Math 251 — Sample Questions for Midterm 1

At least 80% of the points on the real exam will be modifications of problems from Midterm 1 from the last time I taught the course, the problems below, homework problems (particularly written homework), and problems from the sample and real Midterms 0. Note, though, that the exact form of the functions to be differentiated and of the limits to be computed could vary substantially, and the methods required to do them might occur in different combinations.

Midterm 1 from the last time I taught the course should give a reasonable idea of the length to expect.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen the correct notation for limits etc. in the book, in handouts, in files posted on the course website, and on the blackboard; *use it*. The right notation will help you get the mathematics right, and incorrect notation will lose points.

Here is the instruction sheet for Midterm 1:

(1) DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.
(2) Closed book, except for a 3×5 file card.
(3) The following are all prohibited: Calculators (of any kind), cell phones, laptops, iPods, electronic dictionaries, and any other electronic devices or communication devices. All electronic or communication devices you have with you must be turned completely off and put inside something (pack, purse, etc.) and out of sight.
(4) The point values are as indicated in each problem; total 100 points.
(5) Write all answers on the test paper. Use the back of the page with the extra credit problems for long answers or scratch work.
(6) Show enough of your work that your method is obvious. Be sure that every statement you write is correct. Cross out any material you do not wish to have considered. Correct answers with insufficient justification or accompanied by additional incorrect statements will not receive full credit. Correct guesses to problems requiring significant work, and correct answers obtained after a sequence of mostly incorrect steps, will receive no credit.
(7) Be sure you say what you mean, and use correct notation. Credit will be based on what you say, not what you mean.
(8) When exact values are specified, give answers such as $\frac{1}{7}$, $\sqrt{2}$, $\ln(2)$, or $\frac{2\pi}{9}$. Decimal approximations will not be accepted.
(9) Final answers must always be simplified unless otherwise specified.
(10) Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned (usually by the Tuesday after the exam).
(11) Time: 50 minutes.

Problems from Midterm 1 from the previous time I taught this course (not including the extra credit):

1. (a) (5 points) State carefully the definition of the derivative of a function.

(b) (14 points) If $f(x) = 23 - x^2$, compute the derivative $f'(2)$ *directly from the definition*. (You should check your answer using the differentiation formula, but no credit will be given for just using the formula.)

2. (4 points) Let $T(t)$ be the temperature at time t at the Eugene airport. Assume that t is measured in hours after midnight on 31 Dec. 2000, and that $T(t)$ is measured in °C. Do you expect $T'(11)$ to be positive or negative? Why?

3. (8 points/part.) Differentiate the following functions:

(a) $b(t) = \dfrac{q - \sin(t)}{t} + \sqrt{11}$, where q is a constant.

(b) $h(t) = e^{2t^3 - t} + \frac{3}{17}$.

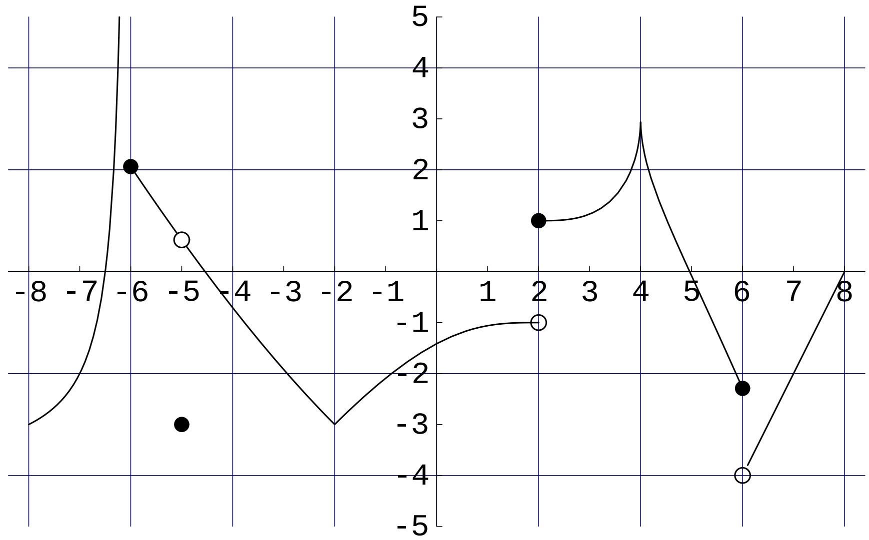
4. (8 points/part) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give reasons in all cases.

