COARSE RICCI CURVATURE WITH APPLICATIONS TO MANIFOLD LEARNING

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Abstract. Consider a sample of \( n \) points taken i.i.d from a submanifold of Euclidean space. This defines a metric measure space. We show that there is an explicit set of scales \( t_n \to 0 \) such that a coarse Ricci curvature at scale \( t_n \) on this metric measure space converges almost surely to the coarse Ricci curvature of the underlying manifold.

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1. Introduction

In [BN08], Belkin and Niyogi show that the graph Laplacian of a point cloud of data samples taken from a submanifold in Euclidean space converges to the Laplace-Beltrami operator on the underlying manifold. It is the goal of this paper, together with [AWb] and [AWa], to demonstrate that this process can be continued to approximate Ricci curvature as well. Singer and Wu [SW12] have developed techniques to

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compute vector Laplacians and other geometric quantities from point clouds. Information about the Ricci curvature allows one to compute the Hodge Laplacian on 1-forms [SW12, pg. 1103]. Thus we expect our result to have applications to manifold learning.

In [AWa] we define a family of coarse Ricci curvatures which depend on a scale parameter $t$, and show that when taken on a smooth embedded submanifold on Euclidean space, these recover the Ricci curvature as $t \to 0$. The goal of this paper is to discuss applications to manifold learning and in particular show that there exists an explicit choice of scales $t_n \to 0$ such that the quantities converge almost surely when computed from a set of $n$ points sampled from a uniform probability distribution on the manifold.

1.1. Background and Motivation. The motivation for the paper stems from both the theory of Ricci lower bounds on metric measure spaces and the theory of manifold learning. For background on the definitions of coarse Ricci and their relation to Ricci curvature lower bounds, see [AWb].

1.1.1. The Manifold Learning Problem. Roughly speaking, the manifold learning problem deals with inferring or predicting geometric information from a manifold if one is only given a point cloud on the manifold, i.e., a sample of points drawn from the manifold at random according to a certain distribution, without any further information. The manifold learning problem is highly relevant to machine learning and to the theory of pattern recognition. An example of an object related to the geometry of an embedded submanifold $\Sigma$ of Euclidean space that one can “learn” or estimate from a point cloud is the rough Laplacian or Laplace-Beltrami operator. Given an embedding $F : \Sigma^d \to \mathbb{R}^N$ consider its induced metric $g$. By the rough Laplacian of $g$ we mean the operator defined on functions by $\Delta_g f = g^{ij} \nabla_i \nabla_j f$ where $\nabla$ is the Levi-Civita connection of $g$. Belkin and Niyogi showed in [BN08] that given a uniformly distributed point cloud on $\Sigma$ there is a 1-parameter family of operators $L_t$, which converge to the Laplace-Beltrami operator $\Delta_g$ on the submanifold. More precisely, the construction of the operators $L_t$ is based on an approximation of the heat kernel of $\Delta_g$, and in particular the parameter $t$ can be interpreted as a choice of scale. In order to learn the rough Laplacian $\Delta_g$ from a point cloud it is necessary to write a sample version of the operators $L_t$. Then, supposing we have $n$ data points that are independent and identically distributed (abbreviated by i.i.d.) one can choose a scale $t_n$ in such a way that the operators $L_{t_n}$ converge almost surely to the rough Laplacian $\Delta_g$. This step follows essentially from applying a quantitative version of the law of large numbers. Thus one can almost surely learn spectral properties of a manifold. While in [BN08] it is assumed that the sample is uniform, it was proved by Coifman and Lafon in [CL06] that if one assumes more generally that the distribution of the data points has a smooth, strictly positive density in $\Sigma$, then it is possible to normalize the operators $L_t$ in [BN08] to recover the rough Laplacian. More generally, the results in [CL06] and [SW12] show that it is possible to recover a whole family of operators that include the Fokker-Planck operator and the weighted Laplacian $\Delta_\rho f = \Delta f - \langle \nabla \rho, \nabla f \rangle$ associated to the smooth metric measure space $(M, g, e^{-\rho}d\text{vol})$. 
where \( \rho \) is a smooth function. Since then, Singer and Wu have developed methods for learning the rough Laplacian of an embedded submanifold on 1-forms using Vector Diffusion Maps (VDM) (see for example [SW12]). The relationship of Ricci curvature to the Hodge Laplacian on 1-forms is given by the Weitzenböck formula.

In this paper we consider the problem of learning the Ricci curvature of an embedded submanifold \( \Sigma \) of \( \mathbb{R}^N \) at a point from a point cloud. The idea is to construct a notion of coarse Ricci curvature that will serve as a sample estimator of the actual Ricci curvature of the embedded submanifold \( \Sigma \). In order to explain our results we provide more background in the next section.

1.2. Definitions. In this section we recall from [AWb] a definition of coarse Ricci curvature on general metric spaces with an operator. When the space is metric measure space, we use a family of operators which are intended to approximate a Laplace operator on the space at scale \( t \). As this definition holds on metric measure spaces constructed from sampling points from a manifold, we can define an empirical or sample version of the Ricci curvature at a given scale \( t \). This last construction will have an application to the Manifold Learning Problem, namely it will serve to predict the Ricci curvature of an embedded submanifold of \( \mathbb{R}^N \) if one only has a point cloud on the manifold and the distribution of the sample has a smooth positive density.

1.2.1. Carré du champ. Consider a metric space \( X \). Given an operator \( L \) we define the Carré du champ as follows.

\[
\Gamma(L, u, v) = \frac{1}{2} (L(uv) - L(u)v - uL(v)).
\]

We will also consider the iterated Carré du Champ introduced by Bakry and Emery, denoted by \( \Gamma_2 \) and defined by

\[
\Gamma_2(L, u, v) = \frac{1}{2} (L(\Gamma(L, u, v)) - \Gamma(L, Lu, v) - \Gamma(L, u, Lv)).
\]

Now define the function for any \( x, y \in X \), define

\[
f_{x,y}(z) = \frac{1}{2} (d^2(x, y) - d^2(y, z) + d^2(z, x))
\]

and define:

**Definition 1.1.** Given an operator \( L \) we define the coarse Ricci curvature for \( L \) as

\[
\text{Ric}_L(x, y) = \Gamma_2(L, f_{x,y}, f_{x,y})(x).
\]

Our goal in this paper is to recover coarse Ricci curvature on a submanifold, via an approximation of the Laplace operator and the extrinsic distance function. However, there is a problem that arises when using the extrinsic distance function: The intrinsic and extrinsic distance squared functions agree only to the first three orders along the diagonal of a submanifold. The classical formula for the Carré du Champ involves three derivatives, so should agree along the diagonal. However, the analogy between Ricci and metric requires that we take two more derivatives to recover the Ricci tensor. This introduces the possibility of the recovered Ricci curvature bringing
along an error depending on the embedding, a situation we would like to avoid. To work around this, we will use the following. For any \( x, y \in X \), define

\[
F_{x,y}(z) = \frac{1}{d(x,y)} \frac{1}{2} \left( d^2(x,y) - d^2(y,z) + d^2(z,x) \right).
\]

**Definition 1.2.** Given an operator \( L \) we define the life-sized coarse Ricci curvature for \( L \) as

\[
\text{RIC}_L(x,y) = \Gamma_2(L, F_{x,y}, F_{x,y})(x).
\]

As we will see, this also can be used to recover the Ricci curvature, without taking any derivatives.

1.2.2. Approximations of the Laplacian, Carré du Champ and its iterate. Following [BN08] and [CL06], we recall how to construct operators which can be thought of as approximations of the Laplacian on metric measure spaces. Consider a metric measure space \((X,d,\mu)\) with a Borel \(\sigma\)-algebra such that \(\mu(X) < \infty\). Given \( t > 0 \), let \( \theta_t \) be given by

\[
\theta_t(x) = \int_X e^{-\frac{d^2(x,y)}{2t}} d\mu(y).
\]

We define a 1-parameter family of operators \( L_t \) as follows: given a function \( f \) on \( X \) define

\[
L_t f(x) = \frac{2}{t\theta_t(x)} \int_X (f(y) - f(x)) e^{-\frac{d^2(x,y)}{2t}} d\mu(y).
\]

With respect to this \( L_t \) one can define a Carré du Champ on appropriately integrable functions \( f, h \) by

\[
\Gamma(L_t, f, h)(x) = \frac{1}{2} (L_t(fh) - (L_t f)h - f(L_t h))
\]

which simplifies to

\[
\Gamma_2(L_t, f, h)(x) = \int_X e^{-\frac{d^2(x,y)}{2t}} (f(y) - f(x))(h(y) - h(x)) d\mu(y).
\]

In a similar fashion we define the iterated Carré du Champ of \( L_t \) to be

\[
\Gamma_2(L_t, f, h) = \frac{1}{2} (L_t(\Gamma(L_t, f, h)) - \Gamma(L_t, L_tf, h) - \Gamma(L_t, f, L_th)).
\]

**Remark 1.3.** This definition of \( L_t \) differs from Belkin-Niyogi operator in that we normalize by \( \theta_t(x) \) instead of \((2\pi t)^{d/2}\) for an assumed manifold dimension \( d \).

1.2.3. Empirical Coarse Ricci Curvature at a given scale. We can also define empirical versions of \( L_t, \Gamma(L_t, \cdot, \cdot) \) and \( \Gamma_2(L_t, \cdot, \cdot) \). On a space which consists of \( n \) points \( \{\xi_1, \ldots, \xi_n\} \) sampled from a manifold, it is natural to consider the empirical measure defined by

\[
\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\xi_i}
\]
where $\delta_{\xi_i}$ is the atomic point measure at the point $\xi_i$ (also called $\delta$-mass). For any function $f: X \to \mathbb{R}$ we will use the notation

$$\mu_n f = \int_X f(y) d\mu_n(y) = \frac{1}{n} \sum_{j=1}^{n} f(\xi_j).$$

**Notation 1.4.** We will use the “hat” notation (for example $\hat{L}_t$) to distinguish those operators, measures, or $t$-densities that have been constructed from a sample of finite points.

