A BERNSTEIN RESULT AND COUNTEREXAMPLE FOR ENTIRE SOLUTIONS TO DONALDSON’S EQUATION

MICAH WARREN
UNIVERSITY OF OREGON

Abstract. We show that convex entire solutions to Donaldson’s equation are quadratic, using a result of Weiyong He. We also exhibit entire solutions to the Donaldson equation that are not of the form discussed by He. In the process we discover some non-trivial entire solutions to complex Monge-Ampère equations.

1. Introduction

In this note we show the following.

Theorem 1. Suppose that $u$ is a convex solution to the Donaldson equation on $\mathbb{R} \times \mathbb{R}^{n-1} = (t, x_2, ..., x_n)$

\[
\tilde{\sigma}_2(D^2 u) = u_{11}(u_{22} + u_{33} + ... + u_{nn}) - u_{12}^2 - ... - u_{1n}^2 = 1.
\]

Then $u$ is a quadratic function.

Donaldson introduced the operator

\[ Q(D^2 u) = u_{tt} \Delta u - |\nabla u_t|^2 \]

arising in the study of the geometry of the space of volume forms on compact Riemannian manifolds [1]. On Euclidean space, [1] becomes an interesting non-symmetric fully nonlinear equation. Weiyong He has studied aspects of entire solutions on Euclidean space, and was able to show that [2, Theorem 2.1] if $u_{11} = \text{const}$, then the solution can be written in terms of solutions to Laplace equations.

Here we show that any convex solution must also satisfy $u_{11} = \text{const}$. It follows quickly that the solution must be quadratic. We also show that, in the absence of the convexity constraint, solutions exists for which $u_{11} = \text{const}$ fails.

Theorem 2. There exists solutions to the Donaldson equation which are not of the form given by He.

In real dimension 3 we note that solutions of [1] can be extended to solutions of the the complex Monge-Ampère equation on $\mathbb{C}^2$

\[ \det (\partial \bar{\partial} u) = 1 \]

and we can conclude the following.

Corollary 3. There exist a nonflat solution of the complex Monge-Ampère equation [2] on $\mathbb{C}^2$ whose potential depends on only three real variables.

The author’s work is supported in part by the NSF via DMS-1161498.
2. Proof of Theorem 1

Lemma 4. Suppose that $K_h$ is the sublevel set $u \leq h$ of a nonnegative solution to
\[
\tilde{\sigma}^2(D^2u) = u_{11}(u_{22} + u_{33} + \ldots + u_{nn}) - u_{12}^2 - \ldots - u_{1n}^2 = 1.
\]
Then for all ellipsoids $E \subset K_h$ such that if $A : E \to B_1$ is affine diffeomorphism with
\[
A = Mx + \vec{b},
\]
we have
\[
\tilde{\sigma}^2(M^2) \geq \frac{1}{4h^2}.
\]
Proof. Consider the function $v$ on $\mathbb{R}^n$ defined by
\[
v(x) = h|A(x)|^2.
\]
On the boundary of $E$, we have
\[
v(x) = h \geq u.
\]
We have
\[
Dv = 2hM \left( Mx + \vec{b} \right)
\]
\[
D^2v = 2hM^2.
\]
Thus
\[
\tilde{\sigma}^2(D^2v) = 4h^2\tilde{\sigma}^2(M^2).
\]
Now suppose that
\[
\tilde{\sigma}^2(M^2) < \frac{1}{4h^2}.
\]
Then
\[
\tilde{\sigma}^2(D^2v) < 1,
\]
so $v$ is a supersolution to the equation, and must lie strictly above the solution $u$. But $v$ must vanish at $A^{-1}(0)$. Because $u$ is nonnegative, this is a contradiction of the strong maximum principle. $\square$

Proposition 5. Suppose that $u$ is an entire convex solution to
\[
\tilde{\sigma}^2(D^2u) = u_{11}(u_{22} + u_{33} + \ldots + u_{nn}) - u_{12}^2 - \ldots - u_{1n}^2 = 1
\]
Then
\[
\lim_{t \to \infty} u_1(t, 0, \ldots, 0) = \infty.
\]
Proof. Assume not. Instead assume that $u_1 \leq A$. Assume that $u(0) = 0$ and $Du(0) = 0$, adjusting $A$ if necessary. Then
\[
u(t, 0, \ldots, 0) = \int_0^t u_1(s)ds \leq \int_0^t Ads \leq At.
\]
Now consider the convex sublevel set $u \leq h$. This must contain the point
\[
\left( \frac{h}{A}, 0, \ldots, 0 \right).
\]
The level set \( u = h \) intersect the other axes at

\[
(0, a_2(h), 0, \ldots)
\]

\[
(0, 0, a_3(h), \ldots, 0)
\]

e tc.

