Reconstruction theorems in noncommutative Riemannian geometry

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A reconstruction theorem for almost-commutative spectral triples,

Erratum


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Real structures on almost-commutative spectral triples,

Erratum

References

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Dirac-type operators

**Definition**

Let
- \((X, g)\) be a compact oriented Riemannian manifold,
- \(E \to X\) be a Hermitian vector bundle.

A *Dirac-type operator* \(D\) on \(E \to (X, g)\) is a symmetric first-order differential operator with \(D^2\) Laplace-type, i.e., locally

\[
D^2 = -g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \text{lower order terms}.
\]

**Example**

- \(\slashed{D}\) on the spinor bundle \(S\) for \(X\) spin;
- \(d + d^*\) on \(\wedge T^*X\) for general \(X\).
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**Definition**

A *spectral triple* is $(A, H, D)$ for

- $H$ a Hilbert space;
- $A$ a unital $\ast$-subalgebra of $B(H)$;
- $D$ a (densely-defined) self-adjoint operator on $H$ such that
  - $D$ has compact resolvent,
  - $[D, a] \in B(H)$ for all $a \in A$.

**Example**

For $(X, g)$ cpct. orient. Riem., $E \to X$ Herm., $D$ Dirac-type on $E$,

$$(A, H, D) := (C^\infty(X), L^2(X, E), D).$$
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The reconstruction theorem


Let \((A, H, D)\) be a \(p\)-dimensional commutative spectral triple. Then there exists an compact oriented \(p\)-manifold \(X\) such that

\[ A \cong C^\infty(X). \]

Moreover, if \(A''\) acts on \(H\) with multiplicity \(2^\lfloor p/2 \rfloor\), then

- \(X\) is spin\(^\mathbb{C}\),
- \(H \cong L^2(X, S)\) for \(S \to X\) a spinor module,
- \(D\) is an ess. self-adjoint Dirac-type operator on \(S\).

See [Gracia-Bondía–Várilly–Figueroa, Ch. 11] for reconstruction of the spinor bundle \(S\) and the Riemannian metric on \(X\).
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Definition (Connes 1996, 2008)

A spectral triple \((A, H, D)\) with \(A\) commutative is called a \(p\)-dimensional commutative spectral triple if it satisfies the following:

**Metric dimension (Weyl’s Law)**

\[ \lambda_k ((D^2 + 1)^{-1/2}) = O(k^{-1/p}); \]

**Order one (First-order differential operator)**

\[ [[D, a], b] = 0 \text{ for all } a, b \in A; \]
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**Strong regularity (Smoothness, ellipticity)**

- \( A + [D, A] \subseteq \bigcap_k \text{Dom} \delta^k \), where \( \delta(T) := [|D|, T] \),
- \( \text{End}_A(\bigcap_k \text{Dom} D^k) \subseteq \bigcap_k \text{Dom} \delta^k \) as well;

**Orientability**

There exists an antisymmetric Hochschild cycle \( c \in Z_p(A, A) \) such that for \( \chi := \pi_D(c) \),

- \( \chi = 1 \) if \( p \) is odd,
- \( \chi^* = \chi, \chi^2 = 1, [\chi, A] = \{0\}, \chi D = -D \chi \) if \( p \) even;
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Finiteness (Vector bundle)

\[ H_\infty := \bigcap_k \text{Dom } D_k \text{ is finitely-generated projective } A\text{-module}; \]

Absolute continuity (Integration, Hermitian metric)

The \( A\)-module \( H_\infty \) admits a Hermitian metric \((\cdot, \cdot)\) defined by

\[ \langle \xi, \eta \rangle := \text{Tr}^+ \left( (\xi, \eta) (D^2 + 1)^{-p/2} \right), \quad \forall \xi, \eta \in H_\infty. \]
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Problems and solutions

Definition is tailored to a spin\(^C\) Dirac operator on a spinor bundle:

- Orientability condition too strong in general;
- Definition too weak in general to recover Riemannian metric.

**Weak orientability**

There exists an antisymmetric Hochschild cycle \(c \in Z_p(A, A)\) such that for \(\chi := \pi_D(c)\),

\[
\chi = \chi^*, \quad \chi^2 = 1, \quad \text{and} \quad \chi a = a \chi \quad \text{for all} \quad a \in A,
\]

\[
\chi [D, a] = (-1)^{p+1} [D, a] \chi \quad \text{for all} \quad a \in A.
\]

**Dirac-type**

For all \(a \in A\), \([D, a]^2 \in A\).
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### Weak orientability

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### Dirac-type

For all \(a \in A\), \([D, a]^2 \in A\).
Theorem (cf. Gracia-Bondía–Várilly–Figueroa, Ch. 11)