To be more precise, we define the operator $\hat{L}_t$ as

$$\hat{L}_t f(x) = \frac{2}{t\theta_t(x)} \int_X (f(y) - f(x)) e^{-\frac{d^2(x,y)}{2t}} d\mu_n(y),$$

where

$$\hat{\theta}_t(x) = \int_X e^{-\frac{d^2(x,y)}{2t}} d\mu_n(y) = \frac{1}{n} \sum_{j=1}^{n} e^{-\frac{d(\xi_j,x)^2}{2t}},$$

and of course

$$\int_X (f(y) - f(x)) e^{-\frac{d^2(x,y)}{2t}} d\mu_n(y) = \frac{1}{n} \sum_{j=1}^{n} e^{-\frac{d(\xi_j,x)^2}{2t}} (f(\xi_j) - f(x)).$$

The sample version of Carré du Champ will be the bilinear form $\Gamma(\hat{L}_t, f, h)$ which from (1.4) takes the form

$$\Gamma(\hat{L}_t, f, h)(x) = \frac{1}{t\hat{\theta}_t(x)} \frac{1}{n} \sum_{j=1}^{n} e^{-\frac{d^2(\xi_j,x)}{2t}} (f(\xi_j) - f(x)) (h(\xi_j) - h(x)).$$

We denote the iterated Carré du Champ corresponding to $\hat{L}_t$ by $\Gamma_2(\hat{L}_t, f, h)$, and by this we mean

$$\Gamma_2(\hat{L}_t, f, h) = \frac{1}{2} \left( \hat{L}_t (\Gamma(\hat{L}_t, f, h)) - \Gamma(\hat{L}_t, \hat{L}_t f, h) - \Gamma(\hat{L}_t, f, \hat{L}_t h) \right).$$

Our notion of empirical coarse Ricci curvature is as follows.

**Definition 1.5.** Let be a sample of points of $X$. The empirical coarse Ricci curvature of $(X, d, \mu)$ at a scale $t$ with respect to a sample $\{\xi_1, \ldots, \xi_n\}$ at $(x, y) \in X \times X$, which will be denoted by $\hat{\text{Ric}}(x, y)$ is

$$\hat{\text{Ric}}(x, y) = \Gamma_2(\hat{L}_t, f_{x,y}, f_{x,y})(x)$$

We are now in position to state our main results.

1.3. **Statement of Results.**
1.3.1. Applications to Manifold Learning. We now show how our notions of coarse Ricci curvature and empirical coarse Ricci curvature at a scale have applications to the Manifold Learning Problem. For the rest of subsection 1.3.1 we will consider a closed, smooth, embedded submanifold $\Sigma$ of $\mathbb{R}^N$, and the metric measure space will be $(\Sigma, \| \cdot \|, d\text{vol})$, where

- $\| \cdot \|$ is the distance function in the ambient space $\mathbb{R}^N$,
- $d\text{vol}_\Sigma$ is the volume element corresponding to the metric $g$ induced by the embedding of $\Sigma$ into $\mathbb{R}^N$.

In addition we will adopt the following conventions

- All operators $L_t$, $\Gamma(L_t, \cdot, \cdot)$ and $\Gamma_2(L_t, \cdot, \cdot)$ will be taken with respect to the distance $\| \cdot \|$ and the measure $d\text{vol}_\Sigma$.
- All sample versions $\hat{L}_t$, $\Gamma(\hat{L}_t, \cdot, \cdot)$ and $\Gamma_2(\hat{L}_t, \cdot, \cdot)$ are taken with respect to the ambient distance $\| \cdot \|$.

The choice of the above metric measure space is consistent with the setting of manifold learning in which no assumption on the geometry of the submanifold $\Sigma$ is made, in particular, we have no a priori knowledge of the geodesic distance and therefore we can only hope to use the chordal distance as a reasonable approximation for the geodesic distance. We will show that while our construction at a scale $t$ involves only information from the ambient space, the limit as $t$ tends to 0 will recover the life-sized Ricci curvature of the submanifold with intrinsic geodesic distance. As pointed out by Belkin-Niyogi [BN08, Lemma 4.3], the chordal and intrinsic distance squared functions on a smooth submanifold disagree first at fourth order near a point, so while much of the analysis is done on submanifolds, the intrinsic geometry will be recovered in the limit.

1.3.2. Summary of previous results. In [AWa] we are able to show the following.

**Theorem 1.6.** Let $\Sigma^d \subset \mathbb{R}^N$ be a closed embedded submanifold, let $g$ be the Riemannian metric induced by the embedding, and let $(\Sigma, \| \cdot \|, d\text{vol}_\Sigma)$ be the metric measure space defined with respect to the ambient distance. Given any $f \in C^5(\Sigma)$ there exists a constant $C_1$ depending on the geometry of $\Sigma$ and the function $f$ such that

$$\sup_{x \in \Sigma} |\Gamma_2(\Delta g, f, f)(x) - \Gamma_2(L_t, f, f)(x)| < C_1(\Sigma, D^5f) t^{1/2}. $$

**Corollary 1.7.** With the hypotheses of Theorem 1.6 we have

$$\text{Ric}_{\Delta g}(x, y) = \lim_{t \to 0} \Gamma_2(L_t, f_{x,y}, f_{x,y})(x).$$

We note that the relation between the coarse Ricci curvature and the Ricci curvature is as follows [AWa].

**Proposition 1.8.** Suppose that $M$ is a smooth Riemannian manifold. Let $V \in T_xM$ with $g(V, V) = 1$. Then

$$\text{Ric}(V, V) = \lim_{\lambda \to 0} \text{RIC}_{\Delta g}(x, \exp_x(\lambda V)).$$
The following bias error estimate from [AWa] will also be useful: For simplicity we will assume that \((\Sigma, d\text{vol}_\Sigma)\) has unit volume. Recall the definitions (1.1), (1.2), (1.3), (1.4) and (1.5).

**Proposition 1.9.** Suppose that \(\Sigma^d\) is a closed, embedded, unit volume submanifold of \(\mathbb{R}^N\). For any \(x\) in \(\Sigma\) and for any functions \(f, h\) in \(C^5(\Sigma)\) we have

\[
\frac{(2\pi t)^{d/2}}{\theta_t(x)} = 1 + tG_1(x) + t^{3/2}R_1(x),
\]

\[
\Gamma_t(f, h)(x) = \langle \nabla f(x), \nabla h(x) \rangle + t^{1/2}G_2(x, J^2(f)(x), J^2(h)(x))
\]

\[
+ tG_3(x, J^3(f)(x), J^3(h)(x)) + t^{3/2}R_2(x, J^4(f)(x), J^4(h)(x)),
\]

\[
L_t f(x) = \Delta_g f(x) + t^{1/2}G_4(x, J^4(f)(x)) + tG_5(x, J^4(f)(x)) + t^{3/2}R_3(x, J^5 f(x)),
\]

where each \(G_i\) is a locally defined function, which is smooth in its arguments, and \(J^k(u)\) is a locally defined \(k\)-jet of the function \(u\). Also, each \(R_i\) is a locally defined function of \(x\) which is bounded in terms of its arguments.

**New Results.** Now we address the problem of choosing a scale depending on the size of the data and the dimension of the submanifold \(\Sigma\), such that the sequence of empirical Ricci curvatures corresponding to the size of the data converge almost surely to the actual Ricci curvature of \(\Sigma\) at a point. In order to simplify the presentation of our results, we start by stating the simplest possible case, which corresponds to a uniformly distributed i.i.d. sample \(\{\xi_1, \ldots, \xi_n\}\). While we do not perform the computation, we expect we should be able to relax this assumption to i.i.d. samples whose distributions have a smooth everywhere positive density.

The following is our main theorem.

**Theorem 1.10.** Consider the metric measure space \((\Sigma, \|\cdot\|, d\text{vol}_\Sigma)\) where \(\Sigma^d \subset \mathbb{R}^N\) is a smooth closed embedded submanifold. Suppose that we have a uniformly distributed i.i.d. sample \(\{\xi_1, \ldots, \xi_n\}\) of points from \(\Sigma\). For \(\sigma > 0\), let

\[
t_n = n^{-\frac{1}{3d+3+\sigma}}.
\]

Then,

\[
\sup_{\xi \in \Sigma} \left| \hat{\Gamma}_2(L_{t_n}, f, f)(\xi) - \Gamma_2(L_{t_n}, f, f)(\xi) \right| \xrightarrow{a.s.} 0,
\]

for fixed \(f \in C^5(\Sigma^d)\).

The proof of Theorem 1.10 requires using ideas from the theory of empirical processes. We will provide the necessary background in Section 2. As pointed out in Section 1.1.1, since we are interested in recovering an object from its sample version, we are forced to consider a law of large numbers in order to obtain convergence in probability or almost surely. The problem is that the sample version of \(\Gamma_2(L_t, \cdot, \cdot)\).
involves a high correlation between the data points, destroying independence and any hope of applying large number results directly. The idea then is to reduce the convergence of the sample version of $\Gamma_L$ to the application of a uniform law of large numbers to certain classes of functions. Theorem 1.10 is proved in Section 2.

**Corollary 1.11.** Let $\Sigma^d \subset \mathbb{R}^N$ be an embedded submanifold and consider the metric measure space $(\Sigma, \| \cdot \|, d\text{vol}_\Sigma)$. Suppose that we have an i.i.d. uniformly distributed sample $\xi_1, \ldots, \xi_n$ drawn from $\Sigma$. Let

$$t_n = n^{-\frac{1}{d+3+\sigma}},$$

for any $\sigma > 0$. Then

$$\sup_{x \in \Sigma} \left| \hat{\Gamma}_2(L_{t_n}, F_{x,y}, F_{x,y})(x) - RIC_{\Delta_d}(x, y) \right| \overset{\text{a.s.}}{\longrightarrow} 0.$$ 

In other words, there is a choice of scale depending on the size of the data and the dimension of the submanifold for which the corresponding empirical life-sized coarse Ricci curvatures converge almost surely to the life-sized coarse Ricci curvature.

**Remark 1.12.** The convergence is better if $t_n$ is chosen to go to zero slower than in (2.74). In particular, if one replaces $d$ with an upper bound on $d$, then Theorem 1.10 and Corollary 1.11 still hold.

We now mention how to obtain similar results for non-uniformly distributed samples and show how we can recover more general objects than the Ricci Curvature, for example the Bakry-Emery tensor if we sample adequately.

