Hessian estimates for the sigma-2 equation in dimension three

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(joint work with Yu Yuan)

We derive an interior a priori Hessian estimate for the \( \sigma_2 \) equation
\[
\sigma_2(D^2u) = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = 1
\]
in dimension three, where \( \lambda_i \) are the eigenvalues of the Hessian \( D^2u \).

**Theorem 1.** Let \( u \) be a smooth solution to \( (1) \) on \( B_R(0) \subset \mathbb{R}^3 \). Then we have
\[
|D^2u(0)| \leq C(3) \exp \left[ C(3) \max_{B_{R/3}(0)} |Du|^3 / R^3 \right].
\]

In the 1950’s, Heinz [H] derived a Hessian bound for the two dimensional Monge-Ampère equation, \( \sigma_2(D^2u) = \lambda_1 \lambda_2 = \det(D^2u) = 1 \), which is equivalent to \( (2) \) with \( n = 2 \) and \( \Theta = \pm \pi/2 \). In the 1970’s Pogorelov [P] constructed his famous counterexamples, namely irregular solutions to three dimensional Monge-Ampère equations \( \sigma_3(D^2u) = \lambda_1 \lambda_2 \lambda_3 = \det(D^2u) = 1 \).

By Trudinger’s [T] gradient estimates for \( \sigma_k \) equations, we can bound \( D^2u \) in terms of \( u \) as
\[
|D^2u(0)| \leq C(3) \exp \left[ C(3) \max_{B_{R/6}(0)} |u|^3 / R^6 \right].
\]

We attack \( (1) \) via its special Lagrangian equation form
\[
\sum_{i=1}^n \arctan \lambda_i = \Theta
\]
with \( n = 3 \) and \( \Theta = \pi/2 \). Equation \( (2) \) stems from the special Lagrangian geometry [HL]. The Lagrangian graph \((x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n\) is called special when the phase or the argument of the complex number \((1 + \sqrt{-1} \lambda_1) \cdots (1 + \sqrt{-1} \lambda_n)\) is constant \( \Theta \), and it is special if and only if \((x, Du(x))\) is a (volume minimizing) minimal surface in \( \mathbb{R}^n \times \mathbb{R}^n \) [HL, Theorem 2.3, Proposition 2.17].

In dimensions two and three, the special Lagrangian equations \( (2) \) can be expressed as
\[
\cos \Theta(\sigma_1 - \sigma_3) + \sin \Theta(\sigma_2 - 1) = 0.
\]

1. **Outline of Proof**

Heuristically, the proof breaks into the following three steps. We may assume by scaling that \( u \) is a solution on \( B_1(0) \subset \mathbb{R}^3 \).

Step 1) Choose a function \( b(D^2u(x)) \) such that, with respect to the induced metric on the graph \((x, Du(x))\), \( b \) satisfies the (weak) Jacobi inequality
\[
\Delta_g b \geq |\nabla_g b|^2_{g_b}.
\]

We choose
\[ b(D^2u) = \ln \sqrt{1 + \lambda_{\text{max}}^2} \]

where \( \lambda_{\text{max}} \) is the largest eigenvalue of the Hessian \( D^2u \).

Step 2) By Michael and Simon’s mean value and Sobolev inequalities for minimal surfaces, it follows from Step 1 (choosing appropriate exponents) that

\[ b(0) \leq C(3) \left[ \int_{B_2} \varphi^2 |\nabla g|^2 V \, dx + \int_{B_2} b |\nabla g\varphi|^2 V \, dx \right] \]

where \( V \) is the volume element on the graph, and \( \varphi \) is an appropriately chosen test function.

Step 3) Finally, we show that the integrals in Step 2 may be bounded by \( \|Du\|_{L^\infty(B_4)} \). In fact, Step 1) implies that

\[ \int_{B_2} \varphi^2 |\nabla g|^2 \, dv_g \leq C(3) \int_{B_2} |\nabla g\varphi|^2 \, dv_g. \]

The identity

\[ g^\mu \nu V = \sigma_1 - \lambda_i \]

yields an estimate

\[ \int_{B_2} |\nabla g\varphi|^2 V \, dx \leq C(3) \int_{B_2} \Delta u \, dx \leq \|Du\|_{L^\infty(B_2)}. \]

Further integration by parts, using the above identity, completes the estimate.

2. Questions

1) Estimates for \( \sigma_2(D^2u) = 1 \) when \( n \geq 4 \)? The special Lagrangian structure is available only when \( n = 2 \) or \( 3 \), so a challenge is to replace the mean value and Sobolev inequalities in Step 2.

2) Estimates for \( \sigma_2(D^2u) = f(x, u, Du) \)? What conditions on \( f(x, u, Du) \)? The mean curvature is not given by a simple expression in terms of derivatives of \( f \), so this generalization does not follow immediately.

3) Similar estimates for special Lagrangian equations (2) with larger phase \( |\Theta| \geq \pi/2 \) in dimension three are obtained in [WY]. The challenging problem is to derive estimates for (2) with lower phase, in particular the equation \( \sigma_1 = \sigma_3 \) corresponding to \( \Theta = 0 \).

References


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