0$ on $L_{i}$. Therefore, we conclude that $\theta_{\iota}$ satisfies (1.1) on $\tilde{L}$.

## 3. Curvature and higher order estimates

3.1. Graphical representation of Lagrangian submanifolds. We begin with rephrasing, for Lagrangian submanifolds, a well known fact about local graphical representation of embedded submanifolds, that gives a precise lower bound, in terms of the length of the second fundamental form, on the size of a ball in the tangent space over which the Lagrangian submanifold is a graph of the gradient of a potential function with uniform Hessian bound. The bounds are written in a convenient for the rotation argument in the proof of Proposition 3.2.

Lemma 3.1. Let $L$ be a properly and smoothly immersed connected Lagrangian submanifold in $\mathbb{C}^{n}$. Suppose that $\|A\|_{\infty} \leq C$ on $\partial L \cap B_{\rho_{0}}^{2 n}(0)=\emptyset$, where $A$ is the second fundamental form of $L$ and $B_{\rho_{0}}^{2 n}(0)$ is the ambient ball with radius $\rho_{0}(C)=\frac{\pi}{12 C}$ and $0 \in L$. Then any embedded connected component $\tilde{L}$ of $B_{\rho_{0}}^{2 n}(0) \cap L$ containing 0 is a gradient graph over a region $\Omega \subset T_{0} \tilde{L}$, that is, there is a function $u: \Omega \rightarrow \mathbb{R}$ such that

$$
\tilde{L}=\{(x, D u(x)), x \in \Omega\}
$$

and $\Omega$ contains the ball $B_{r_{0}}^{n}(0) \subset T_{0} L$, where

$$
\begin{equation*}
r_{0}(C)=\frac{\pi}{12 C} \cos \frac{\pi}{12} \tag{3.1}
\end{equation*}
$$

Further,

$$
\begin{equation*}
\left|D^{2} u\right| \leq \tan \frac{\pi}{12} \text { on } B_{r_{0}}^{n}(0) \tag{3.2}
\end{equation*}
$$

Proof. Locally, any embedded Lagrangian submanifold $\tilde{L}$ is the gradient graph over its tangent space $T_{0} \tilde{L}$ of a function $u$ with $D^{2} u(0)=0$, say over a ball $B_{\sigma_{0}}^{n}(0)$. Let $\lambda_{i}(x)$ be the eigenvalues of $D^{2} u(x)$. First we claim:

$$
\left\|\nabla_{g} \arctan \lambda_{i}\right\| \leq\|A\|_{\infty} \leq C
$$

To see this, note that consider the (3,0)-tensor

$$
A: T \tilde{L} \times T \tilde{L} \times N \tilde{L} \rightarrow \mathbb{R}
$$

defined by

$$
A(X, Y, \vec{n})=D_{X} Y \cdot \vec{n}
$$

with have components

$$
A\left(\partial_{i}, \partial_{j}, n_{k}\right)=u_{j k i}
$$

under a local coordinate frame $\partial_{1}, \ldots, \partial_{n}$, where $n_{k}=J \partial_{k} \in N \tilde{L}$ as $\tilde{L}$ is Lagrangian. Thus

$$
\begin{aligned}
\|A\|^{2} & =\sum_{i j k} g^{i a} g^{j b} g^{k c} u_{i j k} u_{a b c} \\
& =\sum_{i j k} g^{i i} g^{j j} g^{k k} u_{i j k}^{2} \text { when } D^{2} u \text { is diagonalized. }
\end{aligned}
$$

For any $i$,

$$
\left\|\nabla_{g} \arctan \lambda_{i}\right\|^{2}=g^{j j}\left(\frac{1}{1+\lambda_{i}^{2}}\right)^{2} u_{i i j}^{2} \text { when } D^{2} u \text { is diagonalized. }
$$

Now clearly

$$
\sum_{i, j} g^{j j} g^{i i} g^{i i} u_{i i j}^{2} \leq \sum_{i, j} g^{j j} g^{i i} g^{i i} u_{i i j}^{2}+\sum_{i, j, k \neq i} g^{j j} g^{i i} g^{k k} u_{i k j}^{2}=\sum_{i, j} g^{j j} g^{i i} g^{k k} u_{i k j}^{2}=\|A\|^{2}
$$

This proves the claim.
Next, let $v$ be any unit vector in $T_{0} \tilde{L}$ and let $\gamma_{v}(s)=(s v, D u(s v))$ for $s \in\left[0, \sigma_{0}\right)$. Integrating along $\gamma_{v}$ and using the claim, we see that the maximum value of $\arctan \lambda_{i}$ satisfies

$$
\left|\arctan \lambda_{i}\right| \leq C L\left(\gamma_{v}\right)
$$

Thus the maximum slope of $\gamma_{v}$ (precisely, each planar curve $\left.\left(s v, u_{i}(s v)\right), i=1, \ldots, n\right)$ satisfies

$$
\begin{equation*}
\left|\lambda_{i}\right| \leq \tan \left(C L\left(\gamma_{v}\right)\right) \tag{3.3}
\end{equation*}
$$

Therefore

$$
L\left(\gamma_{v}\right)=\int_{0}^{\sigma_{0}} \sqrt{1+|D u(s v)|^{2}} d s \leq \sqrt{1+\tan ^{2}\left(C L\left(\gamma_{v}\right)\right)} \sigma_{0}
$$

That is

$$
L\left(\gamma_{v}\right) \cos \left(C L\left(\gamma_{v}\right)\right) \leq \sigma_{0}
$$

By the slope bound (3.3) and that $v$ can point to any direction and that $L$ is connected with no boundary points inside $B_{\rho_{0}}^{2 n}(0)$, in fact $u$ can be defined as long as $\sigma_{0} \leq \rho_{0}$. Now for

$$
r_{0}=\frac{\pi}{12 C}\left(\cos \frac{\pi}{12}\right)=\frac{\pi}{12 C} \cos \left(C \frac{\pi}{12 C}\right)
$$

it follows that the length $L\left(\gamma_{v}\left(\left[0, r_{0}\right]\right)\right)$ of is no more than $\frac{\pi}{12 C}$ for any unit $v$, which confirms that the slope was never more than $\tan (\pi / 12)$ by (3.3). This also confirms that the graph remains in the ambient ball of radius $\rho_{0}$.
3.2. Smoothness estimates. The local graphical representation in Lemma 3.1 with bounded Hessian for the Lagrangian phase function can be used to construct, by a rotation, a new Lagrangian graph which lies in a region of the Hessian space where the Lagrangian phase function is uniformly concave. Therefore, for Hamiltonian stationary Lagrangian graphs, a priori $C^{2, \alpha}$ estimates applies to the Lagrangian phase and then the bootstrapping procedure in [CW16] leads to higher order estimates on a ball of uniform radius.

