Lisboa Summer School Course on Crossed Product C*-Algebras

N. Christopher Phillips

15 June 2009
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Sign up for the operator algebraist email directory, by emailing: ncp@uoregon.edu.
A brief outline of the lectures:
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- Construction of the crossed product of an action by a discrete group.
- Examples of some elementary computations of crossed products.
- Simplicity of crossed products by minimal homeomorphisms.
- Toward the classification of crossed products by minimal homeomorphisms.
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Let $G$ be a locally compact group, and let $A$ be a C*-algebra. An *action of $G$ on $A$* is a homomorphism $\alpha: G \to \text{Aut}(A)$, usually written $g \mapsto \alpha_g$, such that, for every $a \in A$, the function $g \mapsto \alpha_g(a)$, from $G$ to $A$, is norm continuous. On a von Neumann algebra, substitute the $\sigma$-weak operator topology for the norm topology. The continuity condition is the analog of requiring that a unitary representation of $G$ on a Hilbert space be continuous in the strong operator topology. It is usually much too strong a condition to require that $g \mapsto \alpha_g$ be a norm continuous map from $G$ to the bounded operators on $A$. Of course, if $G$ is discrete, it doesn't matter. In this course, we will concentrate on discrete $G$. 
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Of course, if $G$ is discrete, it doesn’t matter. In this course, we will concentrate on discrete $G$. 
Given $\alpha: G \to \text{Aut}(A)$, we will construct a crossed product C*-algebra $\mathbb{C}^*(G, A, \alpha)$.
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If $A$ is unital and $G$ is discrete, it is a suitable completion of the algebraic skew group ring $A[G]$, with multiplication determined by $gag^{-1} = \alpha_g(a)$ for $g \in G$ and $a \in A$. 

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Motivation for group actions on C*-algebras and their crossed products

Let $G$ be a locally compact group obtained as a semidirect product $G = N \rtimes H$. The action of $H$ on $N$ gives actions of $H$ on the full and reduced group C*-algebras $C^*(N)$ and $C^*_r(N)$, and one has $C^*(G) \cong C^*(H, C^*(N))$ and $C^*_r(G) \cong C^*_r(H, C^*(N))$. Probably the most important group action is time evolution: if a C*-algebra $A$ is supposed to represent the possible states of a physical system in some manner, then there should be an action $\alpha: \mathbb{R} \to \text{Aut}(A)$ which describes the time evolution of the system. Actions of $\mathbb{Z}$, which are easier to study, can be thought of as “discrete time evolution.” Crossed products are a common way of constructing simple C*-algebras. We will see some examples later.
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Crossed products are a common way of constructing simple C*-algebras. We will see some examples later.
Motivation for group actions on $C^*$-algebras and their crossed products (continued)

If one has a homeomorphism $h$ of a locally compact Hausdorff space $X$, the crossed product $C^*(\mathbb{Z}, X, h)$ sometimes carries considerable information about the dynamics of $h$. The best known example is the result of Giordano, Putnam, and Skau on minimal homeomorphisms of the Cantor set: isomorphism of the transformation group $C^*$-algebras is equivalent to strong orbit equivalence of the homeomorphisms.
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For compact groups, equivariant indices take values on the equivariant K-theory of a suitable C*-algebra with an action of the group. When the group is not compact, one usually needs instead the K-theory of the crossed product C*-algebra, or of the reduced crossed product C*-algebra. (When the group is compact, this is the same thing.)
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In other situations as well, the $K$-theory of the full or reduced crossed product is the appropriate substitute for equivariant $K$-theory.
The commutative case

Definition

A continuous action of a topological group $G$ on a topological space $X$ is a continuous function $G \times X \rightarrow X$, usually written $(g, x) \mapsto g \cdot x$ or $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$ and $1 \cdot x = x$ for all $x \in X$. 
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For a continuous action of a locally compact group $G$ on a locally compact Hausdorff space $X$, there is a corresponding action $\alpha: G \to \text{Aut}(C_0(X))$, given by $\alpha_g(f)(x) = f(g^{-1}x)$. 
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(If $G$ is not abelian, the inverse is necessary to get $\alpha_g \circ \alpha_h = \alpha_{gh}$ rather than $\alpha_{hg}$.)
The commutative case (continued)

