NON-POLYNOMIAL ENTIRE SOLUTIONS TO $\sigma_k$ EQUATIONS

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Abstract. For $2k = n+1$, we exhibit non-polynomial solutions to the Hessian equation

$$\sigma_k(D^2u) = 1$$
on all of $\mathbb{R}^n$.

1. INTRODUCTION

In this note, we demonstrate the following.

Theorem 1. For

$$n \geq 2k - 1,$$

there exist non-polynomial elliptic entire solutions to the equation

$$\sigma_k(D^2u) = 1$$
on $\mathbb{R}^n$.

Corollary 2. For all $n \geq 3$, there exist on $\mathbb{R}^n$ non-polynomial entire solutions to

$$\sigma_3(D^2u) = 1.$$  

For $k = 1$, the entire harmonic functions in the plane arising as real parts of analytic functions are classically known. For $k = n$, the famous Bernstein result of Jörgens [5], Calabi [1], and Pogorelov [6] states that all entire solutions to the Monge-Ampère equation are quadratic. Chang and Yuan [2] have shown that any entire convex solution to (2) in any dimension must be quadratic. To the best of our knowledge, for $1 < k < n$, the examples presented here are the first known non-trivial entire solutions to $\sigma_k$ equations.

The special Lagrangian equation is the following

$$\sum_{i=1}^{n} \arctan \lambda_i = \theta$$

(here $\lambda_i$ are eigenvalues of $D^2u$) for

$$\theta \in \left(-\frac{n}{2}\pi, \frac{n}{2}\pi\right)$$
a constant. Fu [3] showed that when $n = 2$ and $\theta \neq 0$ all solutions are quadratic. When $n = 2$ and $\theta = 0$ the equation (3) becomes simply the Laplace equation, which admits well-known non-polynomial solutions. Yuan [8] showed that all convex solutions to special Lagrangian equations are quadratic.

The critical phase for special Lagrangian equations is

$$\theta = \frac{n - 2}{2}\pi.$$  

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Yuan [9] has shown that for values above the critical phase, all entire solutions are quadratic. On the other hand, by adding a quadratic to a harmonic function, one can construct nontrivial entire solutions for phases

$$|\theta| < \frac{n-2}{2} \pi.$$ 

By [4] when $n = 3$, the critical equation

$$\sum_{i=1}^{3} \arctan \lambda_i = \frac{\pi}{2}$$

is equivalent to the equation (2). Thus Corollary 2 answers the critical phase Bernstein question when $n = 3$. In the process, we also show the following.

**Theorem 3.** There exists a special Lagrangian graph in $\mathbb{C}^3$ over $\mathbb{R}^3$ that does not graphically split.

Harvey and Lawson [4], show that a graph

$$(x, \nabla u(x)) \subset \mathbb{C}^n$$

is special Lagrangian and a minimizing surface if and only if $u$ satisfies (3). We say a graph splits graphically when the function $u$ can be written the sum of two functions in independent variables.

There are still many holes in the Bernstein picture for $\sigma_k$ equations. To begin with, when $n = 4$ the existence of interesting solutions to $\sigma_3 = 1$.

For special Lagrangian equations the existence of critical phase solutions when $n \geq 4$ is open.

2. Proof

We will assume that $n$ is odd and

$$2k = n + 1.$$ 

We construct a solution $u$ on $\mathbb{R}^n$. The general result will follow by noting that if we define

$$\tilde{u} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$$

via

$$\tilde{u}(z, w) = u(z)$$

then

$$\sigma_k(D^2 \tilde{u}) = \sigma_k(D^2 u) = 1.$$ 

Consider functions on $\mathbb{R}^{n-1} \times \mathbb{R}$ of the form

$$u(x, t) = r^2 e^t + h(t)$$

where

$$r = \left( x_1^2 + x_2^2 + ... + x_{n-1}^2 \right)^{1/2}.$$
Compute the Hessian, rotating $\mathbb{R}^{n-1}$ so that $x_1 = r$:

$$D^2 u = \begin{pmatrix} 2e^t & 0 & \ldots & 0 & 2re^t \\ 0 & 2e^t & \ldots & 0 & 0 \\ \ldots & 0 & \ldots & 0 & \ldots \\ 0 & \ldots & 0 & 2e^t & 0 \\ 2re^t & 0 & \ldots & 0 & r^2 e^t + h''(t) \end{pmatrix}.$$  

(4)

We then compute. The $k$-th symmetric polynomial is given by the sum of $k$-minors. Let

$$S = \{ \alpha \subset \{1, \ldots, n\} : |\alpha| = k \},$$

and let

$$A = \{ \alpha \in S : 1 \in \alpha \}$$

$$B = \{ \alpha \in S : n \in \alpha \}.$$ 

We express $S$ as a disjoint union

$$S = (A \cap B) \cup (B \setminus A) \cup (S \setminus B).$$

Define

$$\sigma_k^{(\alpha)} = \det \begin{pmatrix} k \times k \text{ matrix with row and columns chosen from } \alpha \end{pmatrix}.$$ 