Proposition 3.2. Suppose that $L=\iota(M)$ is Hamiltonian stationary Lagrangian given by a proper immersion $\iota$. Suppose that $\|A\|_{\infty} \leq C$ on $L \cap B_{\rho_{0}}^{2 n}(0)$ and $0 \in L$ as in Lemma 3.1. Let $\tilde{L}$ be an embedded connected component of $B_{\rho_{0}}^{2 n}(0) \cap L$ containing 0 and let

$$
U_{\pi / 6}\left(T_{0}^{n} \tilde{L}\right)=e^{-i \frac{\pi}{6}} I_{\mathbb{C}^{n}}\left(T_{0}^{n} \tilde{L}\right)
$$

where $e^{-i \frac{\pi}{6}} I_{\mathbb{C}^{n}}$ is the complex multiplication acting on vectors in the (real) subspace $T_{0}^{n} \tilde{L} \subset$ $\mathbb{C}^{n}$. Then, there exists $r_{1}\left(\|A\|_{\infty}\right), C_{4}\left(\alpha,\|A\|_{\infty}\right)$ such that $\tilde{L}$ is a gradient graph over a region $\Omega \subset U_{\pi / 6}\left(T_{0}^{n} \tilde{L}\right)$

$$
\tilde{L}=\{(x, D \bar{u}(x)), x \in \Omega\}
$$

such that $B_{r_{1}}^{n}(0) \subset \Omega$ and we have that

$$
\left\|D^{4} \bar{u}\right\|_{C^{\alpha}\left(B_{r_{1}}\right)} \leq C_{4}\left(\alpha,\|A\|_{\infty}\right) \text { on } B_{r_{1}}^{n}(0)
$$

with

$$
r_{1}=\frac{\pi\left(1-4 \sin ^{2} \frac{\pi}{12}\right) \cos \frac{\pi}{12}}{12\|A\|_{\infty} \times 8}
$$

Proof. First, by Lemma 3.1 we know that $\tilde{L}$ is represented by the gradient graph of a function $u$ over a ball $B_{r_{0}}^{n}(0)$ contained in the tangent space at 0 , and the Hessian of $u$ satisfies (3.2). As in [CW16, Proposition 4.1] we can use a Lewy-Yuan rotation [Yua06, Section 2, Step 1] to rotate the graph up by $\frac{\pi}{6}$ :

$$
\begin{aligned}
\bar{x} & =\cos \frac{\pi}{6} x+\sin \frac{\pi}{6} D u(x) \\
\bar{y} & =-\sin \frac{\pi}{6} x+\cos \frac{\pi}{6} D u(x) .
\end{aligned}
$$

Now by [CW16, Proposition 4.1], the graph of the gradient of new potential function

$$
\begin{equation*}
\bar{u}(x)=u(x)+\sin \frac{\pi}{6} \cos \frac{\pi}{6} \frac{|D u(x)|^{2}-|x|^{2}}{2}-\sin ^{2} \frac{\pi}{6} D u(x) \cdot x \tag{3.4}
\end{equation*}
$$

over the $\bar{x}$-plane represents the same piece of $\tilde{L}$. It follows that all of the eigenvalues now satisfy

$$
\begin{equation*}
\lambda_{i} \in\left(\tan \frac{\pi}{12}, \tan \frac{\pi}{3}\right) \tag{3.5}
\end{equation*}
$$

Thus the Lagrangian phase operator

$$
\begin{equation*}
F\left(D^{2} \varphi\right)=\sum_{\lambda_{j} \text { eigenvalues of } D^{2} \varphi} \arctan \lambda_{j} \tag{3.6}
\end{equation*}
$$

is uniformly concave on this region of Hessian space. We also know that the Jacobian of the rotation map (cf [CW16, 4.4]) is bounded below by

$$
\begin{equation*}
\operatorname{det} \frac{d \bar{x}}{d x} \geq \operatorname{det}\left[\cos \frac{\pi}{6} I-\sin \frac{\pi}{6} \tan \frac{\pi}{12} I\right]>0.7 \tag{3.7}
\end{equation*}
$$

Thus the rotation of coordinates $x \rightarrow \bar{x}$ must give us a radius

$$
\begin{equation*}
\bar{r}_{0}=\left(1-4 \sin ^{2} \frac{\pi}{12}\right) r_{0} \tag{3.8}
\end{equation*}
$$

such that submanifold is graphical over a ball of radius $\bar{r}_{0}$, for a new potential $\bar{u}$ representing the gradient graph over the plane $U_{\pi / 6}\left(T_{0} L^{n}\right)$. Now the Lagrangian phase operator
(3.6) extends to a global (on Hessian space) concave uniformly elliptic operator $\tilde{F}$ (cf. [CW16, Section 5]) which agrees with $F$ on the following region of the Hessian space:

$$
\left\{D^{2} \varphi:\left(\tan \frac{\pi}{12}\right) I \leq D^{2} \varphi \leq\left(\tan \frac{\pi}{3}\right) I\right\} .
$$

In particular

$$
F\left(D^{2} \bar{u}(\bar{x})\right)=\bar{\theta}(\bar{x})=\theta(x)+n \frac{\pi}{6}
$$

A rescaling of $\bar{u}$ gives $\bar{v}$ :

$$
\bar{v}(\bar{x})=\frac{1}{\bar{r}_{0}^{2}} \bar{u}\left(\bar{r}_{0} \bar{x}\right)
$$

which is still a solution of the Hamiltonian stationary equation, since (1.1) only involves the second order derivatives of $\bar{u}$ which are invariant under the rescaling, but now on the ball of radius 1. Note that the range of the Hessian (3.5) does not change under rescaling, in particular, if $\widetilde{\theta}$ is the rescaled $\bar{\theta}$

$$
\widetilde{\theta}(\bar{x})=\bar{\theta}\left(\bar{r}_{0} \bar{x}\right)
$$

then $\widetilde{\theta}$ satisfies in uniformly elliptic equation, with ellipticity constants

$$
\lambda_{0}=\frac{1}{1+\tan ^{2} \frac{\pi}{3}}, \quad \Lambda_{0}=1
$$

according to (3.5). Thus by DeGiorgi-Nash theory, we have a universal interior Hölder bound on $\widetilde{\theta}$

$$
\|\widetilde{\theta}\|_{C^{\alpha}\left(B_{3 / 4}\right)} \leq C_{D N}\left(\lambda_{0}, n\right)
$$

noting that $\bar{\theta}$ is bounded also by (3.5).
Now we can apply [CC03, Theorem 1.2] to obtain

$$
\begin{aligned}
\left\|D^{2} \bar{v}\right\|_{C^{\alpha}\left(B_{1 / 2}\right)} & \leq C_{C C}\left\{\|\widetilde{\theta}\|_{C^{\alpha}\left(B_{3 / 4}\right)}+\|\bar{v}\|_{L^{\infty}\left(B_{1}\right)}\right\} \\
& \leq C_{C C}\left\{C_{D N}\left(\lambda_{0}, n\right)+\|\bar{v}\|_{L^{\infty}\left(B_{1}\right)}\right\} .
\end{aligned}
$$

Now we also have

$$
\|\bar{v}\|_{L^{\infty}\left(B_{1}\right)}=\frac{1}{\bar{r}_{0}^{2}}\|\bar{u}\|_{L^{\infty}\left(B_{\bar{x}_{0}}\right)} .
$$

We were assuming that $D u(0)=0, u(0)=0$ so that with (3.2) we have

$$
\|u\|_{L^{\infty}\left(B_{r_{0}}\right)} \leq\left(\tan ^{2} \frac{\pi}{12}\right) \frac{r_{0}}{2}
$$

which leads to by using (3.4)

$$
\|\bar{u}\|_{L^{\infty}\left(B_{\bar{r}_{0}}\right)} \leq\left\{\tan ^{2} \frac{\pi}{12}+\sin \frac{\pi}{6} \cos \frac{\pi}{6}\left(2 \tan ^{2} \frac{\pi}{12}+1\right)+\sin ^{2} \frac{\pi}{6}\left(\tan ^{2} \frac{\pi}{12}+1\right)\right\} \frac{r_{0}^{2}}{2}
$$

We conclude that

$$
\left\|D^{2} \bar{v}\right\|_{C^{\alpha}\left(B_{1 / 2}\right)} \leq C_{C C}\left\{C_{D N}\left(\lambda_{0}, n\right)+\frac{1}{\bar{r}_{0}^{2}} C_{T}\left(\frac{\pi}{12}\right) r_{0}^{2}\right\}
$$

for some universal trigonometric constant $C_{T}$. Noting that bound (3.8) bounds the ratio between $r_{0}$ and $\bar{r}_{0}$ we see that we have a universal bound (depending only on $\alpha$ ).