Exercise

Let $G$ be a locally compact group, and let $X$ be a locally compact Hausdorff space. Prove that the formulas given above determine a one to one correspondence between continuous actions of $G$ on $X$ and continuous actions of $G$ on $C_0(X)$. (The main point is to show that an action on $X$ is continuous if and only if the corresponding action on $C_0(X)$ is continuous.)
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There are more examples in the notes.
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This gives the trivial action of $G$ on the C*-algebra $\mathbb{C}$. The full and reduced crossed products are the usual full and reduced group C*-algebras $C^*(G)$ and $C^*_{\text{r}}(G)$.
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More generally, if $H \subset G$ is a closed subgroup, then $G$ acts continuously on $G/H$ by translation. The trivial action above is the case $H = G$. 
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More generally, if $H \subset G$ is a closed subgroup, then $G$ acts continuously on $G/H$ by translation. The trivial action above is the case $H = G$.

It turns out that $C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no “twisting”.
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Take $X = S^1 = \{\zeta \in \mathbb{C}: |\zeta| = 1\}$. Taking $G = S^1$, acting by translation, gives a special case of a previous example.
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These are rational rotations (for $\theta \in \mathbb{Q}$) or irrational rotations (for $\theta \notin \mathbb{Q}$). The crossed product for the action of $\mathbb{Z}/n\mathbb{Z}$ turns out to be isomorphic to $C(S^1, M_n)$. (Note that there is no "twisting"). The crossed products for the actions of $\mathbb{Z}$ are the well-known (rational or irrational) rotation algebras. (This will be essentially immediate from the definitions.)
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Take $X = \{0, 1\}^\mathbb{Z}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$. Take $G = \mathbb{Z}$, with action generated by the shift homeomorphism $h(x)_n = x_{n-1}$ for $x \in X$ and $n \in \mathbb{Z}$. Further examples ("subshifts") can be gotten by restricting to invariant subsets of $X$. One can also replace $\{0, 1\}$ by some other compact metric space $S$. 

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Fix a prime $p$, and let $X = \mathbb{Z}_p$, the group of $p$-adic integers. (This group can be defined as the completion of $\mathbb{Z}$ in the metric $d(m, n) = p^{-d}$ when $p^d$ is the largest power of $p$ which divides $n - m$.}
Example 5

**Example**

Fix a prime $p$, and let $X = \mathbb{Z}_p$, the group of $p$-adic integers. (This group can be defined as the completion of $\mathbb{Z}$ in the metric $d(m, n) = p^{-d}$ when $p^d$ is the largest power of $p$ which divides $n - m$. Alternatively, it is $\lim\leftarrow \mathbb{Z}/p^d\mathbb{Z}$.)
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Many generalizations are possible in the inverse limit version of the construction. One need not use a prime, nor even the same number at each stage of the inverse limit.
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Take $X = S^n = \{ x \in \mathbb{R}^{n+1} : \|x\|_2 = 1 \}$. Then the homeomorphism $x \mapsto -x$ has order 2, and so gives an action of $\mathbb{Z}/2\mathbb{Z}$ on $S^n$. 
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The crossed product turns out to be isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber $M_2$. 
Example 7

Take $X = S^1 = \{\zeta \in \mathbb{C}: |\zeta| = 1\}$, and consider the order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$. We get an action of $\mathbb{Z}/2\mathbb{Z}$ on $S^1$. 
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The group $SL_2(\mathbb{Z})$ acts on $S^1 \times S^1$ as follows.
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Notation: If $A$ is a unital C*-algebra and $u \in A$ is unitary, then $\text{Ad}(u)$ is the automorphism $a \mapsto uau^*$ of $A$. 
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Let $G$ be a locally compact group, let $A$ be a unital C*-algebra, and let $g \mapsto z_g$ be a norm continuous group homomorphism from $G$ to the unitary group $U(A)$ of $A$. 
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$$ \alpha_g(a) = \text{Ad}(z_g), $$

for $g \in G$ and $a \in A$, turns out to be isomorphic to the crossed product by the trivial action.
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Actions obtained this way are called inner actions. The crossed product turns out to be isomorphic to the crossed product by the trivial action.
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Example

Let $A = M_2$, let $G = (\mathbb{Z}/2\mathbb{Z})^2$ with generators $g_1$ and $g_2$, and set 

$\alpha_1 = \text{id}_A$, $\alpha_{g_1} = \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\alpha_{g_2} = \text{Ad} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\alpha_{g_1 g_2} = \text{Ad} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

These define an action $\alpha: G \to \text{Aut}(A)$ such that $\alpha_g$ is inner for all $g \in G$, but such that there is no homomorphism $g \mapsto z_g \in U(A)$ such that $\alpha_g = \text{Ad}(z_g)$ for all $g \in G$. The point is that the implementing unitaries for $\alpha_{g_1}$ and $\alpha_{g_2}$ commute up to a scalar, but can't be appropriately modified to commute exactly. The crossed product turns out to be isomorphic to $M_4$. 