**1.3.4. Smooth Metric Measure Spaces and non-Uniformly Distributed Samples.** Consider a smooth metric measure space $(M, g, e^{-\rho}d\text{vol})$ and let $\Delta_\rho$ be the operator

$$\Delta_\rho u = \Delta_g u - \nabla \rho \cdot \nabla u.$$ 

In [CL06], the authors consider a family of operators $L_\alpha^t$ which converge to $\Delta_{2(1-\alpha)\rho}$. Note that a standard computation (cf. [Vil09, Page 384]) gives

$$\Gamma_2(\Delta_{2(1-\alpha)\rho}, f, f) = \frac{1}{2} \Delta_g \| \nabla f \|^2_g - \langle \nabla \rho, \nabla \Delta_g f \rangle_g + 2(1-\alpha) \nabla^2 \rho(\nabla f, \nabla f).$$

We adapt [CL06] to our setting: Recall that

$$\theta_t(x) = \int_X e^{-\frac{d^2(x,y)}{2t}} d\mu(y),$$

and define, for $\alpha \in [0, 1]$

$$\theta_{t,\alpha}(x) = \int_X e^{-\frac{d^2(x,w)}{2t}} \frac{1}{[\theta_t(y)]^\alpha} d\mu(y). \tag{1.16}$$

We can define the operator

$$L_\alpha^t f(x) = \frac{2}{t} \frac{1}{\theta_{t,\alpha}(x)} \int_X e^{-\frac{d^2(x,w)}{2t}} \frac{1}{[\theta_t(y)]^\alpha} (f(y) - f(x)) d\mu(y). \tag{1.17}$$
and again obtain bilinear forms $\Gamma(L^t_\alpha, f, f)$ and $\Gamma_2(L^t_\alpha, f, f)$. For the rest of the section we will consider the metric measure space $(\Sigma, \| \cdot \|, e^{-\rho}d\text{vol}_\Sigma)$ where $\Sigma^d \subset \mathbb{R}^N$ is an embedded submanifold, $\| \cdot \|$ is the ambient distance and $\rho$ is a smooth function in $\Sigma$. We again take all the operators $L_t, \Gamma_t(L_t, \cdot, \cdot)$ and $\Gamma_2(L_t, \cdot, \cdot)$ and their sample counterparts $\hat{L}_t, \Gamma_t(\hat{L}_t, \cdot, \cdot)$ and $\Gamma_2(\hat{L}_t, \cdot, \cdot)$ with respect to the data of $(\Sigma, \| \cdot \|, e^{-\rho}d\text{vol}_\Sigma)$.

Based on estimates in [AWb] and calculations similar to the proof of Theorem 1.10, we make the following conjecture. We note that the relevant bias estimate corresponding to Proposition 1.9 is available in [AWb]. The proof is left to the ambitious reader.

**Conjecture 1.13.** Let $\Sigma^d \subset \mathbb{R}^N$ be an embedded submanifold and consider the metric measure space $(\Sigma, \| \cdot \|, e^{-\rho}d\text{vol}_\Sigma)$. Suppose that we have an i.i.d. sample $\{\xi_1, \ldots, \xi_n\}$ whose distribution has density $e^{-\rho}d\text{vol}_\Sigma$. Letting

\begin{equation}
    t_n = n^{-\frac{1}{4d+4+\sigma}},
\end{equation}

for $\sigma > 0$, it follows that $\Gamma_2(\hat{L}^t_{\alpha}, f_{x,y}, f_{x,y})(x)$ converges almost surely to the coarse Ricci curvature

$\text{Ric}_{\Delta_2(1-\alpha)}(x, y)$.

1.4. **Final remarks.** Our results show that one can give a definition of coarse Ricci curvature at a scale on general metric measure spaces that converges to the actual Ricci curvature on smooth Riemannian manifolds. Moreover, our definition of empirical coarse Ricci curvature at a scale can be thought of as an extension of Ricci curvature to a class of discrete metric spaces namely those obtained from sampling points from a smooth closed embedded submanifold of $\mathbb{R}^N$. Note however, that in order to obtain convergence of the empirical coarse Ricci curvature at a scale to the actual Ricci curvature we need to assume that there is a manifold which fits the distribution of the data. Recently, Fefferman-Mitter-Narayanan in [FMN13] have developed an algorithm for testing the hypothesis that there exists a manifold which fits the distribution of a sample, however, a problem that remains open is how to estimate the dimension of a submanifold from a sample of points.

In another vein, there is much current interest in a converse problem: The development of algorithms for generating point clouds on manifolds or even on surfaces. Recently, there has been progress in this direction by Palais-Palais-Karcher in [KPP14], specifically on methods for generating point clouds on implicit surfaces using Monte Carlo simulation and the Cauchy-Crofton formula.

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2. Empirical Processes and Convergence

The goal of this section is to prove Theorem 1.10. This will be done using tools from the theory of empirical processes in order to establish uniform laws of large numbers in a sense that we will explain in Sections 2.2 through 2.6. For a standard reference in the theory of empirical processes, see [vdVW96]. See also [SW13] for further applications of the theory of empirical processes to the recovery of diffusion operators from a sample.

2.1. Estimators of the Carré Du Champ and the Iterated Carré Du Champ in the uniform case. Let us assume that the measure \( \mu \) is the volume measure \( d\text{vol}_\Sigma \). Recall that our formal definition of the Carré du Champ of \( L_t \) with respect to the uniform distribution is given by

\[
\Gamma(L_t, f, h) = \frac{1}{t} \theta_t(x) \left( \int_{\Sigma} e^{-\frac{|x-y|^2}{2t}} (f(y) - f(x)) (h(y) - h(x)) d\mu(y) \right).
\] (2.1)

It is clear from (2.1) that a sample estimator of the Carré Du Champ at a point \( x \) is given by

\[
\hat{\Gamma}(L_t, f, f)(x) = \frac{1}{t} \hat{\theta}_t(x) \left( \frac{1}{n} \sum_{j=1}^n e^{-\frac{|x-\xi_j|^2}{2t}} (f(\xi_j) - f(x)) (h(\xi_j) - h(x)) \right),
\] (2.2)

and recall that we defined the \( t \)-Laplace operator by

\[
L_t f(x) = \frac{2}{t} \theta_t(x) \int \frac{1}{n} \sum_{j=1}^n e^{-\frac{|x-\xi_j|^2}{2t}} (f(y) - f(x)) d\mu(y),
\] (2.3)

and its sample version is

\[
\hat{L}_t f(x) = \frac{2}{t} \hat{\theta}_t(x) \frac{1}{n} \sum_{j=1}^n e^{-\frac{|x-\xi_j|^2}{2t}} (f(\xi_j) - f(x)).
\] (2.4)

Recall that the iterated Carré du Champ is

\[
\Gamma_2(L_t, f, h) = \frac{1}{2} (L_t \Gamma(L_t, f, h) - \Gamma(L_t, L_t f, f) - \Gamma(L_t, f, L_t h)).
\] (2.5)

For simplicity, we will evaluate \( \Gamma_2(L_t, \cdot, \cdot) \) at a pair \((f, f)\) instead of \((f, h)\) and by symmetry it is clear that we obtain

\[
\Gamma_2(L_t, f, f) = \frac{1}{2} (L_t (\Gamma_t(f, f)) - 2\Gamma_t(L_t f, f)).
\] (2.6)

Combining the sample versions of \( \Gamma_t \) and \( L_t \) we obtain a sample version for \( \Gamma_2(L_t, f, f) \)
(2.7) \[
\hat{\Gamma}_2(L_t, f, f)(x) = \frac{1}{t^2 n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{\theta_t(x)\theta_t(x)} e^{-\frac{\|x-x_j\|^2}{2t} - \frac{\|x-x_k\|^2}{2t}} (f(x) - f(x_j))(f(x) - f(x_k))
\]

(2.8) \[
- \frac{1}{t^2 n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{\theta_t^2(x)} e^{-\frac{\|x-x_j\|^2}{2t} - \frac{\|x-x_k\|^2}{2t}} (f(x) - f(x_j))(f(x) - f(x_k))
\]

(2.9) \[
- \frac{2}{t^2 n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{\theta_t(x)\theta_t(x)} e^{-\frac{\|x-x_j\|^2}{2t} - \frac{\|x-x_k\|^2}{2t}} (f(x) - f(x_j))(f(x) - f(x_k))
\]

(2.10) \[
+ \frac{2}{t^2 n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{1}{\theta_t^2(x)} e^{-\frac{\|x-x_j\|^2}{2t} - \frac{\|x-x_k\|^2}{2t}} (f(x) - f(x))(f(x) - f(x_k))
\]

In principle, the convergence analysis for (2.7)-(2.10) can be done using the following standard result in large deviation theory.

**Lemma 2.1** (Hoeffding’s Lemma). Let \( \xi_1, \ldots, \xi_n \) be i.i.d. random variables on the probability space \((\Sigma, \mathcal{B}, \mu)\) where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra of \( \Sigma \), and let \( f : \Sigma \to [-K, K] \) be a Borel measurable function with \( K > 0 \). Then for the corresponding empirical measure \( \mu_n \) and any \( \varepsilon > 0 \) we have

\[
\Pr \{|\mu_n f - \mu f| \geq \varepsilon\} \leq 2e^{-\frac{\varepsilon^2 n}{2K^2}}.
\]

Observe, however, that (2.7)-(2.10) is a non-linear expression which will involve non-trivial interactions between the data points \( \xi_1, \ldots, \xi_n \). This non-trivial interaction between the points \( \xi_1, \ldots, \xi_n \) will produce a loss of independence and we will not be able to apply Hoeffding’s Lemma directly to (2.7)-(2.10). In order to address this difficulty we will establish several uniform laws of large numbers which will provide us with a large deviation estimate for (2.7)-(2.10).

**Remark 2.2.** We will not use directly the expression (2.7)-(2.10), instead we will write (2.7)-(2.10) schematically in the form

(2.11) \[
\hat{\Gamma}_2(L_t(f, f))(x) = \frac{1}{2} \left( \hat{L}_t \left( \hat{\Gamma}(f, f) \right)(x) - 2\hat{\Gamma}_t(L_t f, f)(x) \right),
\]

which is clearly equivalent to (2.7)-(2.10).

2.2. **Glivenko-Cantelli Classes.** A Glivenko-Cantelli class of functions is essentially a class of functions for which a uniform law of large numbers is satisfied.

**Definition 2.3.** Let \( \mu \) be a fixed probability distribution defined on \( \Sigma \). A class \( \mathcal{F} \) of functions of the form \( f : \Sigma \to \mathbb{R} \) is Glivenko-Cantelli if

(a) \( f \in L^1(d\mu) \) for any \( f \in \mathcal{F} \),

(b) For any i.i.d. sample \( \xi_1, \ldots, \xi_n \) drawn from \( \Sigma \) whose distribution is \( \mu \) we have uniform convergence in probability in the sense that for any \( \varepsilon > 0 \)

(2.12) \[
\lim_{n \to \infty} \Pr^* \left\{ \sup_{f \in \mathcal{F}} |\mu_n f - \mu f| > \varepsilon \right\} = 0.
\]
Remark 2.4. Note that in general we have to consider outer probabilities \( \text{Pr}^* \) instead of \( \text{Pr} \) because the class \( \mathcal{F} \) may not be countable and the supremum \( \sup_{f \in \mathcal{F}} |\mu_n f - \mu f| \) may not be measurable. On the other hand, if the class \( \mathcal{F} \) is separable in \( L^\infty(\Sigma) \), then we can replace \( \text{Pr}^* \) by \( \text{Pr} \). While all of the classes we will encounter in this paper will be separable in \( L^\infty(\Sigma) \), we use \( \text{Pr}^* \) when we deal with a general class.