$$
\left\|D^{2} \bar{v}\right\|_{C^{\alpha}\left(B_{1 / 2}\right)} \leq C_{2}(\alpha)
$$

Now that the Hölder norm of $D^{2} \bar{v}$ is uniformly bounded, we may apply the bootstrapping theory [CW16, Section 5] to obtain

$$
\begin{gathered}
\left\|D^{3} \bar{v}\right\|_{C^{\alpha}\left(B_{1 / 4}\right)} \leq C_{3}\left(C_{2}, \alpha\right) \\
\left\|D^{4} \bar{v}\right\|_{C^{\alpha}\left(B_{1 / 8}\right)} \leq C_{4}\left(C_{3}, C_{2}, \alpha\right)
\end{gathered}
$$

Now we may scale back to $\bar{u}$ and get that

$$
\left\|D^{4} \bar{u}\right\|_{C^{\alpha}\left(B_{\bar{r}_{0} / 8}\right)} \leq C_{4}(\alpha) \bar{r}_{0}^{-2-\alpha}
$$

Choosing $r_{1}=\bar{r}_{0} / 8$ and recalling (3.8) and (3.1) gives us the result.
3.3. Curvature estimates with small total extrinsic curvature. The next result establishes the key pointwise curvature estimates of a Hamiltonian stationary submanifold under the assumption that the total extrinsic curvature $\|A\|_{L^{n}}$ is small. This is in analogue to minimal surfaces and harmonic maps (cf. [CS85], [And86], [SU82], [Sh17]). The main difference here from the minimal surfaces case is the lack a useful Simon's type inequality in the Hamiltonian stationary case. The $C^{4, \alpha}$ estimate for the scalar potential function $u$ allows us to carry through an argument similar to that in [CS85].

Proposition 3.3. Suppose that $L$ is a smooth Lagrangian Hamiltonian stationary manifold in $B_{1}(0)$ with $\partial L \cap B_{1}(0)=\emptyset$. Then, there exists an $\varepsilon_{0}$ such that if $r_{0} \leq 1$ and

$$
\int_{B_{r_{0}}(0) \cap L}|A|^{n}<\varepsilon_{0}
$$

then for all $0<\sigma \leq r_{0}$ and $y \in B_{r_{0}-\sigma}$

$$
\sigma^{2}|A(y)|^{2} \leq\left(\frac{\pi}{24}\right)^{2}
$$

Proof. Without loss of generality let $r_{0}=1$. We will deduce the general case by rescaling at the end. Consider the nonnegative function

$$
(1-|x|)^{2}|A(x)|^{2}
$$

This function attains its maximum somewhere inside $B_{1}(0)$, say at $x_{0}$. We assume the maximum is positive, otherwise the result is trivial. Thus

$$
(1-|x|)^{2}|A(x)|^{2} \leq\left(1-\left|x_{0}\right|\right)^{2}\left|A\left(x_{0}\right)\right|^{2}
$$

in particular, for $x \in B_{\frac{1-\left|x_{0}\right|}{2}}\left(x_{0}\right)$

$$
\begin{aligned}
|A(x)|^{2} & \leq \frac{\left(1-\left|x_{0}\right|\right)^{2}}{(1-|x|)^{2}}\left|A\left(x_{0}\right)\right|^{2} \\
& \leq \frac{\left(1-\left|x_{0}\right|\right)^{2}}{\left(\frac{1-\left|x_{0}\right|}{2}\right)^{2}}\left|A\left(x_{0}\right)\right|^{2} \\
& =4\left|A\left(x_{0}\right)\right|^{2}
\end{aligned}
$$

Rescaling the graph over the ball $B_{\frac{1-\left|x_{0}\right|}{2}}\left(x_{0}\right)$ by $\left|A\left(x_{0}\right)\right|$, we get a Hamiltonian stationary manifold on a ball of radius

$$
R_{0}=\frac{1-\left|x_{0}\right|}{2}\left|A\left(x_{0}\right)\right|
$$

such that the second fundamental form $\tilde{A}$ satisfies

$$
|\tilde{A}(0)|=1 \text { and }|\tilde{A}| \leq 4
$$

First, we suppose that (this will be contradicted) both

$$
R_{0}>\frac{\pi}{48}
$$

and

$$
\int_{B_{r_{0}}(0) \cap L}|A|^{n}<\varepsilon_{0} .
$$

We have a Hamiltonian stationary Lagrangian submanifold on a ball of radius $\frac{\pi}{48}$ with $|\tilde{A}| \leq 4$. It follows that there is an interior ball of radius $r_{1}$ (4) (from Lemma 3.2) such that $L$ is represented as the gradient graph of a function with

$$
\begin{aligned}
\left\|D^{2} u\right\|_{C^{\alpha}\left(B_{r_{1}}\right)} & \leq \tan \frac{\pi}{3} \\
\left\|D^{4} u\right\|_{C^{\alpha}\left(B_{r_{1}}\right)} & \leq C_{4}(4)
\end{aligned}
$$

In particular, we have

$$
\|\nabla \tilde{A}\|_{C^{0}\left(B_{r_{1}}\right)} \leq C_{5}
$$

Therefore, as $|\tilde{A}(0)|=1$ we have

$$
|\tilde{A}|>\frac{1}{2} \quad \text { on } B_{\frac{1}{2 C_{5}}}(0)
$$

Then integration leads to

$$
\int_{B_{\frac{1}{2 C_{5}}}(0)}|\tilde{A}|^{n} \geq\left(\frac{1}{2 C_{5}}\right)^{n}\left(\frac{1}{2}\right)^{n}=\frac{1}{4^{n} C_{5}^{n}}
$$

Take

$$
\varepsilon_{0}=\frac{1}{4^{n} C_{5}^{n}}
$$

So we have

$$
\int_{B_{\frac{1}{2 C_{5}}}(0)}|\tilde{A}|^{n} \geq \varepsilon_{0}
$$

which contradicts, by the scaling invariance of the total curvature, the assumption

$$
\int_{B_{1}(0)}|A|^{n}<\varepsilon_{0}
$$

So we reject our assumption that $R_{0}>\frac{\pi}{48}$ and conclude that

$$
R_{0} \leq \frac{\pi}{48}
$$

In this case, we have

$$
\frac{1-\left|x_{0}\right|}{2}\left|A\left(x_{0}\right)\right| \leq \frac{\pi}{48}
$$

which in turn implies

$$
(1-|x|)^{2}|A(x)|^{2} \leq\left(1-\left|x_{0}\right|\right)^{2}\left|A\left(x_{0}\right)\right|^{2} \leq\left(\frac{\pi}{24}\right)^{2}
$$

It follows that, for $|x| \leq r$ we have

$$
|A(x)|^{2} \leq \frac{1}{(1-r)^{2}}\left(\frac{\pi}{24}\right)^{2}
$$

Now suppose $r_{0}<1$. Rescaling the manifold by a factor of $\frac{1}{r_{0}}$ the first condition still holds, and we obtain

$$
r_{0}^{2}|A(x)|^{2}=\left|\tilde{A}\left(\frac{x}{r_{0}}\right)\right|^{2} \leq \frac{1}{\left(1-\frac{r}{r_{0}}\right)^{2}}\left(\frac{\pi}{24}\right)^{2}
$$

That is

$$
|A(x)|^{2} \leq \frac{1}{\left(r_{0}-r\right)^{2}}\left(\frac{\pi}{24}\right)^{2}
$$

which is the conclusion.

## 4. Extension of Hamiltonian stationary Lagrangians across a small set

4.1. Extending Hamiltonian stationary sets under volume constrains. The following extendibility result will be used in the proof of Theorem 1.1 to conclude the limiting varifold of a sequence of smooth Hamiltonians stationary Lagrangian immersions is Hamiltonian stationary including singular points; there, in fact we will only need the special case that the singular set is of zero dimension.