(a) $\displaystyle\lim_{x\to 3} \frac{x-3}{x^2 + x - 12}$.

(b) $\displaystyle\lim_{x\to -\infty} \frac{3x^2 - 2}{7x^2 - 2\sin(x^3)}$.

(c) $\displaystyle\lim_{x\to 4} \frac{\sqrt{f(x)}}{x-6}$, given that $\displaystyle\lim_{x\to 4} f(x) = 9$.

5. (4 points/part.) For the function $y = h(x)$ graphed below, answer the following questions:



(a) List all numbers a in $(-8, 0)$ such that $\lim_{x\to a} h(x)$ does not exist. Give brief reasons. (This interval is only **part** of what is shown in the graph.)

(b) Which of the following best describes $h'(5)$?

(1) $h'(5)$ does not exist.
(2) $h'(5)$ is close to 0.
(3) $h'(5)$ is positive and not close to 0.
(4) $h'(5)$ is negative and not close to 0.

6. (10 points) Let c be a constant. Find the equation of the tangent line to the graph of $q(x) = 8/x^2 + c$ at $x = 2$. You need not calculate the derivative directly from the definition.

7. (4 points/part) A drone equipped with a camera is hovering over somebody's backyard, spying on it. Its height above the ground varies. During the period from 3:00 pm to 3:05 pm, its height $y(t)$ above the ground, measured in meters (m), at time t, measured in minutes (min) after 3:00 pm, is given by $y(t) = t^3 - 4t^2 + 20$.

(a) Is the drone falling or rising 2 minutes after 3:00 pm? How fast?

(b) What is the average upwards velocity of the drone between 3:00 pm and 3:03 pm?

8. (11 points) If $x\sin(y) + \arctan(19) = x^7 + y$, find $\dfrac{dy}{dx}$ by implicit differentiation. (You must solve for $\dfrac{dy}{dx}$.)

Additional sample problems:

9. (5 points/part) The following questions refer to the function whose graph is shown in problem 5.

(a) List all numbers a in $(-8, 8)$ such that h is not differentiable at a. Give reasons.

(b) List all numbers a in $(-8, 8)$ such that h is continuous at a but not differentiable at a. Give brief reasons.

(c) List all numbers a in $(-8, 8)$ such that h is differentiable at a but not continuous at a. Give brief reasons.

(d) Which of the following best describes $h'(7)$?
(1) $h'(7)$ does not exist.
(2) $h'(7)$ is close to 0.
(3) $h'(7)$ is positive and not close to 0.
(4) $h'(7)$ is negative and not close to 0.

(e) Find the largest interval containing 5 on which h is continuous.

10. (11 points) Find $\dfrac{dy}{dx}$ by implicit differentiation. (You must solve for $\dfrac{dy}{dx}$.)

(a) $x\sin(y) = (xy + 1)^7$.

(b) $x^7 y = x + \cos(ky) + \pi^3$, where k is a constant.

11. (16 points; point values of parts as shown) A small spacecraft takes off from the surface of a planet, reaches a maximum height, and then crashes. Its position at time t is given by $y(t) = 9t^2 - 4t^3$, where $y(t)$ is measured in kilometers (km) above the surface and t is measured in minutes (min). Answer the following questions, being careful to give correct units when called for.

(a) (4 points) Find the speed of the spacecraft at time $t = 1$.

(b) (4 points) How long will it take for the spacecraft to reach its maximum height?

(c) (8 points) At time $t = 2$, what is the acceleration of the spacecraft? Would a person in the craft feel himself speeding up or slowing down at this moment? Explain.

12. (10 points/part) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give reasons in all cases.

(a) $\displaystyle\lim_{x\to -2} \frac{x+2}{x^2 - 3x - 10}$.