Let \( \mathcal{F} \) be a class of functions defined on \( \Sigma \) and totally bounded in \( L^\infty(\Sigma) \). Given \( \delta > 0 \) we let \( \mathcal{N}(\mathcal{F}, \delta) \) be the \( L^\infty(\Sigma) \)-covering number of \( \mathcal{F} \), i.e.,

\[
\mathcal{N}(\mathcal{F}, \delta) = \inf \{ m : \mathcal{F} \text{ is covered by } m \text{ balls of radius } \delta \text{ in the } L^\infty \text{ norm} \}.
\]

Lemma 2.5. Let \( \mathcal{F} \) be an equicontinuous class of functions in \( L^\infty(\Sigma) \) and satisfies \( \sup_{f \in \mathcal{F}} \{ \| f \|_{L^\infty(\Sigma)} \} \leq M < \infty \) for some \( M > 0 \). Then for any distribution \( \mu \) which is absolutely continuous with respect to \( d\text{vol}_\Sigma \) the class \( \mathcal{F} \) is \( \mu \)-Glivenko-Cantelli. Moreover, if \( \xi_1, \ldots, \xi_n \) is an i.i.d. sample drawn from \( \Sigma \) with distribution \( \mu \) we have

\[
\text{Pr}^* \left\{ \sup_{f \in \mathcal{F}} |\mu_n f - \mu f| \geq \varepsilon \right\} \leq 2^m \left( \mathcal{F}, \frac{\varepsilon}{4} \right) e^{\frac{-\varepsilon^2 n}{8M^2}}.
\]

Proof. By equicontinuity of \( \mathcal{F} \), it follows from the Arzelà-Ascoli theorem that \( \mathcal{F} \) is precompact in the \( L^\infty(\Sigma) \) norm and hence totally bounded in \( L^\infty(\Sigma) \). In particular for every \( \delta > 0 \), the number \( \mathcal{N}(\mathcal{F}, \delta) \) is finite. Let \( \mathcal{G} \) be a finite class such that the union of all balls with center in \( \mathcal{G} \) and radius \( \delta \) covers \( \mathcal{F} \) and \( |\mathcal{G}| = \mathcal{N}(\mathcal{F}, \delta) \). For any \( f \in \mathcal{F} \) there exists \( \phi \in \mathcal{G} \) such that \( \| f - \phi \|_{L^\infty(\Sigma)} < \delta \) and we obtain

\[
|\mu_n f - \mu f| \leq 2\delta + |\mu_n \phi - \mu \phi|,
\]

and clearly

\[
\sup_{f \in \mathcal{F}} |\mu_n f - \mu f| \leq 2\delta + \max_{\phi \in \mathcal{G}} |\mu_n \phi - \mu \phi|.
\]

Fixing \( \varepsilon > 0 \) and choosing \( \delta = \varepsilon/4 \) we observe that

\[
\text{Pr}^* \left\{ \sup_{f \in \mathcal{F}} |\mu_n f - \mu f| \geq \varepsilon \right\} \leq \text{Pr} \left\{ \max_{\phi \in \mathcal{G}} |\mu_n \phi - \mu \phi| \geq \frac{\varepsilon}{2} \right\},
\]

and by Hoeffding’s inequality we have

\[
\text{Pr} \left\{ \max_{\phi \in \mathcal{G}} |\mu_n \phi - \mu \phi| \geq \frac{\varepsilon}{2} \right\} \leq 2^m \left( \mathcal{F}, \frac{\varepsilon}{4} \right) e^{\frac{-\varepsilon^2 n}{8M^2}},
\]

which implies the lemma.

For a Lipschitz function \( f \) defined on the ambient space \( \mathbb{R}^N \) we will write \( \| f \|_{\text{Lip}} \) to denote the Lipschitz norm of \( f \) with respect to the ambient distance \( \| \cdot \| \), i.e.

\[
\| f \|_{\text{Lip}} = \inf_{x,y \in \mathbb{R}^N, x \neq y} \left\{ \frac{|f(x) - f(y)|}{\|x - y\|} \right\}.
\]
For a function $f \in C^k(\mathbb{R}^N)$ we will use $\|f\|_{C^k}$ to denote the following

$$\|f\|_{C^k} = \|f\|_{L^\infty(\Sigma)} + \sum_{j=1}^k \|D_j f\|_{L^\infty(\Sigma)}.$$  

In particular, when in (2.19) we write $\|D_j f\|_{C^k}$ we mean

$$\|D_j f\|_{C^k} = \|D_j f\|_{L^\infty(\Sigma)} = \sup_{x \in \Sigma} \|D_j f(x)\|,$$

i.e., the norm $\|D_j f(x)\|$ is the norm in the ambient space $\mathbb{R}^N$.

We introduce another norm ($t$-almost Lipschitz) weaker than the Lipschitz norm.

$$\|f\|_{A-t-Lip} = \inf \{ A + B : |f(x) - f(y)| \leq A \|x - y\| + B t^{1/2} \text{ for all } x, y \in \mathbb{R}^N \}$$

We will frequently use the following classes of functions

$$\mathcal{F}^t_{f,h} = \left\{ \phi_t(\xi, \zeta) = t^{-1/2} e^{-\frac{1}{2t} \|\xi - \zeta\|^2} (f(\xi) - f(\zeta))(h(\xi) - h(\zeta)) : \xi \in \Sigma \right\},$$

$$\mathcal{G}^t = \left\{ \psi_t(\xi, \zeta) = t^{1/2} e^{-\frac{1}{2t} \|\xi - \zeta\|^2} : \xi \in \Sigma \right\},$$

where $f, h$ in (2.20) are fixed functions. We will also use $\mathcal{F}^t_f$ to denote the class $\mathcal{F}^t_{f,f}$.

Observe that for the classes in (2.20) and (2.21) one has

$$M_{\mathcal{F}^t_{f,h}} = \sup_{\phi \in \mathcal{F}^t_{f,h}} \{ \|\phi\|_{L^\infty(\Sigma)} \} \leq \left( \frac{2}{e} \right) t^{1/2} \|f\|_{Lip} \|h\|_{Lip},$$

$$M_{\mathcal{F}^t_{f,h}} \leq t^{1/2} \|f\|_{A-t-Lip} \|h\|_{Lip}$$

$$M_{\mathcal{G}^t} = \sup_{\psi \in \mathcal{G}^t} \{ \|\psi\|_{L^\infty(\Sigma)} \} = t^{1/2}.$$  

2.3. Ambient Covering Numbers. In this subsection we show that the computation of covering numbers of the classes of functions $\mathcal{F}^t_{f,h}$ and $\mathcal{G}^t$ introduced in (2.20), (2.21) reduces to the computation of covering numbers of submanifolds of $\mathbb{R}^N$. For this purpose we will need the notion of ambient covering number of $\Sigma$ defined by

$$A(\Sigma, \delta) = \inf \{ m : \exists A \text{ with } |A| = m \text{ and } \Sigma \subset \bigcup_{a \in A} B_{\mathbb{R}^N,\delta}(a) \}.$$  

where the ball $B_{\mathbb{R}^N,\delta}(a)$ is taken with respect to the ambient distance $\|\cdot\|$.

**Lemma 2.6.** For any $\xi, \tilde{\xi}, \zeta \in \mathbb{R}^N$ we have

$$\left| t^{1/2} e^{-\frac{1}{2t} \|\xi - \zeta\|^2} - t^{1/2} e^{-\frac{1}{2t} \|\xi - \tilde{\xi}\|^2} \right| \leq e^{-1/2} \|\xi - \tilde{\xi}\|.$$  

In particular

$$\mathcal{N}(\mathcal{G}^t, \varepsilon) \leq A(\Sigma, e^{1/2} \varepsilon).$$
Proof. Observe that the function \( \psi_t(\xi, \zeta) = t^{1/2}e^{-\frac{\|\zeta - \xi\|^2}{2t}} \) satisfies
\[
D_\xi \psi_t(\xi, \zeta) = t^{1/2} \frac{(\zeta - \xi)}{t} e^{-\frac{\|\zeta - \xi\|^2}{2t}},
\]
and therefore
\[
\sup_{\xi, \zeta \in \mathbb{R}^N} \|D_\xi \psi_t(\xi, \zeta)\| \leq \sqrt{2} \sup_{\rho > 0} \left\{ \rho e^{-\rho^2} \right\} = e^{-\frac{1}{2}},
\]
and therefore we have the Lipschitz estimate
\[
|\psi_t(\xi, \zeta) - \psi_t(\tilde{\xi}, \zeta)| \leq e^{-1/2} \|\xi - \tilde{\xi}\|.
\]

**Corollary 2.7.** Fix a function \( 0 \neq h \in L^\infty(\Sigma) \) with \( \|h\|_{L^\infty(\Sigma)} \leq C \) and consider the class of functions
\[
\mathcal{H}_h^t = \{ \psi_t(\zeta, \cdot)h(\cdot) : \zeta \in \Sigma \}.
\]
Then for every \( \varepsilon > 0 \) we have
\[
\mathcal{N}(\mathcal{H}_h^t, \varepsilon) \leq A \left( \frac{e^{1/2}}{C} \varepsilon \right).
\]

Proof. We use Lemma 2.6 to obtain the estimate
\[
\left\| \psi_t(\zeta, \cdot)h(\cdot) - \psi_t(\zeta', \cdot)h(\cdot) \right\|_{L^\infty(\Sigma)} \leq C \|\psi_t(\zeta, \cdot) - \psi_t(\zeta', \cdot)\|_{L^\infty(\Sigma)} \leq C e^{-1/2} \|\zeta - \zeta'\|,
\]
from which the corollary follows. \( \Box \)

**Lemma 2.8.** For any \( \phi_t(\xi, \cdot), \phi_t(\xi', \cdot) \in \mathcal{F}_f h \) we have
\[
\sup_{\zeta \in \mathbb{R}^d} |\phi_t(\xi, \zeta) - \phi_t(\xi', \zeta)| \leq C(f, h) \|\xi - \xi'\|,
\]
where
\[
C(f, h) = C_0 \left( \|f\|_{\text{Lip}} \|h\|_{\text{Lip}} + \|f\|_{C^1} \|h\|_{\text{Lip}} + \|h\|_{C^1} \|f\|_{\text{Lip}} \right)
\]
and \( C_0 \) is a universal constant. Thus
\[
\mathcal{N}(\mathcal{F}_f h, \delta) \leq A \left( \frac{\delta}{C(f, h)} \right).
\]