Theorem 4.1. Let $N$ be a compact set in a domain $\Omega \subset \mathbb{R}^{2 n}$ with finite $k$-dimensional Hausdorff measure for $k \leq n-2$ which satisfies the local $k$-noncollapsing property

$$
\begin{equation*}
\inf _{x \in N}\left|N \cap B_{\varepsilon}(x)\right| \geq C_{3} \varepsilon^{k} \tag{4.1}
\end{equation*}
$$

for all $\varepsilon \in(0, \delta)$ for some $\delta$ and a constant $C_{3}>0$ independent of $\varepsilon$. Let $L$ defined by

$$
\iota: M \rightarrow \Omega \backslash N
$$

be a proper Hamiltonian stationary Lagrangian immersion of a connected manifold $M$ in $\Omega \backslash N$ satisfying
(i) $\int_{L}|\bar{H}|^{n} d \mu_{L}<C_{1}$, where $\bar{H}$ is the weak mean curvature vector of $\iota$ as in (2.1);
(ii) There exists a decreasing sequence $\varepsilon_{i} \rightarrow 0$ such that

$$
\int_{B_{\varepsilon_{i}}(y)} d \mu_{L}<C_{2} \varepsilon_{i}^{k+\frac{n}{n-1}}
$$

for all $y \in N$ with $C_{2}$ independent of $y$.
Then $\bar{L}$ is Hamiltonian stationary in $\Omega$ : the closure $\bar{L}$ of $L$ admits a generalized mean curvature $\mathcal{H}$ in $\Omega$ such that for any $f \in C_{0}^{\infty}(\Omega)$ it holds

$$
\int_{\bar{L}}\langle J \nabla f, \mathcal{H}\rangle=0 .
$$

Proof. Define the $\varepsilon$-neighborhood of the compact set $N$ by

$$
U_{\varepsilon}=\left\{x \in \mathbb{R}^{2 n}: \min _{y \in N}|x-y|<\varepsilon\right\} .
$$

Since $N$ is compact, we may assume $U_{\varepsilon}$ is contained in the open domain $\Omega$ by choosing $\varepsilon$ small. For simplicity of notations, we will assume (ii) holds for $3 \varepsilon_{i}$ 's.

Step 1. Volume estimate of $M \cap U_{\varepsilon_{j}}$.
For any fixed large $j$, let $\left\{B_{\varepsilon_{j}}\left(x_{1}\right), \ldots, B_{\varepsilon_{j}}\left(x_{\ell\left(\varepsilon_{j}\right)}\right)\right\}$ be the maximal family of disjoint balls in $\Omega \subset \mathbb{R}^{2 n}$ centered at $x_{i} \in N$ of radius $\varepsilon_{j}$. Compactness of $N$ ensures the number $\ell\left(\varepsilon_{j}\right)$ well defined. By maximality,

$$
N \subseteq \bigcup_{i=1}^{\ell\left(\varepsilon_{j}\right)} B_{2 \varepsilon_{j}}\left(x_{i}\right)
$$

To estimate $\ell\left(\varepsilon_{j}\right)$, summing the $k$-dimensional volumes over the disjoint balls and using the local $k$-noncollapsing assumption (4.1), we have

$$
\ell\left(\varepsilon_{j}\right) C_{3} \varepsilon_{j}^{k} \leq \sum_{i=1}^{\ell\left(\varepsilon_{j}\right)}\left|N \cap B_{\varepsilon_{j}}\left(x_{i}\right)\right| \leq|N|
$$

Therefore

$$
\ell\left(\varepsilon_{j}\right) \leq \frac{|N|}{C_{3}} \varepsilon_{j}^{-k}
$$

Next, we claim

$$
U_{\varepsilon_{j}} \subset \bigcup_{i=1}^{\ell\left(\varepsilon_{j}\right)} B_{3 \varepsilon_{j}}\left(x_{i}\right)
$$

This can be seen from that for any point $p \in U_{\varepsilon_{j}}$ there is a $q \in N$ with $|p-q| \leq \varepsilon_{j}$ and $q \in B_{2 \varepsilon_{j}}\left(x_{i}\right)$ for some $i$, and it follows $p \in B_{\varepsilon_{j}}\left(x_{i}\right)$. Now by the assumption (ii), we have

$$
\begin{align*}
\int_{B_{\varepsilon_{j}}(y)} \mathrm{d} \mu_{L} & \leq \sum_{i=1}^{\ell\left(\varepsilon_{j}\right)} \int_{B_{\varepsilon_{3 j}}(y)} \mathrm{d} \mu_{L}  \tag{4.2}\\
& \leq \ell\left(\varepsilon_{j}\right) C_{2}\left(3 \varepsilon_{j}\right)^{k+\frac{n}{n-1}} \\
& \leq \frac{|N|}{C_{3}} C_{2} 3^{k+\frac{n}{n-1}} \varepsilon_{j}^{\frac{n}{n-1}} \\
& =C_{4}(N) \varepsilon_{j}^{\frac{n}{n-1}} .
\end{align*}
$$

Step 2. Existence of the generalized mean curvature $\mathcal{H}$ of $\bar{L}$ in $\Omega$.
Let $X$ be a $C^{1}$ vector field on $\Omega$ with compact support. Our goal is to verify [Sim83, Definition 16.5]

$$
\begin{equation*}
\int_{\bar{L}} \operatorname{div}_{\bar{L}} X=-\int_{\bar{L}}\langle\mathcal{H}, X\rangle . \tag{4.3}
\end{equation*}
$$

Let $\phi_{\varepsilon_{j}}$ be a cut-off function satisfying $\phi_{\varepsilon_{j}}=0$ on $U_{\varepsilon_{j} / 2}, \phi_{\varepsilon_{j}}=1$ on $\Omega \backslash U_{\varepsilon_{j}}, 0 \leq \phi_{\varepsilon_{j}} \leq 1$ and $\left|D \phi_{\varepsilon_{j}}\right|<C / \varepsilon_{j}$. The existence of such $\phi_{\varepsilon_{j}}$ is given, for example, in Lemma 2.2 in [HP70] and also due to Bochner [Bo56]. Then $\phi_{\varepsilon_{j}} X$ is a $C^{1}$ vector field which vanishes on $U_{\varepsilon_{j} / 2}$. By the standard first variation formula, we have

$$
\begin{gather*}
\int_{M}\left\langle H_{\iota}(x), \phi_{\varepsilon_{j}}(\iota(x)) X(\iota(x))\right\rangle \mathrm{d} \mu_{M}=-\int_{M} \operatorname{div}_{M} \phi_{\varepsilon_{j}}(\iota(x)) X(\iota(x)) \mathrm{d} \mu_{M}  \tag{4.4}\\
=-\int_{M}\left(\left\langle\nabla \phi_{\varepsilon_{j}}(\iota(x)), X(\iota(x))\right\rangle-\phi_{\varepsilon_{j}}(\iota(x)) \operatorname{div}_{M} X(\iota(x))\right) \mathrm{d} \mu_{M}
\end{gather*}
$$

From the volume estimate (4.2),

$$
\left|\int_{M}\left(\left\langle\nabla \phi_{\varepsilon_{j}}(\iota(x)), X(\iota(x))\right\rangle\right) \mu_{M}\right|=\left|\int_{L}\left\langle\nabla \phi_{\varepsilon_{j}}, X\right\rangle \mathrm{d} \mu_{L}\right| \leq C(X) \varepsilon_{j}^{-1} \int_{U_{\varepsilon_{j}} \backslash U_{\varepsilon_{j} / 2}} \mathrm{~d} \mu_{L} \rightarrow 0
$$