Let \((A, H, D)\) be a \(p\)-dimensional (weakly orientable!) Dirac-type commutative spectral triple. Then there exist:

- a compact oriented Riemannian \(p\)-manifold \(X\),
- a Hermitian vector bundle \(\mathcal{E} \to X\),

such that, up to unitary equivalence,

\[(A, H, D) = (C^\infty(X), L^2(X, \mathcal{E}), D)\]

for \(D\) symmetric Dirac-type on \(\mathcal{E}\).
The key technical lemma

Lemma

Let \((A, H, D)\) be a \(p\)-dimensional commutative spectral triple, and let \(\chi = \pi_D(c)\) be its chirality operator. Then there exists a self-adjoint \(M \in \text{End}_A(\cap_k \text{Dom } D^k)\) such that for

\[
D' = \begin{cases} 
D - M & \text{if } p \text{ is even}, \\
\chi(D - M) & \text{if } p \text{ is odd},
\end{cases}
\]

\((A, H, D')\) is an orientable \(p\)-dimensional commutative spectral triple. In particular, \(\text{Dom } D^k = \text{Dom}(D')^k\) for each \(k\) with comparable Sobolev norms, and

\[
(D^2 + 1)^{-p/2} - ((D')^2 + 1)^{-p/2} \in \mathcal{L}^1(H).
\]
The conventional definition

Definition
Let:

- $X$ be a compact spin $n$-manifold with spinor bundle $S$, Dirac operator $\slashed{D}$, and chirality $\chi$;
- $(A_F, H_F, D_F)$ be a finite spectral triple.

Then the spectral triple $X \times F$ is called an almost-commutative spectral triple, where, e.g.,

$$X \times F := (C^\infty(X) \otimes A_F, L^2(X, S) \otimes H_F, \slashed{D} \otimes 1 + \chi \otimes D_F)$$

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when $n$ is even.
This definition explicitly requires the base manifold to be spin.

This definition does not accommodate “non-trivial fibrations”:

- Topologically non-trivial NCG Einstein–Yang–Mills (Boeijink–v. Suijlekom 2011);
- Noncommutative tori for $\theta$ rational;
- *Projective spectral triples* on compact oriented Riemannian manifolds (Zhang 2009, cf. Mathai–Melrose–Singer);
- NCG cosmological models with topologically nontrivial coupling to matter (Ć.–Marcolli–Teh 2012);

In particular, inner fluctuations of the metric generally break almost-commutativity!
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In particular, inner fluctuations of the metric generally break almost-commutativity!
An almost-commutative spectral triple is a triple of the form

\[(C^\infty(X, \mathcal{A}), L^2(X, \mathcal{E}), \mathcal{D}),\]

where:
- \(X\) is a compact oriented Riemannian manifold;
- \(\mathcal{E}\) is a Hermitian vector bundle on \(X\);
- \(\mathcal{D}\) is a Dirac-type operator on \(\mathcal{E}\);
- \(\mathcal{A}\) is a \(*\)-algebra sub-bundle of \(\text{End}_{\text{Cl}(X)}^+(\mathcal{E})\).
An abstract definition

**Definition**

Let $(A, H, D)$ be a spectral triple, and let $B$ be a central unital $*$-subalgebra of $A$. Then $(A, H, D)$ is called an *abstract almost-commutative spectral triple* with base $B$ if:

1. $(B, H, D)$ is a Dirac-type commutative spectral triple;
2. $A$ is a finitely-generated projective $B$-module-$*$-subalgebra of $\text{End}_B(\cap_k \text{Dom} D^k)$;
3. $[[D, b], a] = 0$ for all $a \in A$, $b \in B$. 

Branimir Čačić

Reconstruction theorems in noncommutative Riemannian geometry
Theorem

Let \((A, H, D)\) be a \(p\)-dimensional abstract almost-commutative spectral triple with base \(B\). Then, up to unitary equivalence,

\[(A, H, D) = (C^\infty(X, A), L^2(X, E), D),\]

where

- \(X\) is a cpct. orient. Riemannian \(p\)-maniflod with \(B = C^\infty(X)\);
- \(E \to X\) is a Hermitian vector bundle;
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Real almost-commutative spectral triples

**Definition**

Let \((A, H, D, J)\) be a real spectral triple of \(KO\)-dimension \(n \mod 8\). Then \((A, H, D, J)\) is called a *real almost-commutative spectral triple* if \((A, H, D)\) is an abstract almost-commutative spectral triple with base

\[
\tilde{A}_J := \{ a \in A \mid Ja^* J^* = a \}.
\]

- Can obtain the corresponding concrete definition.
- A manifold \(X\) admits real (almost-)commutative spectral triples of arbitrary \(KO\)-dimension.
- Metric dimension and \(KO\)-dimension are thus independent in general.
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