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Let $A$ be a simple unital C*-algebra, and let $\alpha: \mathbb{Z}/n\mathbb{Z} \to \text{Aut}(A)$ be an action such that each automorphism $\alpha_g$, for $g \in \mathbb{Z}/n\mathbb{Z}$, is an inner automorphism.

Problem

Find a counterexample when $A$ is not assumed simple. (I presume that a counterexample exists, but I do not know of one.)

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Prove the statements made in the example on the previous slide.
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$$\alpha_n(u) = \exp(\pi in_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}} \quad \text{and} \quad \alpha_n(v) = \exp(\pi in_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$
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The group $SL_2(\mathbb{Z})$ has finite subgroups of orders 2, 3, 4, and 6. Restriction of the action gives actions of these groups on rotation algebras. The crossed products by these actions have been intensively studied. Recently, it has been proved that for $\theta \notin \mathbb{Q}$ they are all AF.
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\alpha(\zeta_1, \zeta_2)(u) = \zeta_1 u \quad \text{and} \quad \alpha(\zeta_1, \zeta_2)(v) = \zeta_2 v.
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Checking continuity of the action requires a $3\varepsilon$ argument. If we fix $\zeta_1, \zeta_2 \in S^1$, then $\alpha(\zeta_1, \zeta_2)$ generates an action of $\mathbb{Z}$. If both have finite order, we get an action of a finite cyclic group. For example, there is an action of $\mathbb{Z}/n\mathbb{Z}$ generated by the automorphism which sends $u$ to $\exp(2\pi i/n) u$ and $v$ to $v$. 
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If we fix $\zeta_1, \zeta_2 \in S^1$, then $\alpha(\zeta_1, \zeta_2)$ generates an action of $\mathbb{Z}$. 
Example 13

Example

Let $A_\theta$ be generated by unitaries $u$ and $v$ as in the previous example. For $\zeta_1, \zeta_2 \in S^1$, the unitaries $\zeta_1 u$ and $\zeta_2 v$ satisfy the same commutation relation. Therefore there is an action $\alpha: S^1 \times S^1 \to \text{Aut}(A_\theta)$ determined by

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If we fix $\zeta_1, \zeta_2 \in S^1$, then $\alpha(\zeta_1, \zeta_2)$ generates an action of $\mathbb{Z}$. If both have finite order, we get an action of a finite cyclic group. For example, there is an action of $\mathbb{Z}/n\mathbb{Z}$ generated by the automorphism which sends $u$ to $\exp(2\pi i/n)u$ and $v$ to $v$. 
Example 14

Example

Recall that the Cuntz algebra $\mathcal{O}_n$ is the universal unital C*-algebra on generators $s_1, s_2, \ldots, s_n$, subject to the relations $s_j^*s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} s_j s_j^* = 1$. (It is in fact simple.)
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There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha(\zeta_1, \zeta_2, \ldots, \zeta_n)(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$. 
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In fact, regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $\mathcal{O}_n$, defined as follows.
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Recall that the Cuntz algebra $\mathcal{O}_n$ is the universal unital $\mathrm{C}^*$-algebra on generators $s_1, s_2, \ldots, s_n$, subject to the relations $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^n s_j s_j^* = 1$. (It is in fact simple.)

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$$\alpha_u(s_j) = \sum_{k=1}^n u_{k,j} s_k.$$
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$$\alpha_u(s_j) = \sum_{k=1}^n u_{k,j}s_k.$$ 

This determines a continuous action of the compact group $U(M_n)$ on $O_n$. 

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N. Christopher Phillips ()

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Example 14

Example

Recall that the Cuntz algebra $O_n$ is the universal unital $C^*$-algebra on generators $s_1, s_2, \ldots, s_n$, subject to the relations $s_j^*s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^n s_j s_j^* = 1$. (It is in fact simple.)

There is an action of $(S^1)^n$ on $O_n$ such that $\alpha(\zeta_1, \zeta_2, \ldots, \zeta_n)(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$.