For $\alpha \in (A \cap B)$ we have

$$\sigma_k^{(\alpha)} = \det \left( e^t \begin{pmatrix} 2 & 0 & \ldots & 0 & 2r \\ 0 & 2 & 0 & \ldots & 0 \\ \ldots & 0 & \ldots & 0 & \ldots \\ 0 & \ldots & 0 & 2 & 0 \\ 2r & 0 & \ldots & 0 & r^2 + e^{-t} h'' \end{pmatrix} \right),$$

that is

$$\sigma_k^{(\alpha)} = e^{kt} 2^{k-2} \left( 2r^2 + 2e^{-t} h'' - 4r^2 \right).$$

Next, for $\alpha \in B \setminus A$,

$$\sigma_k^{(\alpha)} = \det \left( e^t \begin{pmatrix} 2 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 \\ \ldots & 0 & 2 & \ldots \\ 0 & \ldots & r^2 + e^{-t} h'' \end{pmatrix} \right),$$

that is

$$\sigma_k^{(\alpha)} = e^{kt} 2^{k-1} \left( r^2 + e^{-t} h'' \right).$$

Finally, for $\alpha \in (S \setminus B)$ we have

$$\sigma_k^{(\alpha)} = \det \left( e^t \begin{pmatrix} 2 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 2 \end{pmatrix} \right),$$
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that is
\[ \sigma_k^{(\alpha)} = e^{k \alpha^2}. \]

We sum these up:
\[ \sigma_k (D^2 u) = \sum_{\alpha \in (A \cap B)} \sigma_k^{\alpha} + \sum_{\alpha \in (B \setminus A)} \sigma_k^{\alpha} + \sum_{\alpha \in (S \setminus B)} \sigma_k^{\alpha}. \]

Counting, we get
\[ \sigma_k (D^2 u) = \left( n - 2 \right) e^{k t} 2^{k-1} \left( e^{-t} h'' - r^2 \right) + \left( n - 2 \right) e^{k t} 2^{k-1} \left( r^2 + e^{-t} h'' \right) + \left( n - 1 \right) e^{k t} 2^k. \]

Grouping the terms, we see
\[ \sigma_k (D^2 u) = e^{k t} 2^{k-1} \left[ \frac{n - 2}{k - 2} + \frac{n - 2}{k - 1} \right] r^2 + e^{k t} 2^{k-1} \left[ \frac{n - 2}{k - 2} + \frac{n - 2}{k - 1} \right] e^{-t} h'' + e^{k t} 2^{k-2} \left( n - 1 \right) e^{k t} 2^k. \]

Now
\[ - \left( \frac{n - 2}{k - 2} + \frac{n - 2}{k - 1} \right) = - \frac{(n - 2)!}{(n - k)!(k - 2)!} + \frac{(n - 2)!}{(n - k - 1)!(k - 1)!}. \]

This vanishes if and only if
\[ 1 = \frac{(n - k)!(k - 2)!}{(n - k - 1)!(k - 1)!} = \frac{(n - k)}{(k - 1)}, \]

or precisely when
\[ n - k = k - 1 \]
or
\[ 2k = n + 1. \]

Thus for this choice of \( k \), (6) becomes
\[ \sigma_k (D^2 u) = A_{n,k} e^{(k-1)t} h'' + B_{n,k} e^{kt} \]

for some constants \( A_{n,k}, B_{n,k} \). Setting to this expression to 1, we solve for \( h''(t) \)
\[ (7) \]
\[ h''(t) = \frac{1 - B_{n,k} e^{kt}}{A_{n,k} e^{(k-1)t}}, \]

noting the right-hand side is a smooth function in \( t \). Integrating twice in \( t \) yields solutions to (7) and hence to (1).

To see that the equation is elliptic, we first note that inspecting (4) the \( n - 2 \) eigenvalues in the middle must be positive. Of the remaining two, at least one must be positive as the diagonal (of the \( 2 \times 2 \) matrix) contains at least one positive entry. We then note the following.
Lemma 4. Suppose that 
\[ \sigma_k(D^2 u) > 0 \]
and \( D^2 u \) has at most 1 negative eigenvalue. Then \( D^2 u \in \Gamma_k^+ \).

Proof. Diagonalize \( D^2 u \) so that \( D^2 u = \text{diag} \{ \lambda_1, \ldots, \lambda_n \} \) with \( 0 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \). Clearly
\[
\frac{d}{ds} \sigma_k(\text{diag} \{ \lambda_1 + s, \lambda_2, \ldots, \lambda_n \}) \geq 0
\]
so we may deform \( D^2 u \) to a positive definite matrix \( D^2 u + sM \), with \( \sigma_k(D^2 u + sM) > 0 \) for \( s \geq 0 \). Thus \( D^2 u \) is in the component of \( \sigma_k > 0 \) containing the positive cone, that is, \( D^2 u \in \Gamma_k^+ \). 

Example 5. When \( n = 3 \) the function
\[
u(x, y, t) = (x^2 + y^2)e^t + \frac{1}{4}e^{-t} - e^t
\]
solves
\[ \sigma_2(D^2 u) = 1. \]

Remark 6. This method allows one to construct solutions to complex Monge-Ampère equations as well. See [7].

References


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