This level set is convex. It must contain the simplex with the above points as vertices, and this simplex must contain an ellipsoid \( E \) which has an affine transformation to the unit ball of the following form

\[
A = Mx + \vec{b}
\]

\[
M = c_n \begin{pmatrix} \frac{A}{h} & \frac{1}{a_2} & \frac{1}{a_3} & \ldots \end{pmatrix}
\]

Thus

\[
M^2 = c_n^2 \begin{pmatrix} \left( \frac{A}{h} \right)^2 & \left( \frac{1}{a_2} \right)^2 & \left( \frac{1}{a_3} \right)^2 & \ldots \end{pmatrix}
\]

and

\[
\bar{\sigma}_2(M^2) = c_n^2 \left( \frac{A}{h} \right)^2 \left( \frac{1}{a_2} \right)^2 + \ldots + \left( \frac{1}{a_n} \right)^2 \geq \frac{1}{4} \frac{1}{h^2}
\]

with the latter inequality following from the previous lemma.

Thus

\[
\left( \frac{1}{a_2} \right)^2 + \left( \frac{1}{a_3} \right)^2 + \ldots + \left( \frac{1}{a_n} \right)^2 \geq \frac{1}{c_n^2 A^2}
\]

It follows that for some \( i \),

\[
\frac{1}{a_i^2} \geq \frac{1}{4(n-1)c_n^2 A^2}
\]

That is

\[
a_i \leq 2\sqrt{n-1}c_n A.
\]

Now to finish the argument, let

\[
R = 2\sqrt{n-1}c_n A.
\]

On a ball of radius \( R \), there is some bound on the function (not a priori but depending on \( u \)) say \( \bar{U} \). That is

\[
u(x) \leq \bar{U} \text{ on } B_R.
\]

Now by convexity for any large enough \( h \) the level set \( u = h \) is non-empty and convex. Choose \( h > \bar{U} \). According to the above argument, this level set must intersect some axis at a point less than \( R \) from the origin, which is a contradiction.

\[\square\]
Now using this Proposition, we may repeat the argument of He [2, section 3]: Letting $z = u_1(t, x)$ the map

$$
\Phi : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}
$$

$$
\Phi(t, x) = (z, x)
$$

is a diffeomorphism. Thus for $x$ fixed, there exists a unique $t = t(z, x)$ such that $z = u_1(t, x)$. Defining

$$
\theta(z, x) = t(z, x)
$$

the computations in [2, section 3] yield that $\theta$ is a harmonic function. It follows that $\frac{\partial \theta}{\partial z} = 1/u_{11}$ is a positive harmonic function, so must be constant. Now we have

$$
u(t, x) = at^2 + tb(x) + g(x)
$$

which satisfies [2, section 2]

$$
\Delta b = 0
$$

$$
\Delta g = \frac{1}{2a} \left( 1 + |\nabla b|^2 \right).
$$

Letting $t = 0$ we conclude that $g$ is convex. Letting $t \to \pm \infty$ we conclude that $b$ is convex and concave, so must be linear. It follows that $|\nabla b|$ is constant, and

$$
\Delta g - c |x|^2
$$

is a semi-convex harmonic function, which must be a quadratic.

3. Counterexamples

We use the method described in [3] and restrict to $n = 3$. Consider

$$
u(t, x) = r^2 e^t + h(t)
$$

where $r = (x_2^2 + x_3^2)^{1/2}$. At any point we may rotate $\mathbb{R}^2$ so that $x_2 = r$ and get

$$
D^2 u = \begin{pmatrix}
  r^2 e^t + h''(t) & 2re^t & 0 \\
  2re^t & 2e^t & 0 \\
  0 & 0 & 2e^t
\end{pmatrix}.
$$

We compute

$$
\tilde{\sigma}_2 (D^2 u) = 4e^t \left( r^2 e^t + h''(t) \right) - 4r^2 e^{2t} = 4e^t h''(t).
$$

Then

$$
u = r^2 e^t + \frac{1}{4} e^{-t}
$$

is a solution.

Now defining complex variables

$$
z_1 = t + is
$$

$$
z_2 = x + iy
$$

we can consider the function

$$
u = (x^2 + y^2) e^t + \frac{1}{4} e^{-t}.
$$

The function satisfies the equation complex Monge-Ampère equation

$$
(\partial_{z_1} \partial_{\bar{z}_1} u) (\partial_{z_2} \partial_{\bar{z}_2} u) - (\partial_{z_1} \partial_{\bar{z}_2} u) (\partial_{z_2} \partial_{\bar{z}_1} u) = 1.
$$
One can check that the induced Ricci-flat complex metric
\[ g_{i\bar{j}} = \partial_{z_i} \partial_{\bar{z}_j} u \]
on \mathbb{C}^2 is neither complete complete nor flat.

References