(b) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{2x} - \frac{1}{2x + \sqrt{x}}\right)$.

(c) $\displaystyle\lim_{x\to -\infty} \frac{x^2 - x + 17}{7x^2 + 9x + 19}$.

(d) $\displaystyle\lim_{x\to -\infty} \frac{x}{\sqrt{5x^2 + 9}}$.

(e) $\displaystyle\lim_{x\to 2^+} \frac{x^2 - 5x + 4}{x - 2}$.

(f) $\displaystyle\lim_{x\to -\infty} \frac{3x - 2}{37x^2 - 6\sin(x)}$.

13. (9 points/part) Differentiate the following functions:

(a) $f(y) = \left(7y^3 + \frac{1}{2}\right)\left(\frac{1}{8}y^7 - 16\sqrt{y} + \frac{2}{3}\right)$.

(b) $h(t) = \dfrac{\sqrt[3]{t} - 2\pi}{\sqrt[3]{t} + 2\pi}$.

(c) Given that $h'(x) = 3h(x)$, find $\dfrac{d}{dx}\left(\dfrac{x}{h(x)}\right)$. (Your answer might involve the function h.)

(d) $g(t) = \dfrac{t^2 - 27}{k + e^t} - \sqrt{17}$, where k is a constant.

(e) $q(x) = \sin(cxe^x + \pi^2)$, where c is a constant.

(f) $f(x) = \sin\left((x^2 - k)^{17}\right)$, where k is a constant.

14. (6 points/part) The function $P(t) = (6t+1)e^{k(t-1)}$ models the population of a colony of bacteria at time t, where P is measured in hundreds of bacteria and t is measured in hours. Observations indicate that after one hour there are 700 bacteria, and at that time the colony is growing at a rate of 200 bacteria per hour.

(a) Find k.

(b) What happens to the population of bacteria in the long run, as $t \to \infty$? Explain.

14. Let f and g be functions such that:

$$f(-3) = -5, \quad f'(-3) = 12, \quad g(-3) = 2, \quad \text{and} \quad g'(-3) = -3$$

and

$$f(2) = 7, \quad f'(2) = 3, \quad g(2) = -3, \quad \text{and} \quad g'(2) = 2.$$

Let $h(x) = f(g(x))$.

(a) (2 points) Find $h(2)$. (You will not need to use all the information provided.)

(b) (8 points) Find $h'(2)$. (You will not need to use all the information provided.)

16. (6 points) Express the following statement in terms of calculus. Be sure to define everything that appears in your formulas.

"The population of fire-breathing monsters on the planet Yuggxth was increasing throughout the period 1900–2000."

17. (4 points/part) Let $T(h)$ be the temperature at height h above the Eugene airport at noon on 17 July 2001. Assume that h is measured in kilometers, and that $T(h)$ is measured in °C.

(a) What are the units of $T'(h)$?

(b) For $0 \le h \le 10$, do you expect $T'(h)$ to be positive or negative? Why?

(c) Explain the practical significance of the statement $T'(30) = 5$.

18. (10 points/part) Let f be a function such that $\lim_{x\to 2} f(x) = 7$. Calculate the following expressions, or explain why there is not enough information to do so:

(a) $\displaystyle\lim_{x\to 2} \frac{f(x)}{\sqrt{x}}$.

(b) $\displaystyle\lim_{x\to 2} \frac{f(x)}{x-2}$.

(c) $\displaystyle\lim_{x\to 2^-} \frac{f(x)}{x-2}$.

(d) $\displaystyle\lim_{x\to 7} f(x)$.

(e) $f(2)$.

(f) $\lim_{x \to 2} \frac{f(x) - 7}{x - 2}$.

(g) $\lim_{x \to 2} \sqrt{f(x)}$.

19. (10 points) The population of the Kingdom of Oggchobb grows at a rate proportional to the existing population. If the population was 9.73 million in 1960 and 11.31 million in 1990, help the Royal Statistical Office determine the population in 2000.

20. (15 points) Prove that there exists a real solution to the equation $x^9 + 2x + 2 = 0$. Give a complete justification for any theorems that you use, in particular being sure to check that the hypotheses hold.