Proof. Let \( \phi_t(\xi, \zeta) \in \mathcal{F}_f h \), then we have
\[
D_\xi \phi_t(\xi, \zeta) = t^{-1/2} \frac{(\zeta - \xi)}{2t} e^{-\frac{\|\zeta - \xi\|^2}{2t}} \left( f(\xi) - f(\zeta) \right)(h(\xi) - h(\zeta)) + t^{-1/2} e^{-\frac{\|\zeta - \xi\|^2}{2t}} D_\xi f(\xi)(h(\xi) - f(\zeta)) + t^{-1/2} e^{-\frac{\|\zeta - \xi\|^2}{2t}} D_\xi h(\xi)(f(\xi) - f(\zeta)),
\]
and then
\[\|D\phi_t(\xi, \zeta)\| \leq \|f\|_{\text{Lip}}\|h\|_{\text{Lip}} \frac{\|\xi - \zeta\|^3}{t^{3/2}} e^{-\frac{\|\xi - \zeta\|^2}{2t}} \]
\[+ \frac{\|\xi - \zeta\|}{t^{1/2}} e^{-\frac{\|\xi - \zeta\|^2}{2t}} \|f\|_{C^1} \|h\|_{\text{Lip}} \]
\[+ \frac{\|\xi - \zeta\|}{t^{1/2}} e^{-\frac{\|\xi - \zeta\|^2}{2t}} \|h\|_{C^1} \|f\|_{\text{Lip}} \]
\[\leq C_0 (\|f\|_{\text{Lip}}\|h\|_{\text{Lip}} + \|f\|_{C^1} \|h\|_{\text{Lip}} + \|f\|_{\text{Lip}}\|h\|_{C^1}), \]
where
\[C_0 = \max \left( \sup_{\rho > 0} \{ \rho^3 e^{-\rho^2/2} \}, \sup_{\rho > 0} \{ \rho e^{-\rho/2} \} \right). \]
It follows that for any \(\xi \in \Sigma\) we have the Lipschitz estimate
\[|\phi_t(\xi) - \phi_t(\xi')| \leq C(f, h)\|\xi - \xi'\|. \]

The following can be obtained in a similar fashion.

**Lemma 2.9.** Let \(f \in C^1\) and let \(v_t\) be given by
\[v_t(\xi, \zeta) = e^{-\frac{\|\xi - \zeta\|^2}{2t}} (f(\xi) - f(\zeta)).\]
We then have the following estimate
\[|v_t(\xi, \zeta) - v_t(\xi', \zeta)| \leq \left( \frac{2t}{e} \|f\|_{\text{Lip}} + \|f\|_{C^1} \right) \|\xi - \xi'\|. \]

**Proof.** As before,
\[|D_x v_t(\xi, \zeta)| \leq \frac{\|\xi - \zeta\|}{t} e^{-\frac{\|\xi - \zeta\|^2}{2t}} \|f(\xi) - f(\zeta)\| + e^{-\frac{\|\xi - \zeta\|^2}{2t}} \|D_\xi f(\xi)\| \]
\[\leq \frac{\|\xi - \zeta\|^2}{t} e^{-\frac{\|\xi - \zeta\|^2}{2t}} \|f\|_{\text{Lip}} + \sup_{x \in \Sigma} \|D_\xi f(\xi)\|, \]
and
\[\sup_{\rho > 0} \rho^2 e^{-\rho^2} = \frac{2}{e}. \]

Lemmas 2.6 and 2.8 say that we can relate covering numbers of the classes \(\mathcal{F}_{f,h}^t, \mathcal{G}^t\) to ambient covering numbers of the submanifold \(\Sigma\). In order to estimate \(\mathcal{A}(\Sigma, \delta)\) we need to introduce the notion of reach of an embedded submanifold of \(\mathbb{R}^N\). For every \(\varepsilon > 0\) we can consider the \(\varepsilon\) neighborhood of \(\Sigma\)
\[(2.49) \quad \Sigma_\varepsilon = \{ x \in \mathbb{R}^N : d(x, \Sigma) < \varepsilon \}, \]
where \(d(\cdot, \Sigma)\) measures the distance from points in \(\mathbb{R}^N\) to \(\Sigma\) with respect to the ambient norm \(\| \cdot \|\). If \(\Sigma\) is a smooth, embedded submanifold of \(\mathbb{R}^N\), for \(\varepsilon > 0\) sufficiently small we can define a smooth map \(\varphi : \Sigma_\varepsilon \to \Sigma\) such that
\( \varphi \) is smooth,

(2) \( \varphi(x) \) is the unique point in \( \Sigma \) such that \( \| \varphi(x) - x \| = d(x, \Sigma) \) for all \( x \in \Sigma_\epsilon \).

(3) \( x - \varphi(x) \in T_{\varphi(x)}^\perp \Sigma \),

(4) \( \varphi(y + z) \equiv \varphi(y) \) for all \( y \in \Sigma \) and \( z \in (T_y \Sigma)^\perp \) with \( \| z \| < \epsilon \),

(5) For any vector \( V \in \mathbb{R}^N \), \( D_V \varphi(x) = p_{\varphi(x)}(V) \), where \( p_{\varphi(x)}(V) \) is the orthogonal projection of \( V \) onto \( T_{\varphi(x)} \Sigma \).

See for example [Sim96, Theorem 1]. The map \( \varphi \) is called \textit{nearest point projection} onto \( \Sigma \).

\textbf{Definition 2.10.} Let \( \Sigma \) be an embedded submanifold of \( \mathbb{R}^N \). The reach of \( \Sigma \) is the number

\[ \tau = \sup_{\epsilon > 0} \{ \text{There exists a nearest point projection in } \Sigma_\epsilon \} . \]

We quote following result

\textbf{Theorem 2.11 ([FMN13] Corollary 6).} Suppose that \( \Sigma \) is a \( d \)-dimensional embedded submanifold of \( \mathbb{R}^N \) with volume \( V \) and reach \( \tau > 0 \) and let \( U : \mathbb{R}_+ \to \mathbb{R}_+ \) be the function

\[ U(r) = V \left( \frac{1}{\tau^d} + r^d \right) , \]

then for any \( \epsilon > 0 \) there is an \( \epsilon \)-net of \( \Sigma \) with respect to the ambient distance \( \| \cdot \| \) of no more than \( C_d U(\epsilon^{-1}) \) points where \( C_d \) is a dimensional constant. In particular,

\[ \mathcal{A}(\Sigma, \epsilon) \leq C_d U(\epsilon^{-1}) . \]

\textbf{Corollary 2.12.} We have the following bounds

(a)

\[ N(\mathcal{F}_f, \epsilon) \leq C_d V \left( \frac{1}{\tau^d} + \left( \frac{C(f, h)}{\epsilon} \right)^d \right) \]

(b)

\[ N(\mathcal{G}_f, \epsilon) \leq C_d V \left( \frac{1}{\tau^d} + \left( \frac{1}{\epsilon^{1/2} \epsilon} \right)^d \right) , \]

(c)

\[ N(\mathcal{H}_h, \epsilon) \leq C_d V \left( \frac{1}{\tau^d} + \left( \frac{C}{\epsilon^{1/2} \epsilon} \right)^d \right) . \]

where \( C \) is an upper bound for \( \| h \|_{L^\infty(\Sigma)} \).
2.4. Sample Version of the Carré du Champ. In this section we are still assuming that the distribution of the sample \(\xi_1, \ldots, \xi_n\) in \(\Sigma\) is uniform, i.e., \(d\mu = d\text{vol}_\Sigma\). In this case we know that \(\lim_{t \to 0} t^{-d/2}\theta_t = (2\pi)^{d/2}\) uniformly in \(L^\infty(\Sigma)\) and therefore there exists \(t_0 > 0\) such that for \(0 < t < t_0\) we have
\[
2(2\pi)^{d/2} \geq \theta_t \geq \frac{1}{2}(2\pi)^{d/2}.
\]
If we let
\[
\lambda_0 = \frac{(2\pi)^{d/2}}{4},
\]
we have for \(0 < t < t_0\) the inequality
\[
\theta_t(x) \geq 2t^{d/2}\lambda_0,
\]
which will be a convenient normalization for us in the sequel. The main lemma in this section is the following.

**Lemma 2.13.** Suppose \(\mathcal{F}\) is a class of functions totally bounded in \(L^\infty(\Sigma)\) of the form
\[
f(x, \cdot) \text{ for } x \in \Sigma
\]
for a fixed \(f \in L^\infty(\Sigma \times \Sigma)\) and
\[
M = \|f\|_{L^\infty(\Sigma \times \Sigma)}.
\]
Suppose also \(0 < t < t_0\) and \(\varepsilon\) is small enough so that
\[
\varepsilon t^{d+1/2}\lambda_0 < M.
\]
Then
\[
\Pr^* \left\{ \sup_{x \in \Sigma} \left| t^{-1/2}\mu_n f(x, \cdot) - t^{-1/2}\mu f(x, \cdot) \right| \theta_t(x) \geq \varepsilon \right\}
\]
\[
\leq 2N \left( \mathcal{G}^t, \varepsilon t^{d+1} \lambda_0^2 \right) \exp \left( \frac{-\varepsilon^2 t^{2d+1} \lambda_0^4 n}{8M^2} \right)
\]
\[
2N \left( \mathcal{F}, \varepsilon \lambda_0 t^{(d+1)/2} \right) \exp \left( \frac{-\varepsilon^2 \lambda_0^2 t^{d+1} n}{8M^2} \right).
\]

Before proving Lemma 2.13 we will prove the following elementary lemma.