Now letting $\varepsilon_{j} \rightarrow 0$ in (4.4)

$$
\begin{equation*}
\int_{M}\left\langle H_{\iota}(x), X(\iota(x))\right\rangle \mathrm{d} \mu_{M}=-\int_{M} \operatorname{div}_{L} X(\iota(x)) \mathrm{d} \mu_{M} \tag{4.5}
\end{equation*}
$$

Since $L$ has no manifold boundary points in $\Omega$, and $N$ is dimension $k \leq n-2$, we have $(\bar{L} \backslash L) \cap N \subseteq N$ has zero $n$-dimensional measure. So $\bar{L}=L \cup(\bar{L} \backslash L)$ is a rectifiable $n$-varifold. The divergence operator $\operatorname{div}_{\bar{L}}$ is defined as $\operatorname{div}_{L}$, by noting that $\bar{L} \backslash L$ has zero measure (cf. [Sim83, 16.2]). Then by (4.5) and (2.2)

$$
\begin{align*}
\int_{\bar{L}} \operatorname{div}_{\bar{L}} X \mathrm{~d} \mu_{\bar{L}} & =\int_{L} \operatorname{div} X \mathrm{~d} \mu_{L} \\
& =\int_{M} \operatorname{div}_{L} X(\iota(x)) \mathrm{d} \mu_{M} \\
& =-\int_{M}\left\langle H_{\iota}(x), X(\iota(x))\right\rangle \mathrm{d} \mu_{M}  \tag{4.6}\\
& =-\int_{L}\langle\bar{H}, X\rangle \mathrm{d} \mu_{L} \\
& =-\int_{\bar{L}}\langle\mathcal{H}, X\rangle \mathrm{d} \mu_{\bar{L}}
\end{align*}
$$

where $\mathcal{H}$ equals $\bar{H}$ on $L$ and zero on $\bar{L} \backslash L$, so it is locally $\mu^{n}$-integrable on $\bar{L}$, in turn $\mathcal{H}$ is the generalized mean curvature of $\bar{L}$ in $\Omega$ since $X$ is arbitrary.

Step 3. $\bar{L}$ is Hamiltonian stationary in $\Omega$.
Our goal is to show that

$$
\begin{equation*}
\int_{\bar{L}}\langle J \nabla f, \mathcal{H}\rangle \mathrm{d} \mu_{L}=0 \tag{4.7}
\end{equation*}
$$

for all $f \in C_{0}^{\infty}(\Omega)$. For any smooth function $f$ with compact support in $\Omega, J D\left(\phi_{\varepsilon_{j}} f\right)$ is a Hamiltonian vector field on $\Omega$ with compact support, in particular it vanishes on $U_{\varepsilon_{j} / 2}$ containing $N$. Applying (4.6) with $X=J \nabla f$, we see

$$
\begin{align*}
\int_{\bar{L}}\langle J \nabla f, \mathcal{H}\rangle \mathrm{d} \mu_{L} & =\int_{L}\langle J \nabla f, \bar{H}\rangle \mathrm{d} \mu_{L} \\
& =\int_{L \cap U_{\varepsilon_{j}}}\langle J \nabla f, \bar{H}\rangle \mathrm{d} \mu_{L}+\int_{L \backslash U_{\varepsilon_{j}}}\langle J \nabla f, \bar{H}\rangle \mathrm{d} \mu_{L} . \tag{4.8}
\end{align*}
$$

Since $L$ is Hamiltonian stationary in $\Omega \backslash N$, we have

$$
\left|\int_{L \backslash U_{\varepsilon_{j}}}\langle J \nabla f, \bar{H}\rangle \mathrm{d} \mu_{L}\right|=\left|\int_{L}\left\langle J \nabla\left(\phi_{\varepsilon_{j}} f\right), H\right\rangle \mathrm{d} \mu_{L}-\int_{L \cap U_{\varepsilon_{j}}}\left\langle J \nabla\left(\phi_{\varepsilon_{j}} f\right), H\right\rangle \mathrm{d} \mu_{L}\right|
$$

$$
=\left|0-\int_{L \cap\left(U_{\varepsilon_{j}} \backslash U_{\varepsilon_{j} / 2}\right)}\left(\left\langle\phi_{\varepsilon_{j}} J \nabla f, H\right\rangle+\left\langle f J \nabla \phi_{\varepsilon_{j}}, H\right\rangle\right) \mathrm{d} \mu_{L}\right|
$$

$$
\leq C(f)\left(1+\varepsilon_{j}^{-1}\right) \int_{L \cap\left(U_{\varepsilon_{j}} \backslash U_{\varepsilon_{j}} / 2\right.}|\bar{H}| \mathrm{d} \mu_{L}
$$

$$
\begin{equation*}
\leq C(f)\left(1+\varepsilon_{j}^{-1}\right)\left(\int_{L \cap\left(U_{\varepsilon_{j}} \backslash U_{\varepsilon_{j} / 2}\right)}|\bar{H}|^{n} \mathrm{~d} \mu_{L}\right)^{\frac{1}{n}}\left(\int_{U_{\varepsilon_{j}} \backslash U_{\varepsilon_{j} / 2}} \mathrm{~d} \mu_{L}\right)^{\frac{n-1}{n}} \tag{4.9}
\end{equation*}
$$

by Hölder's inequality, where $C(f)$ depends on $f$ and $|D f|$ as $\nabla f$ is the tangential projection of $D f$ along $L$ so $|J \nabla f|=|\nabla f| \leq|D f|$. Similarly

$$
\begin{equation*}
\left|\int_{L \cap U_{\varepsilon_{j}}}\langle J \nabla f, \bar{H}\rangle \mathrm{d} \mu_{L}\right| \leq C(f)\left(\int_{L \cap U_{\varepsilon_{j}}}|\bar{H}|^{n} \mathrm{~d} \mu_{L}\right)^{\frac{1}{n}}\left(\int_{U_{\varepsilon_{j}}} \mathrm{~d} \mu_{L}\right)^{\frac{n-1}{n}} \tag{4.10}
\end{equation*}
$$

It then follows from the assumption (i), and the volume estimate (4.2) that both terms (4.9) and (4.10) vanish as $\varepsilon_{j} \rightarrow 0$. Combining with (4.8) we conclude (4.7).
4.2. Volume estimate via the monotonicity formula. The local $k$-noncollapsing property is automatically satisfied if $N$ is a compact manifold of dimension no larger than $n-2$.

Corollary 4.2. Let $N$ be a compact submanifold in a domain $\Omega \subset \mathbb{R}^{2 n}$ of dimension $k \leq n-2$. Let $L$ be Hamiltonian stationary immersion in $\Omega \backslash N$ satisfying
(i) $\int_{L}|\bar{H}|^{n}<C_{1}$, where $\bar{H}$ is the weak mean curvature vector of $L$;
(ii) There exists a decreasing sequence $\varepsilon_{i} \rightarrow 0$ such that

$$
\int_{B_{\varepsilon_{i}}(y)} d \mu_{L}<C_{2} \varepsilon_{i}^{k+\frac{n}{n-1}}
$$

for all $y \in N$ with $C_{2}$ independent of $y$.
Then $L$ is Hamiltonian stationary in $\Omega$. The closure $\bar{L}$ is Hamiltonian stationary in $\Omega$ in the sense: there exists a generalized mean curvature $\mathcal{H}$ of $\bar{L}$ in $\Omega$ such that

$$
\int_{\bar{L}}\langle J \nabla f, \mathcal{H}\rangle d \mu_{L}=0
$$

for any $f \in C_{0}^{\infty}(\Omega)$.
The following volume upper estimate is a direct consequence of the standard monotonicity formula for volumes. In particular, it implies that the assumption (ii) in Theorem 4.1 holds when $N$ is finite set of points $(k=0)$ and $H \in L^{n}$.
Proposition 4.3. Let $L$ be an integral n-rectifiable varifold, with generalized mean curvature $\mathcal{H}$ in $L^{n}(L, \mu)$ where $\mu$ is the Radon measure associated with $L$. Then $\mu\left(B_{r}(x)\right) \leq$ $C(|\ln \rho|+1)^{n} \rho^{n}$. In particular when $n \geq 2$, for any $0 \leq k \leq n-2$ it holds $\mu_{L}\left(B_{\rho}(x)\right) \leq$ $C \rho^{k+\frac{n}{n-1}}$ for small $\rho$.