In fact, regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $O_n$, defined as follows. If $u = (u_{j,k})_{j,k=1}^n \in M_n$ is unitary, then define an automorphism $\alpha_u$ of $O_n$ by the following action on the generating isometries $s_1, s_2, \ldots, s_n$:

$$\alpha_u(s_j) = \sum_{k=1}^n u_{k,j} s_k.$$

This determines a continuous action of the compact group $U(M_n)$ on $O_n$. Any individual automorphism from this action gives an action of $\mathbb{Z}$ on $O_n$. 
Example 14 (continued)

Exercise

Verify that the formula above does in fact define a continuous action of $U(M_n)$ on $O_n$. 
Exercise

Verify that the formula above does in fact define a continuous action of $U(M_n)$ on $O_n$.

(Check that the elements $\zeta_js_j$ satisfy the required relations. Use a $3\varepsilon$ argument to prove continuity.)
Let $k_1, k_2, \ldots$ be integers with all $k_n \geq 2$. Consider the UHF algebra $A$ of type $\prod_{n=1}^\infty k_n$. 
Example 15

Example

Let $k_1, k_2, \ldots$ be integers with all $k_n \geq 2$. Consider the UHF algebra $A$ of type $\prod_{n=1}^{\infty} k_n$. We construct it as $\bigotimes_{n=1}^{\infty} M_{k_n}$.
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Let $k_1, k_2, \ldots$ be integers with all $k_n \geq 2$. Consider the UHF algebra $A$ of type $\prod_{n=1}^{\infty} k_n$. We construct it as $\bigotimes_{n=1}^{\infty} M_{k_n}$, or, in more detail, as $\lim A_n$ with $A_n = M_{k_1} \otimes M_{k_2} \otimes \cdots \otimes M_{k_n}$. 
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Let $G$ be a locally compact group, and let $\beta^{(n)} : G \to \text{Aut}(M_{k_n})$ be an action of $G$ on $M_{k_n}$.
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$$\alpha_g^{(n)}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \beta_g^{(1)}(a_1) \otimes \beta_g^{(2)}(a_2) \otimes \cdots \otimes \beta_g^{(n)}(a_n).$$
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$$\alpha^{(n)}_g(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \beta^{(1)}_g(a_1) \otimes \beta^{(2)}_g(a_2) \otimes \cdots \otimes \beta^{(n)}_g(a_n).$$

One checks immediately that $\varphi_n \circ \alpha^{(n-1)}_g = \alpha^{(n)}_g \circ \varphi_n$ for all $n \in \mathbb{Z}_{>0}$ and $g \in G$, and

\[ (1, \ldots, 1) \cdot (1, \ldots, 1) = (1, \ldots, 1). \]
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Let $k_1, k_2, \ldots$ be integers with all $k_n \geq 2$. Consider the UHF algebra $A$ of type $\prod_{n=1}^{\infty} k_n$. We construct it as $\bigotimes_{n=1}^{\infty} M_{k_n}$, or, in more detail, as $\lim A_n$ with $A_n = M_{k_1} \otimes M_{k_2} \otimes \cdots \otimes M_{k_n}$. Note that $A_n = A_{n-1} \otimes M_{k_n}$, and the map $\varphi_n : A_{n-1} \to A_n$ is given by $a \mapsto a \otimes 1_{M_{k_n}}$.

Let $G$ be a locally compact group, and let $\beta^{(n)} : G \to \text{Aut}(M_{k_n})$ be an action of $G$ on $M_{k_n}$. Then define an action $\alpha^{(n)} : G \to \text{Aut}(A_n)$ by

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Let $G$ be a locally compact group, and let $\beta^{(n)}: G \rightarrow \text{Aut}(M_{k_n})$ be an action of $G$ on $M_{k_n}$. Then define an action $\alpha^{(n)}: G \rightarrow \text{Aut}(A_n)$ by

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One checks immediately that $\varphi_n \circ \alpha^{(n-1)}_g = \alpha^{(n)}_g \circ \varphi_n$ for all $n \in \mathbb{Z}_{>0}$ and $g \in G$, so there is a direct limit action $g \mapsto \alpha_g$ of $G$ on $A = \lim_{\rightarrow} A_n$. (One needs a $3\varepsilon$ argument to prove continuity.)
The easiest way to get such an action is to choose a unitary representation \( g \mapsto u_n(g) \) on \( \mathbb{C}^{k_n} \), and set \( \beta_g^{(n)}(a) = u_n(g)au_n(g)^* \) for \( g \in G \) and \( a \in M_{k_n} \).
Example 15 (continued)

The easiest way to get such an action is to choose a unitary representation $g \mapsto u_n(g)$ on $\mathbb{C}^{k_n}$, and set $\beta^{(n)}_g(a) = u_n(g)a u_n(g)^*$ for $g \in G$ and $a \in M_{k_n}$. In this case, the resulting action is called a \textit{product type action}. 