**Lemma 2.14.** Let \(\xi, \zeta\) be positive random variables. For any \(\varepsilon > 0\) we have
\[
\Pr \left\{ \left| \frac{1}{\xi} - \frac{1}{\zeta} \right| \geq \varepsilon \right\} \leq \Pr \left\{ |\zeta - \xi| \geq \varepsilon \xi^2 \frac{1 + \varepsilon \xi}{1 + \varepsilon} \right\}.
\]
Proof. Assume $0 < \zeta < \xi$

\[
\frac{1}{\xi} - \frac{1}{\zeta} = \frac{\xi - \zeta}{\xi \zeta} = \frac{\xi - \zeta}{\xi \zeta - \xi^2 + \xi^2} = \frac{\zeta - \xi}{\xi (\zeta - \xi) + \xi^2} = \frac{|\zeta - \xi|}{-\xi |\zeta - \xi| + \xi^2},
\]

and from

\[
\frac{|\zeta - \xi|}{-\xi |\zeta - \xi| + \xi^2} \leq \varepsilon
\]

we have

\[
|\zeta - \xi| \leq \frac{\varepsilon \xi^2}{1 + \varepsilon \xi}.
\]

For the case $0 < \xi < \zeta$ we have

\[
\frac{1}{\zeta} - \frac{1}{\xi} = \frac{\zeta - \xi}{\xi (\zeta - \xi) + \xi^2} = \frac{|\xi - \zeta|}{\xi |\zeta - \xi| + \xi^2},
\]

and \(\frac{|\xi - \zeta|}{\xi |\zeta - \xi| + \xi^2} \geq \varepsilon\) implies

\[
(1 + \varepsilon \xi) |\xi - \zeta| \geq (1 - \varepsilon \xi) |\xi - \zeta| \geq \varepsilon \xi^2.
\]

\(\square\)

Proof of Lemma 2.13. It is easy to prove from Lemma 2.14 and (2.57) that for any $\delta > 0$ we have

(2.62) \[\Pr \left\{ \sup_{x \in \Sigma} \left| \frac{1}{\tilde{\theta}_t(x)} - \frac{1}{\theta_t(x)} \right| \geq \delta \right\} \]

(2.63) \[\leq \Pr \left\{ \sup_{x \in \Sigma} \left| \theta_t(x) - \tilde{\theta}_t(x) \right| \geq \frac{4\delta t^d \lambda_0^2}{1 + 2\delta t^d/2 \lambda_0} \right\}.
\]

Let us write

\[
t^{-1/2} \left( \frac{\mu_n f(x, \cdot)}{\tilde{\theta}_t(x)} - \frac{\mu f(x, \cdot)}{\theta_t(x)} \right) = t^{-1/2} \mu_n f(x, \cdot) \left( \frac{1}{\tilde{\theta}_t(x)} - \frac{1}{\theta_t(x)} \right)
\]

\[+ t^{-1/2} \frac{1}{\theta_t(x)} [\mu_n f(x, \cdot) - \mu f(x, \cdot)].\]
Thus
\[
\Pr \left\{ \sup_{x \in \Sigma} \left| \frac{\mu_n f(x, \cdot)}{\hat{\theta}_t(x)} - \frac{\mu f(x, \cdot)}{\theta_t(x)} \right| \geq \varepsilon \right\} 
\leq \Pr \left\{ \sup_{x \in \Sigma} \left| \frac{1}{\hat{\theta}_t(x)} - \frac{1}{\theta_t(x)} \right| \geq \frac{\varepsilon t^{1/2}}{2 M} \right\} 
+ \Pr \left\{ \sup_{x \in \Sigma} \left| \mu_n f(x, \cdot) - \mu f(x, \cdot) \right| \geq \varepsilon \lambda_0 t^{(d+1)/2} \right\}. 
\]

Analyzing the first term using (2.58) and (2.62)-(2.63) leads us to the inequality
\[
\Pr \left\{ \sup_{x \in \Sigma} \left| \frac{1}{\hat{\theta}_t(x)} - \frac{1}{\theta_t(x)} \right| \geq \frac{\varepsilon t^{1/2}}{2 M} \right\} 
\leq \Pr \left\{ \sup_{x \in \Sigma} \left| \theta_t(x) - \hat{\theta}_t(x) \right| \geq \frac{2\varepsilon t^{d+1/2} \lambda_0^2}{M + \varepsilon t^{d+1/2} \lambda_0} \right\} 
\leq \Pr \left\{ \sup_{x \in \Sigma} \left| \theta_t(x) - \hat{\theta}_t(x) \right| \geq \frac{\varepsilon \lambda_0 t^{d+1/2} \lambda_0^2}{M} \right\} 
\leq 2 \mathcal{N} \left( G^t, \frac{\varepsilon \lambda_0 t^{d+1} \lambda_0}{4M} \right) \exp \left( -\frac{\varepsilon^2 t^{2d+1} \lambda_0^4}{8M^2 n} \right), 
\]
where we have applied Lemma 2.5 to the class $G^t$ (in particular we have used (2.24)).

Finally,
\[
\Pr \left\{ \sup_{x \in \Sigma} \left| \mu_n f(x, \cdot) - \mu f(x, \cdot) \right| \geq \varepsilon \lambda_0 t^{(d+1)/2} \right\} \leq 2 \mathcal{N} \left( F, \frac{\varepsilon \lambda_0 t^{(d+1)/2}}{4} \right) \exp \left( -\frac{\varepsilon^2 \lambda_0^2 t^{(d+1)/2} n}{8M^2} \right). 
\]

In view of Lemma 2.13, we introduce the following notation

**Definition 2.15.** Given a class of functions $F$ as in the statement of Lemma 2.13 and positive numbers $t, \varepsilon, M$ we define for compactness of notation the following function

\[
Q_t(F, \varepsilon, M, n) = 2 \mathcal{N} \left( G^t, \frac{\varepsilon \lambda_0 t^{d+1} \lambda_0}{4t^{1/2}M} \right) \exp \left( -\frac{\varepsilon^2 t^{2d+1} \lambda_0^4}{8(t^{1/2}M)^2 n} \right) 
+ 2 \mathcal{N} \left( F, \frac{\varepsilon \lambda_0 t^{(d+1)/2}}{4} \right) \exp \left( -\frac{\varepsilon^2 \lambda_0^2 t^{(d+1)/2} n}{8(t^{1/2}M)^2} \right).
\]

**Remark 2.16.** Because each one of these function classes satisfies a bound of the form (2.22), we include the $t^{1/2}$ factor in the expression for $M$. 

Corollary 2.17. For each of the classes of functions defined by (2.20), (2.21), (2.31), we have

\[
\Pr^* \left\{ \sup_{x \in \Sigma} \left| t^{-1/2} \frac{\mu_n f(x, \cdot)}{\hat{\theta}_t(x)} - t^{-1/2} \frac{\mu f(x, \cdot)}{\theta_t(x)} \right| \geq \varepsilon \right\} \leq Q_t(\mathcal{F}, \varepsilon, C, n)
\]

for some constant C depending on \(\Sigma\), and \(\|f\|_{\text{Lip}}\|h\|_{\text{Lip}}, \Sigma\), and \(\|h\|\) respectively.

As a corollary we obtain the rate of convergence in probability of the sample Carré du Champ to its expected value.

Corollary 2.18. Letting

\[
K = \inf \left\{ \left( \frac{2}{\varepsilon} \right) \|f\|_{\text{Lip}}\|h\|_{\text{Lip}}, \|f\|_{A-t-Lip}\|h\|_{\text{Lip}} \right\}
\]

we have

\[
\Pr \left\{ \sup_{x \in \Sigma} \left| \hat{\Gamma}_t(f, h)(x) - \Gamma_t(f, h)(x) \right| \geq \varepsilon \right\} \leq Q_t(\mathcal{F}^t_{f,h}, \varepsilon, K, n).
\]

Proof. Recall that

(2.67) \[
\hat{\Gamma}_t(L_t, f, h)(x) = \frac{1}{t} \sum_{j=1}^{n} e^{-\frac{|x-x_j|^2}{2t}} (f(x_j) - f(x))(h(x_j) - h(x))
\]

(2.68) \[
= \frac{t^{-1/2}}{\hat{\theta}_t(x)} \sum_{j=1}^{n} \phi_t(x, x_j) = t^{-1/2} \mu_n (\phi_t(x, \cdot)) \frac{\hat{\theta}_t(x)}{\theta_t(x)},
\]

where \(\phi_t\) is given by the definition of the class \(\mathcal{F}^t_{f,h}\) in (2.20) and therefore we can apply Lemma 2.13 to the class \(\mathcal{F}^t_{f,h}\) and use the bound

\[
\sup_{\phi \in \mathcal{F}^t_{f,h}} \|\phi\|_{L^\infty(\Sigma)} \leq t^{1/2} K.
\]

which follows from (2.22). \(\square\)

If we now consider the \(t\)-Laplacian \(L_t\) and its sample version \(\hat{L}_t\), we see that the deviation of \(\hat{L}_t\) from \(L_t\) on a function \(h \in L^\infty(\Sigma)\) with \(\|h\|_{L^\infty(\Sigma)} \leq M\) simplifies to

(2.69) \[
\hat{L}_t h(x) - L_t h(x) = \frac{2}{t \hat{\theta}_t(x)n} \sum_{j=1}^{n} e^{-\frac{|x-x_j|^2}{2t}} (h(x_j) - h(x)) - \frac{2}{t \theta_t(x)} \int_\Sigma e^{-\frac{|x-x|^2}{2t}} (h(\xi) - h(x)) d\mu(\xi)
\]

(2.70) \[
= \frac{2}{\hat{\theta}_t(x)tn} \sum_{j=1}^{n} e^{-\frac{|x-x_j|^2}{2t}} h(x_j) - \frac{2}{\theta_t(x)} \int_\Sigma e^{-\frac{|x-x|^2}{2t}} h(\xi) d\mu(\xi)
\]

(2.71) \[
= \frac{2\mu_n \psi_t(x, \cdot) h(\cdot)}{t^{3/2} \hat{\theta}_t(x)} - \frac{2\mu \psi_t(x, \cdot) h(\cdot)}{t^{3/2} \theta_t(x)}.
\]

(2.72)
Observe that
\[ Pr \left\{ \sup_{x \in \Sigma} \left\| \hat{L} t h(x) - L_t h(x) \right\| \geq \varepsilon \right\} \leq Pr \left\{ t^{-1/2} \sup_{\eta \in \mathcal{H}_t^h} \left| \frac{\mu_n \eta}{\theta_t} - \frac{\mu \eta}{\theta_t} \right| \geq t \varepsilon \right\}. \]

We have obtained

Corollary 2.19. Fix a function \( h \in L^\infty(\Sigma) \). If we set \( M = \| h \|_{L^\infty(\Sigma)} \) we have
\[ Pr \left\{ \sup_{x \in \Sigma} \left\| \hat{L} t h(x) - L_t h(x) \right\| \geq \varepsilon \right\} \leq Q_t(\mathcal{H}_t^h, \varepsilon t, M, n). \]

Proof. The proof follows from combining Lemma 2.13 with (2.73) and the fact that
\[ \sup_{\eta \in \mathcal{H}_t^h} \| \eta \|_{L^\infty(\Sigma)} \leq t^{1/2} \| h \|_{L^\infty(\Sigma)} = t^{1/2} M. \]

\[ \square \]

2.5. Subexponential Decay and Almost Sure Convergence. The goal of this subsection is to demonstrate that the decay rate for the quantities \( Q_t(F, \varepsilon, M, n) \) implies the almost sure convergence. We illustrate the Borel-Cantelli type proof in this section for the purpose of introducing some notation. This notation will be used in later sections.