Proof. Recall the monotonicity formula [Sim83, 17.3 p. 84]

$$
\begin{align*}
\frac{d}{d \rho}\left(\rho^{-n} \mu\left(B_{\rho}(x)\right)\right) & =\frac{d}{d \rho} \int_{B_{\rho}(x)} \frac{\left|D^{\perp} r\right|^{2}}{r^{n}} \mathrm{~d} \mu+\rho^{-1-n} \int_{B_{\rho}(x)}\langle y-x, \mathcal{H}\rangle \mathrm{d} \mu  \tag{4.11}\\
& \geq \rho^{-1-n} \int_{B_{\rho}(x)}\langle y-x, \mathcal{H}\rangle \mathrm{d} \mu \\
& \geq-\rho^{-1-n} \int_{B_{\rho}(x)} \rho|\mathcal{H}| \mathrm{d} \mu \\
& \geq-\rho^{-n}\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n} \mu\left(B_{r}(x)\right)^{\frac{n-1}{n}}
\end{align*}
$$

Now let

$$
w(\rho)=\frac{\mu\left(B_{\rho}(x)\right)^{1 / n}}{\rho}
$$

in which case we have

$$
\frac{d}{d \rho}[w(\rho)]^{n} \geq-\frac{1}{\rho}\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n} w^{n-1}
$$

That is

$$
\begin{aligned}
n w^{n-1} \frac{d}{d \rho} w & \geq-\frac{1}{\rho}\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n} w^{n-1} \\
\frac{d}{d \rho} w & \geq-\frac{1}{\rho n}\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n}
\end{aligned}
$$

Integrating over $\left(\rho, \rho_{0}\right)$,

$$
w\left(\rho_{0}\right)-w(\rho) \geq\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n} \frac{1}{n}\left[\ln \rho-\ln \rho_{0}\right]
$$

that is

$$
w(\rho) \leq w\left(\rho_{0}\right)+\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n} \frac{1}{n}\left(-\ln \rho+\ln \rho_{0}\right)
$$

or

$$
\frac{\mu\left(B_{\rho}(x)\right)}{\rho^{n}} \leq\left\{\mu\left(B_{\rho_{0}}(x)\right)+\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n} \frac{1}{n}\left(|\ln \rho|+\ln \rho_{0}\right)\right\}^{n}
$$

that is

$$
\mu\left(B_{\rho}(x)\right) \leq \rho^{n}\left\{\mu\left(B_{\rho_{0}}(x)\right)+\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n} \frac{1}{n}\left(|\ln \rho|+\ln \rho_{0}\right)\right\}^{n}
$$

In particular we have

$$
\rho^{-k-\frac{n}{n-1}} \mu\left(B_{\rho}(x)\right) \leq \rho^{\frac{n(n-2)}{n-1}-k}\left\{\mu\left(B_{\rho_{0}}(x)\right)+\left(\int_{B_{\rho}(x)}|\mathcal{H}|^{n} \mathrm{~d} \mu\right)^{1 / n} \frac{1}{n}\left(|\ln \rho|+\ln \rho_{0}\right)\right\}^{n}
$$

and the term on the right hand side tends to zero as $\rho \rightarrow 0$ when $n>2$, as $k \leq n-2$ by assumption; however, when $n=2$, this term becomes unbounded.

For $n=2, k$ must be 0 , and the desired result follows from [KS04, (A.6)] (cf. [Sim93]): for any $0<\rho<\rho_{0}$,

$$
\rho^{-2} \mu\left(B_{\rho}\left(x_{i}\right)\right) \leq C \mu\left(B_{\rho_{0}}(x)\right)+C \int_{B_{\rho_{0}}(x)}|\mathcal{H}|^{2} \mathrm{~d} \mu<\infty
$$

## 5. Sequential Convergence of Hamiltonian stationary Lagrangians

Convergence of a sequence of embedded manifolds in $C^{k}$ topology has been used in [CS85] and then in [And86] and recently in [Sh17] via local graphical representations of the manifolds. Along the same line, we write down a definition of $C^{k}$ convergence of manifolds to a varifold that will be sufficient for our purposes.
Definition 5.1. Let $\left\{S_{j}\right\}$ be a sequence of finite sets of embedded $n$-dimensional submanifolds $M_{j, i}$ in an open subset $U$ of $\mathbb{R}^{n+l}$, where $S_{j}=\left\{M_{j, 1}, \ldots, M_{j, m}\right\}$ for some positive integer $m$. Suppose that for each $i \in\{1, \ldots, m\}$ there is a point $x(i)$ and an $n$-plane $P(i)$ containing $x(i)$ such that $\left\{M_{j, i}\right\}_{j=1, \ldots \infty}$ is a sequence of graphs over a connected domain $\Omega(i)$ in $P(i)$. If for each $j$ there is a $\sigma_{j}$ in the permutation group of $\{1, \ldots, m\}$, such that for each $i$, the graphs $\left\{M_{j, \sigma_{j}(i)}\right\}$ converge uniformly in the $C^{k}$ topology to a graph $M_{\infty, i}$ over $\Omega(i)$, we say that $\left\{S_{j}\right\}$ converges uniformly in $C^{k}$ topology to the integral varifold on $U$

$$
V=\sum_{i \in\{1, \ldots, m\}} M_{\infty, i}
$$

Definition 5.2. Given an open set $U$ in $\mathbb{R}^{n+l}$, we say that a sequence of immersed submanifolds $L_{j}$ in $U$ converges uniformly to a varifold $V$ in the $C^{k}$ topology in $U$, if for every point $x \in \operatorname{supp}(V)$ there is a neighborhood $U_{x}$ in $U$ such that

$$
S_{j}=\left\{\text { embedded connected components of } L_{j} \cap U_{x}\right\}
$$

converges uniformly in $C^{k}$ topology to $V$ restricted to $U_{x}$.
Proof of Theorem 1.1. First, we claim that the manifolds $L_{i}$ remain in a bounded region in $\mathbb{C}^{n}$. Fixing an $L_{i}$, by the Wiener Covering Lemma [KP08, Lemma 4.1.1], we may choose a finite collection of balls $B_{1}\left(x_{k}\right)$, for $x_{k} \in L_{i}$ that cover $L_{i}$ such that $B_{1 / 3}\left(x_{k}\right)$ are disjoint. Now for each $x_{k}$ either

$$
\begin{equation*}
\int_{B_{1 / 3}\left(x_{k}\right)}|A|^{n}<\varepsilon_{0} \tag{5.1}
\end{equation*}
$$

or

$$
\begin{equation*}
\int_{B_{1 / 3}\left(x_{k}\right)}|A|^{n} \geq \varepsilon_{0} . \tag{5.2}
\end{equation*}
$$

In the first case, by Lemma 3.3, we have a uniform bound on the curvature on $B_{1 / 6}\left(x_{k}\right)$ in particular

$$
|A| \leq \sqrt{\frac{3 \pi}{2}}
$$

Lemma 3.1 then guarantees there is a fixed minimum radius

$$
r_{1}=\frac{\sqrt{\pi}}{12} \sqrt{\frac{2}{3}} \cos \left(\frac{\pi}{12}\right)
$$

such that a connected component of $L_{i} \cap B_{r_{1}}\left(x_{k}\right)$ is graphical is over the tangent plane at $x_{k}$, which implies

$$
\operatorname{Vol}\left(B_{1 / 6}\left(x_{k}\right) \cap L_{i}\right) \geq \omega_{n} r_{1}^{n}
$$