Exercise

Prove that the actions above really are continuous.
Example 15 (continued)

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As a specific example, take $G = \mathbb{Z}/2\mathbb{Z}$, and for every $n$ take $k_n = 2$ and take $\beta^{(n)}$ to be generated by

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As a specific example, take $G = \mathbb{Z}/2\mathbb{Z}$, and for every $n$ take $k_n = 2$ and take $\beta^{(n)}$ to be generated by $\text{Ad} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$. 

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Prove that the actions above really are continuous.
Example 16

Example

Let $A$ be a unital $\text{C}^*$-algebra. The tensor flip is the automorphism

$$\phi(a \otimes b) = b \otimes a$$

for $a, b \in A$.

This gives an action of $\mathbb{Z}/2\mathbb{Z}$ on $A \otimes \text{max} A$.

Similarly, the same formula defines a tensor flip action of $\mathbb{Z}/2\mathbb{Z}$ on $A \otimes \text{min} A$.

In a similar manner, the symmetric group $S_n$ acts on the $n$-fold maximal and minimal tensor products of $A$ with itself.

There is also a "tensor shift", a noncommutative analog, defined on $\bigotimes_{n \in \mathbb{Z}} A$, of the shift on $S\mathbb{Z}$.
Example 16

Let $A$ be a unital C*-algebra. The *tensor flip* is the automorphism $\varphi \in \text{Aut}(A \otimes_{\text{max}} A)$ of order 2 determined by the formula

$$\varphi(a \otimes b) = b \otimes a$$

for $a, b \in A$. (Use the universal property of $A \otimes_{\text{max}} A$.)

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(Choose an injective representation $\pi: A \to \mathcal{L}(H)$, and consider $\pi \otimes \pi$ as a representation of $A \otimes_{\text{min}} A$ on $H \otimes H$.)

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There is also a “tensor shift”, a noncommutative analog, defined on $\bigotimes_{n \in \mathbb{Z}} A$, of the shift on $S^\mathbb{Z}$.
Covariant representations

To define the crossed product, we need:

**Definition**

Let $\alpha : G \to \text{Aut}(A)$ be an action of a locally compact group $G$ on a $C^\ast$-algebra $A$. A covariant representation of $(G, A, \alpha)$ on a Hilbert space $H$ is a pair $(v, \pi)$ consisting of a unitary representation $v : G \to U(H)$ (the unitary group of $H$) and a representation $\pi : A \to \mathcal{L}(H)$ (the algebra of all bounded operators on $H$), satisfying the covariance condition $v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$ for all $g \in G$ and $a \in A$.

It is called nondegenerate if $\pi$ is nondegenerate. By convention, unitary representations are strong operator continuous. Representations of $C^\ast$-algebras, and of other *-algebras are *-representations (and, similarly, homomorphisms are *-homomorphisms).
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$$\nu(g) \pi(a) \nu(g)^* = \pi(\alpha_g(a))$$

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Let $\alpha : G \to \text{Aut}(A)$ be an action of a locally compact group $G$ on a C*-algebra $A$. A **covariant representation** of $(G, A, \alpha)$ on a Hilbert space $H$ is a pair $(\nu, \pi)$ consisting of a unitary representation $\nu : G \to U(H)$ (the unitary group of $H$) and a representation $\pi : A \to L(H)$ (the algebra of all bounded operators on $H$), satisfying the covariance condition

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\nu(g)\pi(a)\nu(g)^* = \pi(\alpha_g(a))
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for all $g \in G$ and $a \in A$. It is called **nondegenerate** if $\pi$ is nondegenerate.

By convention, unitary representations are strong operator continuous. Representations of C*-algebras, and of other *-algebras are *-representations (and, similarly, homomorphisms are *-homomorphisms).
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Using Haar measure in the integral, we define multiplication by the following "twisted convolution":

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Suppose $A = C_0(X)$, and $\alpha$ comes from an action of $G$ on $X$. 

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