Theorem 2.20. Consider the metric measure space \((\Sigma, \| \cdot \|, d\text{vol}_\Sigma)\) where \( \Sigma^d \subset \mathbb{R}^N \) is a smooth closed embedded submanifold. Suppose that we have a uniformly distributed i.i.d. sample \( \{\xi_1, \ldots, \xi_n\} \) of points from \( \Sigma \). For \( \sigma > 0 \), let
\[ t_n = n^{-\frac{1}{2d+\sigma}}. \]

Then we have for fixed \( f, h \in \text{Lip}(\Sigma) \)
\[ \left\| \hat{\Gamma}(L_{t_n}, f, h)(\xi) - \Gamma(L_{t_n}, f, h)(\xi) \right\| \xrightarrow{a.s.} 0 \]
as \( n \to \infty \).

Proof. For a fixed \( n \) and \( \varepsilon \), we have that
\[ Pr \left\{ \sup_{x \in \Sigma} \left| \hat{\Gamma}(L_{t_n}, f, h)(x) - \Gamma(L_{t_n}, f, h)(x) \right| \geq \varepsilon \right\} \leq Q_{t_n}(\mathcal{F}_{f,h}^n, \varepsilon, K, n). \]

Now plugging in the expression for \( t_n \) we observe a bound of the form
\[ Q_{t_n}(\mathcal{F}_{f,h}^n, \varepsilon, K) \leq p \left( n^{-\frac{1}{2d+\sigma}}, \frac{1}{\varepsilon} \right) \exp \left( -c_1 \varepsilon^2 n^{\sigma/(2d+\sigma)} \right) \]
where \( p \) is a fixed polynomial bound and \( c_1 > 0 \) is a constant. Thus with \( \varepsilon \) fixed, we have
\[ \sum_{n=1}^{\infty} Q_{t_n}(\mathcal{F}_{f,h}^n, \varepsilon, K, n) < \infty. \]

Applying the Borel-Cantelli Lemma gives the almost sure convergence. \[ \square \]
Now we see that for almost sure convergence, one requires a bound of the form (2.75). For this reason, we introduce notation for use in the sequel: consider a function
\[ Q : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{N} \to \mathbb{R}^+. \]

We say that \( Q(t, \varepsilon, n) \in O_{BC}(\beta) \) if
\[ \sum_{n=1}^{\infty} Q(n^{-\sigma}, \varepsilon, n) < \infty \]
for all \( \sigma, \varepsilon > 0 \). Clearly the definition gives
\[ O_{BC}(\beta) \subset O_{BC}(\beta') \]
for \( \beta' > \beta \). We also observe that
\[ O_{BC}(\beta) + O_{BC}(\beta') \in O_{BC}(\max\{\beta, \beta'\}). \]

**Lemma 2.21.** For the classes of functions defined by (2.20), (2.21), (2.31) and fixed \( \varepsilon, M > 0 \) we have
\[ Q_t(F, \varepsilon, M, n) \in O_{BC}(2d). \]

**Proof.** Plugging in
\[ t = n^{-\sigma/2} \]
the dominant exponential factor in (2.64) becomes
\[ \exp \left( -\frac{\varepsilon^2 \lambda^4}{8C} n^{\frac{\beta+\sigma-2d}{\beta+\sigma}} \right). \]

Clearly, for
\[ \beta + \sigma - 2d > 0 \]
we have
\[ \sum_{n=1}^{\infty} Q(n^{-\sigma/2}, \varepsilon, n) < \infty. \]

We record the following, which is clear from the definitions and Corollary 2.12.

**Corollary 2.22.** Given any of the classes above, let \( Q(t, \varepsilon, n) = Q_t(F, \varepsilon t^\alpha, Kt^\delta, n) \).

Then
\[ Q(t, \varepsilon, n) \in O_{BC}(2d + 2\alpha - 2\delta). \]
2.6. **Proof of Theorem 1.10.** Recall that

\[ \Gamma_2(L_t, f, f) = \frac{1}{2} (L_t (\Gamma_t(f, f)) - 2 \Gamma_t(L_t f, f)), \]

and that from Remark 2.2

\[ \hat{\Gamma}_2(L_t, f, f) = \frac{1}{2} \left( \hat{L}_t (\hat{\Gamma}_t(f, f)) - 2 \hat{\Gamma}_t(\hat{L}_t f, f) \right), \]

so we will start by estimating the difference \( \hat{L}_t (\hat{\Gamma}_t(f, f)) - L_t (\Gamma_t(f, f)) \) which we write as

\[ \left[ \hat{L}_t (\hat{\Gamma}_t(f, f)) - L_t (\Gamma_t(f, f)) \right] (x) = \hat{L}_t (\hat{\Gamma}_t(f, f) - \Gamma_t(f, f)) (x) + (\hat{L}_t - L_t) \Gamma_t(f, f)(x) \]

\[ = A_1(x) + A_2(x), \]

and observe that

\[ \|A_1\|_\infty = \left| \hat{L}_t \left( \hat{\Gamma}_t(f, f) - \Gamma_t(f, f) \right) \right| \leq \frac{4}{t} \sup_{x \in \Sigma} \left| \hat{\Gamma}_t(f, f)(\xi) - \Gamma_t(f, f)(\xi) \right| . \]

In order to estimate \( A_2 = (\hat{L}_t - L_t) (\Gamma_t(f, f)) \) we note from Proposition 1.9 that \( \Gamma_t(f, f) \) converges to \( |\nabla f|^2 \) in \( L_\infty(\Sigma) \) as \( t \to 0 \) and we can choose \( t_1 \) with \( 0 < t < t_1 < t_0 \) so that

\[ \|\Gamma_t(f, f)\|_{L_\infty(\Sigma)} \leq C_d \|f\|_{\text{Lip}}^2, \]

and the idea will be to use Lemma 2.19. We now estimate the difference

\[ \hat{\Gamma}_t(\hat{L}_t f, f)(x) - \Gamma_t(L_t f, f)(x) = \hat{\Gamma}_t(\hat{L}_t f - L_t f, f)(x) + (\hat{\Gamma}_t - \Gamma_t) (L_t f, f)(x) \]

\[ = A_3(x) + A_4(x), \]

and we note

\[ \|A_3\|_\infty = \sup_{x \in \Sigma} \left| \hat{\Gamma}_t(\hat{L}_t f - L_t f, f)(x) \right| \]

\[ \leq \sup_{x \in \Sigma} \frac{2}{t} \left( \sup_{\xi \in \Sigma} \left| \hat{L}_t(f)(\xi) - L_t(f)(\xi) \right| \right) \frac{1}{n^\theta_t(x)} \sum_{j=1}^n e^{-\frac{\|\xi_j - x\|^2}{2t}} |f(\xi_j) - f(x)| \]

\[ \leq 2 e^{-1/2} \|f\|_{\text{Lip}} \sup_{x \in \Sigma} \frac{1}{\theta_t(x)} \left( \sup_{\xi \in \Sigma} \left| (\hat{L}_t(f)(\xi) - L_t(f)(\xi)) \right| \right) , \]
where we have used the fact that for functions $u, v$

$$\left| \hat{\Gamma}_t(u, v)(x) \right| \leq \frac{2}{t} \sup_{\xi \in \Sigma} |u(\xi)| \left( \frac{1}{n\theta_t(x)} \sum_{j=1}^{n} e^{-|x-\xi_j|^2/2\pi} |v(x) - v(\xi_j)| \right)$$

(2.86)

$$\leq \frac{2}{t} \sup_{\xi \in \Sigma} |u(\xi)| \left( \frac{\|v\|_{Lip}}{n\theta_t(x)} \sup_{\rho > 0} \rho e^{-\frac{\rho^2}{2\pi}} \right),$$

(2.87)

and

$$\sup_{\rho > 0} \rho e^{-\frac{\rho^2}{2\pi}} = t^{1/2} e^{-1/2} \leq t^{1/2}.$$  

In view of Lemma (2.13), we will write the bound (2.85) on $\|A_3\|_{\infty}$ as

$$\|A_3\|_{\infty} \leq 2 e^{-1/2} \|f\|_{Lip} \sup_{x \in \Sigma} \frac{1}{\theta_t(x)} \left( \sup_{\xi \in \Sigma} \left| \hat{L}_t(f)(\xi) - L_t(f)(\xi) \right| \right)$$

(2.88)

$$+ \frac{2e^{-1/2} \|f\|_{Lip}}{t^{1/2}} \sup_{x \in \Sigma} \left( \frac{1}{\theta_t(x)} - \frac{1}{\theta_t(x)} \right) \left( \sup_{\xi \in \Sigma} \left| \hat{L}_t(f)(\xi) - L_t(f)(\xi) \right| \right).$$

(2.89)

In order to estimate $|A_4|$, i.e.

$$\|A_4\|_{\infty} = \left| \sup_{x \in \Sigma} \left( \hat{\Gamma}_t - \hat{\Gamma}_t \right)(L_t(f, f))(x) \right|,$$

we consider the class of functions $\mathcal{F}_{L_tf,f}^i$ introduced in (2.20) so that we can use Lemma 2.18, however, this requires a Lipschitz or almost Lipschitz estimate on the functions $L_t f$. Considering the formula (1.14) we see that

$$\|L_t f\|_{\Lambda-t-Lip} \leq C_0$$

(2.90)

where $C_0$ depends on $\Sigma$, $f$ and $t_0$.