It follows that the number of $\left\{x_{k}\right\}$ for which (5.1) hold is bounded by

$$
\begin{equation*}
\#\left\{x_{k}: \int_{B_{1 / 3}\left(x_{k}\right)}|A|^{n}<\varepsilon_{0}\right\} \leq \frac{C_{1}}{\omega_{n} r_{1}^{n}} \tag{5.3}
\end{equation*}
$$

On the other hand, it is clear that

$$
\#\left\{x_{k}: \int_{B_{1 / 3}\left(x_{k}\right)}|A|^{n} \geq \varepsilon_{0} .\right\} \leq \frac{C_{2}}{\varepsilon_{0}}
$$

It follows that there are at most

$$
R_{0}=\frac{C_{1}}{\omega_{n} r_{1}^{n}}+\frac{C_{2}}{\varepsilon_{0}}
$$

balls of radius 1 in this cover. Immediately we conclude (recall $L_{i}$ are connected):

$$
L_{i} \subset B_{R_{0}}(0)=\left\{x \in \mathbb{R}^{2 n}:|x| \leq R_{0}\right\}
$$

Next, define $\mathcal{C}_{k}=\left\{B_{r_{k}}\left(y_{k, j}\right)\right\}$ to be a finite cover of $B_{R_{0}}(0)$ by balls $B_{r_{k}}\left(y_{k, j}\right)$ in $\mathbb{R}^{2 n}$, where $r_{k}=2^{-k} \varepsilon_{0}$ and $\varepsilon_{0}$ is the constant in Proposition 3.3, with the property that each point in $B_{R_{0}}(0)$ is covered by at most $b$ balls in $\mathcal{C}_{k}$ and $\left\{B_{r_{k} / 2}\left(y_{k, j}\right)\right\}$ still covers $B_{R_{0}}(0)$. This can be done with $b$ independent of $r_{k}, y_{k, j}$, by Besicovitch's covering theorem (cf. [KP08, Theorem 4.2.1]). Now we observe

$$
\sum_{j} \int_{L_{i} \cap B_{r_{k}}\left(y_{k, j}\right)}\left|A_{i}\right|^{n} \mathrm{~d} \mu_{i} \leq b \int_{L_{i}}\left|A_{i}\right|^{n} \mathrm{~d} \mu_{i} \leq b C_{2}
$$

where $L_{i} \cap B_{r_{k}}\left(y_{k, j}\right) \neq \emptyset$. It then follows that for each $i$ and each $k$ there are $J_{k}^{i}$ balls of radius $r_{k}$ such that the integral of $\left|A_{i}\right|^{n}$ on each of these balls is not smaller than $\varepsilon_{0}$, for an integer $J_{k}^{i}$ with $J_{k}^{i} \leq b C / \varepsilon_{0}$. By reindexing, we may denote the centers of these balls by $\mathcal{B}_{k}(i)=\left\{y_{k, 1}(i), \cdots, y_{k, J_{k}^{i}}^{i}(i)\right\}$. Letting

$$
J_{k}=\lim \sup _{i \rightarrow \infty} J_{k}^{i} \leq \frac{b C}{\varepsilon_{0}}
$$

we may choose a subsequence $\left\{L_{i}\right\}$ (here and in the sequel, we will use the same indices for subsequences for simplicity) such that $J_{k}^{i}=J_{k}$ for all $i$. We may then assume, by switching to a subsequence if necessary, the sequence $y_{k, j}(i) \rightarrow x_{k, j}$ as $i \rightarrow \infty$ for each $1 \leq j \leq J_{k}^{i} \leq b C_{2} / \varepsilon_{0}$. Next, letting

$$
J=\lim \sup _{k \rightarrow \infty} J_{k} \leq \frac{b C_{2}}{\varepsilon_{0}}
$$

we may select a subsequence $\mathcal{K} \subset \mathbb{N}$ such that $|\mathcal{K}|=\infty$ and $J_{k}^{i}=J$ for all $i$ and $k \in \mathcal{K}$. By choosing yet another subsequence we further assume that $x_{k, j} \rightarrow x_{j}$ for each $j=1, \ldots, J$ as $k \in \mathcal{K} \rightarrow \infty$, and let $S=\left\{x_{1}, \ldots, x_{J}\right\}$, and $S$ may be empty.

We assume there is no subsequence of $\left\{L_{i}\right\}$ that converges to a single point, otherwise we are done. We construct a sequence of nested open sets

$$
U_{0} \subset U_{1} \ldots \subset B_{R_{0}} \backslash S
$$

such that

$$
\bigcup_{l} U_{l}=B_{R_{0}} \backslash S
$$

and show that there is a subsequence $\left\{L_{i}\right\}$ that converges in $C^{m}$ in the sense of Definition 5.2, uniformly on each $U_{l}$ to a Hamiltonian stationary varifold.

Let $\tau_{0}>0$ be smaller than the minimum distance between points in $S$ and the minimum distance from points in $S$ to $\partial B_{R_{0}}$ and let $\tau_{l+1}=3^{-l} \tau_{1}$. For each $l$, choose $k=k(l) \in \mathcal{K}$ so that

$$
\begin{gathered}
\left\|x_{k, j}-x_{j}\right\|<\tau_{l} / 4 \text { for all } j \in\{1, \ldots, J\} \\
r_{k}<\tau_{l} / 8
\end{gathered}
$$

in particular the balls $B_{\tau_{l} / 2}\left(x_{k, j}\right)$ are disjoint and contained in $B_{\tau_{l}}\left(x_{j}\right)$ respectively. Let

$$
U_{l}=B_{R_{0}}(0) \backslash \bigcup_{x_{j} \in S} \overline{B_{\tau_{l}}\left(x_{j}\right)}
$$

For a fixed $l$, we may choose $i \geq i(l)$ large enough so that

$$
\left\|y_{k, j}-x_{k, j}\right\|<\frac{\tau_{l}}{4}
$$

It then follows that

$$
U_{l} \subset B_{R_{0}}(0) \backslash \bigcup_{y_{k, j} \in \mathcal{B}_{k}(i)} B_{r_{k}}\left(y_{k, j}\right)
$$

In particular, for each $i$ the set $U_{l}$ is covered by the balls $\mathcal{C}_{k} \backslash \mathcal{B}_{k}$ and $d\left(\overline{U_{l}}, S\right) \geq 3 \tau_{l} / 8$. Then, for a ball $B_{r_{k}}\left(y_{k, j}\right)$ with $L_{i} \cap U_{l} \cap B_{r_{k}}\left(y_{k, j}\right) \neq \emptyset$, we conclude that $y_{k, j} \notin \mathcal{B}_{k}(i)$ and thus

$$
\left\|A_{i}\right\|_{L^{n}\left(L_{i} \cap B_{r_{k}}\left(y_{k, j}\right)\right)}<\varepsilon_{0}
$$

and we have a curvature bound

$$
\begin{equation*}
\|A\|(y) \leq \frac{3 \times 2^{k}}{\varepsilon_{0}} \frac{\pi}{24} \tag{5.4}
\end{equation*}
$$

for points $y \in L_{i} \cap B_{2 r_{k} / 3}\left(y_{k, j}\right)$. This must hold uniformly on each point of $U_{l}$. Now consider the components of $L_{i} \cap U_{l} \cap B_{r_{k}}\left(y_{k, j}\right)$ that intersect $B_{r_{k} / 2}\left(y_{k, j}\right)$. There are a finite number of these, by the same reasoning leading to (5.3). Applying Lemma 3.1, we see that for any point on one of these components, the manifold stays graphical over a ball in the tangent plane of radius

$$
\frac{2 \varepsilon_{0}}{3 \times 2^{k}} \cos \frac{\pi}{12}>\frac{r_{k}}{2}
$$

with Lagrangian potential $u$ satisfying

$$
\begin{equation*}
\left|D^{2} u\right| \leq \tan \frac{\pi}{12} \quad \text { on } B_{r_{k+1}}\left(y_{k, j}\right) \tag{5.5}
\end{equation*}
$$

Every embedded connected component of $L_{i} \cap B_{r_{k} / 2}\left(y_{k, j}\right)$ is contained in an embedded connected component of $L_{i} \cap B_{r_{k}}\left(y_{k, j}\right)$. We may choose a subsequence of $\left\{L_{i}\right\}$ so that for each $j$ the number $m\left(y_{k, j}\right)$ of components of $L_{i} \cap B_{r_{k}}\left(y_{k, j}\right)$ that intersect $B_{r_{k+1}}\left(y_{k, j}\right)$ with $y_{k, j} \notin \mathcal{B}_{k}(i)$, is independent of $i$, again by the same reasoning leading to (5.3). For such chosen $L_{i}$, each embedded connected component of $L_{i} \cap B_{r_{k+1}}\left(y_{k, j}\right)$ is graphical over an $n$-plane in the Lagrangian Grassman, so using (5.5) we may choose further subsequence such that each sequence of components remains graphical over a fixed Lagrangian $n$-plane, the bound (5.5) together with Proposition 3.2 gives uniform $C^{m}$ bounds for each graphing function for each positive integer $m$; by Arzelà-Ascoli theorem, the graphs converge uniformly to a limit. We therefore conclude that $\left\{L_{i} \cap U_{l}\right\}$ converges uniformly in $C^{m}$ to a varifold (or vacates $U_{l}$ completely) in the sense of Definition 5.2, and the limit is locally the sum of finitely many immersed submanifolds, possibly with multiplicity. Because every compact set $K \subset B_{R_{0}}(0) \backslash S$ must eventually be contained in some $U_{l}$ we see that $\left\{L_{i}\right\}$ converges uniformly on $K$. The $C^{m}$ convergence also implies that each of these limiting immersed submanifolds satisfies the Hamiltonian stationary equation (1.1), since by Proposition 2.4 each graph satisfies the (1.1). Now, take a diagonal sequence $\left\{L_{i}\right\}$ to get a sequence which converges on each open set $U_{l}$ in the $C^{m}$ topology to a varifold, or vacates every $U_{l}$. By the definition of this limit, the $n$-varifolds must be nested. In particular, the limit will be nonempty unless a subsequence satisfies (as $L_{i}$ is connected) $L_{i} \subset B_{\tau_{l}}\left(x_{j}\right)$ for arbitrary small $\tau_{l}$ and some point $x_{j} \in S$. We are assuming that $\left\{L_{i}\right\}$ does not converge to a point, so we conclude that the limit is a nonempty varifold on $B_{R_{0}}(0) \backslash S$, and we call its support $L$.

From the construction, the limit $L$ is covered by a countable open cover $\left\{U_{l}\right\}$ and recall $U_{l} \subset B_{R_{0}} \backslash \cup_{x_{j} \in S} \overline{B_{\tau_{l}}\left(x_{j}\right)}$ inside which $L$ is the limit of a fixed number of smooth graphs, say $M_{1}\left(U_{l}\right), \ldots, M_{m(l)}\left(U_{l}\right)$. Let

$$
\iota_{l}: \coprod_{i=1}^{m(l)} M_{i}\left(U_{l}\right) \rightarrow \mathbb{B}^{2 n}
$$

be a map from the disjoint union $M\left(U_{l}\right)$ that is defined by taking the inclusion on each $M_{i}\left(U_{l}\right)$. Now $\iota_{l}$ is a proper immersion of the manifold $M(l)$ (disjoint if $\left.m(l)>1\right)$ which is Hamiltonian stationary Lagrangian in $B_{R_{0}} \backslash S$, because $\theta_{\iota l}$ is harmonic on each $M_{i}\left(U_{l}\right)$ and with $\bar{H}_{\iota_{l}} \in L^{n}$, from the smooth convergence. By Proposition 4.3 and Theorem 4.1 with $k=0, L \cap U_{l}$ is Hamiltonian stationary in $\mathbb{R}^{2 n}$. For $L$, let $\left\{\varphi_{l}\right\}$ be a partition of unity subordinate to the family $\left\{U_{l}\right\}$. Now for any $f \in C_{0}^{\infty}\left(\mathbb{R}^{2 n}\right)$, we have

$$
\begin{aligned}
\int_{L}\langle J D f, \mathcal{H}\rangle \mathrm{d} \mu & =\int_{L}\left\langle J D\left(\sum_{l} \varphi_{l} f\right), \mathcal{H}\right\rangle \mathrm{d} \mu \\
& =\sum_{l} \int_{L \cap U_{l}}\left\langle J D\left(\varphi_{l} f\right), \bar{H}_{\iota_{l}}\right\rangle \mathrm{d} \mu \\
& =0
\end{aligned}
$$

We point out that the argument above works when $S$ is empty as well, in that case $U_{0}=U_{l}=B_{R_{0}}(0)$.

Finally, we show that $L$ is connected. Suppose that $L \subset U_{1} \cup U_{2}$ for nonempty open disjoint bounded sets $U_{1}$ and $U_{2}$. Because $L \subset B_{R_{0}}(0)$ is closed, $L \cap U_{1}$ and $L \cap U_{2}$ are
compact, so there is a minimum distance

$$
\delta_{1}=\min \left\{d(p, q): p \in L \cap U_{1}, q \in \partial U_{2}\right\}>0 .
$$

Now we may define

$$
U_{1}^{\prime}=\left\{p \in U_{1}: d(p, L)<\frac{\delta_{1}}{4}\right\}
$$

It follows that $U_{1}^{\prime}$ and $U_{2}$ also disconnect $L$. Now take $p \in L \cap U_{1}^{\prime}$ and $q \in L \cap U_{2}$, there must be a path $\gamma_{i}$ in $L_{i}$ from an $\varepsilon$-neighborhood of $p$ to an $\varepsilon$-neighborhood of $q$ where $\varepsilon_{i}<\frac{\delta_{1}}{4}$. For each path $\gamma_{i}$ and each value $\sigma \in\left(\frac{\delta_{1}}{4}, \frac{\delta_{1}}{2}\right)$, the intermediate value theorem insures there will be a point $z(i, \sigma) \in \gamma_{i}$ such that $d\left(z(i, \sigma), L \cap U_{1}^{\prime}\right)=\sigma$. In particular, for each $\sigma \in\left(\frac{\delta_{1}}{4}, \frac{\delta_{1}}{2}\right)$, there is a sequence $z(i, \sigma)$ that converges to some $z(\sigma)$. These points must lie outside both $U_{1}^{\prime}$ and $U_{2}$. By varying $\sigma$ there are clearly infinitely many $z(\sigma)$, we can choose a limit point $z\left(\sigma_{0}\right)$ not in $S$. The point $z\left(\sigma_{0}\right)$ has a positive distance $d_{0}$ to the set $S$ of singular points, so we conclude that for $\varepsilon \ll d_{0}$ the $L_{i}$ converge smoothly near $z\left(\sigma_{0}\right)$ and $z\left(\sigma_{0}\right) \in L$. But

$$
d\left(z\left(\sigma_{0}\right), L \cap U_{1}^{\prime}\right) \geq \frac{\delta_{1}}{4}
$$

and

$$
d\left(z\left(\sigma_{0}\right), L \cap U_{2}\right) \geq \delta_{1}-\frac{\delta_{1}}{2}=\frac{\delta_{1}}{2}
$$

together imply $z\left(\sigma_{0}\right) \notin L$.

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