Let $A_i = \sup_{x \in \Sigma} |A_i(x)|$ for $i = 1, 2, 3, 4$. We now have from (2.76) and (2.81)

$$\Pr \left\{ \sup_{x \in \Sigma} \left| \hat{\Gamma}_2(L_t, f, f) - \Gamma_2(L_t, f, f) \right| (x) \geq \varepsilon \right\}$$

(2.91)

$$\leq \Pr \left\{ \|A_1\|_{\infty} \geq \frac{\varepsilon}{4} \right\} + \Pr \left\{ \|A_2\|_{\infty} \geq \frac{\varepsilon}{4} \right\} + \Pr \left\{ \|A_3\|_{\infty} \geq \frac{\varepsilon}{4} \right\} + \Pr \left\{ \|A_4\|_{\infty} \geq \frac{\varepsilon}{4} \right\}$$

(2.92)

$$= P_1 + P_2 + P_3 + P_4.$$

From (2.79) and Corollary 2.18 we have.

$$P_1 \leq \Pr \left\{ \sup_{x \in \Sigma} \left| \hat{\Gamma}_1(f, f)(\xi) - \Gamma_1(f, f)(\xi) \right| \geq \frac{t\varepsilon}{16} \right\} \leq Q_t \left( \mathcal{F}_{f, t\varepsilon/16}^i, K, n \right),$$

(2.94)

$$\in O_{BC} (2d + 2).$$

The statement about the convergence order follows from Corollary 2.22.
For $A_2 = \left( \hat{L}_t - L_t \right) (\Gamma_t(f, f))$ we apply Corollary 2.19 and the estimate (2.63) to the class $\mathcal{H}_{t}(f, f)$ and obtain

$$P_2 \leq Q_t \left( \mathcal{H}_{t}(f, f), \varepsilon t, 2\|f\|_{\text{Lip}}, n \right) \in O_{BC}(2d + 2).$$

Using (2.89) we have

$$P_3 \leq \Pr \left\{ \varepsilon \leq 2 \frac{e^{-1/2}\|f\|_{\text{Lip}}}{t^{1/2} \theta_t} \sup_{x \in \Sigma} \frac{1}{\theta_t(x)} \left( \sup_{\xi \in \Sigma} \left| (\hat{L}_t(f)(\xi) - L_t(f)(\xi)) \right| \right) \right\}$$

$$+ \Pr \left\{ \varepsilon \leq 2 \frac{e^{-1/2}\|f\|_{\text{Lip}}}{t^{1/2}} \sup_{x \in \Sigma} \left( \frac{1}{\theta_t(x)} - \frac{1}{\theta_t(x)} \right) \left( \sup_{\xi \in \Sigma} \left| (\hat{L}_t(f)(\xi) - L_t(f)(\xi)) \right| \right) \right\}$$

and choosing $t > 0$ small enough we obtain from (2.57) and Corollary 2.19 the estimate

$$\Pr \left\{ \varepsilon \leq 2 \frac{e^{-1/2}\|f\|_{\text{Lip}}}{t^{1/2} \theta_t} \left( \sup_{\xi \in \Sigma} \left| (\hat{L}_t(f)(\xi) - L_t(f)(\xi)) \right| \right) \right\}$$

$$\leq \Pr \left\{ \varepsilon \lambda_0 t^{\frac{d+2}{2}} e^{1/2} \frac{1}{\|f\|_{\text{Lip}}} \leq \sup_{\xi \in \Sigma} \left| (\hat{L}_t(f)(\xi) - L_t(f)(\xi)) \right| \right\}$$

$$\leq Q_t \left( \mathcal{H}_{t}(f), \varepsilon \lambda_0 t^{\frac{d+1}{2}} e^{1/2} \frac{1}{\|f\|_{\text{Lip}}} t, \|f\|_{L^\infty}, n \right) \in O_{BC}(3d + 3).$$

Here the increased order is $t^{\frac{d+1}{2}}$ and $t$ which is $2d + d + 1 + 2$.

On the other hand, we will make use of the estimate

$$\sup_{\xi \in \Sigma} \left| (\hat{L}_t(f)(\xi) - L_t(f)(\xi)) \right| \leq \frac{4}{t} \|f\|_{L^\infty},$$

for any $t > 0$ to obtain

$$\Pr \left\{ \varepsilon \leq 2 \frac{e^{-1/2}\|f\|_{\text{Lip}}}{t^{1/2}} \left( \frac{1}{\theta_t(x)} - \frac{1}{\theta_t(x)} \right) \left( \sup_{\xi \in \Sigma} \left| (\hat{L}_t(f)(\xi) - L_t(f)(\xi)) \right| \right) \right\}$$

$$\leq \Pr \left\{ \varepsilon t^{3/2} e^{1/2} \frac{1}{64\|f\|_{\text{Lip}}} \leq \sup_{x \in \Sigma} \left| \frac{1}{\theta_t(x)} - \frac{1}{\theta_t(x)} \right| \right\}$$

and from (2.62)-(2.63) and Lemma 2.1 we observe that

$$\Pr \left\{ \varepsilon \leq 2 \frac{e^{-1/2}\|f\|_{\text{Lip}}}{t^{1/2}} \left( \frac{1}{\theta_t} - \frac{1}{\theta_t} \right) \left( \sup_{\xi \in \Sigma} \left| (\hat{L}_t(f)(\xi) - L_t(f)(\xi)) \right| \right) \right\} \in O_{BC}(d + 3).$$

We then conclude that

$$P_3 \in O_{BC}(3d + 3).$$
Finally, in view of (2.90) and Corollary 2.18
\[
\Pr \left\{ \sup_{x \in \Sigma} \left| \hat{\Gamma}_t (L_t f, f) (x) - \Gamma_t (L_t f, f) (x) \right| \geq \varepsilon \right\} \leq Q_t (F_{L_t f, f}, \varepsilon, K, n).
\]
where
\[
K = \|L_t f\|_{\Lambda-t-Lip} \|f\|_{Lip}
= C_0 \|f\|_{Lip}
\]
so
\[
(2.105) \quad P_4 = \Pr \left\{ \|A_4\|_{\infty} \geq \frac{\varepsilon}{4} \right\} \leq Q_t (F_{L_t f, f}, \frac{\varepsilon}{4}, K, n) \in O_{BC} (2d).
\]
having used
\[
\sup_{l \in F_{L_t f, f}} \|l\|_{L^\infty (\Sigma)} \leq C_0.
\]
Using again
\[
\Pr \left\{ \left| \hat{\Gamma}_2 (L_t, f, f) - \Gamma_2 (L_t, f, f) \right| \geq \varepsilon \right\} \leq P_1 + P_2 + P_3 + P_4,
\]
and from (2.95), (2.96), (2.104) and (2.105) we have
\[
\Pr \left\{ \left| \hat{\Gamma}_2 (L_t, f, f) - \Gamma_2 (L_t, f, f) \right| \geq \varepsilon \right\} \in O_{BC} (3d + 3),
\]
and it follows from the Borel-Cantelli argument given in Section 2.5 that for any sequence of the form \( t_n = n^{-\gamma} \) where \( \gamma = \frac{1}{3d+3+\sigma} \) and \( \sigma \) is any positive number we have
\[
\sup_{\xi \in \Sigma} \left| \hat{\Gamma}_2 (L_{t_n}, f, f) - \Gamma_2 (L_{t_n}, f, f) \right| \overset{a.s.}{\longrightarrow} 0.
\]
Now with the convergence for each fixed function \( f \) we can prove Corollary 1.11.

**Proof.** We work on a compact smooth submanifold of Euclidean space. Thus with the ambient distance function, there is no cut locus, and the set of functions given by
\[
R = \{ F_{x,y} : (x, y) \in \Sigma \times \Sigma \}
\]
is uniformly bounded in \( C^5 \). The map
\[
\Sigma \times \Sigma \to C^5 (\Sigma)
(x, y) \to F_{x,y}
\]
is a Lipschitz map. It follows that we can take a finite net \( G \) with respect to the \( C^5 \) topology, and the net size will grow at worst polynomially. That is for a given
\[
f \in R
\]
there exists
\[
f^* \in G
\]
such that
\[
\|f - f^*\|_{C^5} < \delta.
\]
We want to estimate the probability that
\[ P = \Pr \left\{ \sup_{f \in \mathcal{R}} \sup_{\xi \in \Sigma} \left| \hat{\Gamma}_2(L_t, f, f)(\xi) - \Gamma_2(L_t, f, f)(\xi) \right| \geq \varepsilon \right\}. \]

Then we have
\[
\hat{\Gamma}_2(L_t, f, f)(\xi) - \Gamma_2(L_t, f, f)(\xi) = \hat{\Gamma}_2(L_t, f^*, f^*)(\xi) - \Gamma_2(L_t, f^*, f^*)(\xi) \\
+ \Gamma_2(L_t, f^*, f^*)(\xi) - \Gamma_2(L_t, f, f)(\xi).
\]
Thus,
\[
P \leq \Pr \left\{ \sup_{f \in \mathcal{R}} \sup_{\xi \in \Sigma} \left| \hat{\Gamma}_2(L_t, f, f)(\xi) - \hat{\Gamma}_2(L_t, f^*, f^*)(\xi) \right| \geq \frac{\varepsilon}{3} \right\} \\
+ \Pr \left\{ \sup_{f \in \mathcal{R}} \sup_{\xi \in \Sigma} \left| \hat{\Gamma}_2(L_t, f^*, f^*)(\xi) - \Gamma_2(L_t, f^*, f^*)(\xi) \right| \geq \frac{\varepsilon}{3} \right\} \\
+ \Pr \left\{ \sup_{f \in \mathcal{R}} \sup_{\xi \in \Sigma} \left| \Gamma_2(L_t, f^*, f^*)(\xi) - \Gamma_2(L_t, f, f)(\xi) \right| \geq \frac{\varepsilon}{3} \right\} \\
\leq P_1 + P_2 + P_3.
\]
This requires two estimates:

If
\[ \|f - f^*\|_{C^5} \leq \delta \]
then
\[
|\Gamma_2(L_t, f^*, f^*)(\xi) - \Gamma_2(L_t, f, f)(\xi)| \leq C_d \delta + C_d M_5 t^{1/2}
\]
and
\[
|\hat{\Gamma}_2(L_t, f^*, f^*)(\xi) - \hat{\Gamma}_2(L_t, f, f)(\xi)| \leq C_d M_5 \frac{1}{t^2}
\]
So we choose
\[ \delta = \frac{\varepsilon}{6C_d t^{2+\sigma}} \]
and only consider the case where
\[ t^{1/2} < \frac{\varepsilon}{6C_d M_5} \]
then we have
\[ P_1 = P_3 = 0. \]

We are left to estimate
\[
\Pr \left\{ \sup_{f \in \mathcal{R}} \sup_{\xi \in \Sigma} \left| \hat{\Gamma}_2(L_t, f^*, f^*)(\xi) - \Gamma_2(L_t, f^*, f^*)(\xi) \right| \geq \frac{\varepsilon}{3} \right\} \\
\in \mathcal{N} \left( \mathcal{R}, \frac{\varepsilon}{6C_d t^{2+\sigma}} \right) O_{BC}(3d + 3).
\]
To finish the proof, we need only to not that \( \mathcal{N} \left( \mathcal{R}, \frac{\varepsilon}{6C_d t^{2+\sigma}} \right) \) is a polynomial in \( \frac{1}{t} \).
The first estimate (2.106) follows from Theorem (1.6).

References


[AWb] ———, *Coarse ricci curvature as a function on m x m*.


