Math 251 Solutions to the Sample Questions for Midterm 1

At least 80% of the points on the real exam will be modifications of problems from Midterm 1 from the last time I taught the course, the problems below, homework problems (particularly written homework), and problems from the sample and real Midterms 0. Note, though, that the exact form of the functions to be differentiated and of the limits to be computed could vary substantially, and the methods required to do them might occur in different combinations.

Midterm 1 from the last time I taught the course should give a reasonable idea of the length to expect.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen the correct notation for limits etc. in the book, in handouts, in files posted on the course website, and on the blackboard; *use it*. The right notation will help you get the mathematics right, and incorrect notation will lose points.

Here is the instruction sheet for Midterm 1:

(1) DO NOT TURN THIS PAGE UNTIL YOU ARE TOLD TO DO SO.
(2) Closed book, except for a 3×5 file card.
(3) The following are all prohibited: Calculators (of any kind), cell phones, laptops, iPods, electronic dictionaries, and any other electronic devices or communication devices. All electronic or communication devices you have with you must be turned completely off and put inside something (pack, purse, etc.) and out of sight.
(4) The point values are as indicated in each problem; total 100 points.
(5) Write all answers on the test paper. Use the back of the page with the extra credit problems for long answers or scratch work.
(6) Show enough of your work that your method is obvious. Be sure that every statement you write is correct. Cross out any material you do not wish to have considered. Correct answers with insufficient justification or accompanied by additional incorrect statements will not receive full credit. Correct guesses to problems requiring significant work, and correct answers obtained after a sequence of mostly incorrect steps, will receive no credit.
(7) Be sure you say what you mean, and use correct notation. Credit will be based on what you say, not what you mean.
(8) When exact values are specified, give answers such as $\frac{1}{7}$, $\sqrt{2}$, $\ln(2)$, or $\frac{2\pi}{9}$. Decimal approximations will not be accepted.
(9) Final answers must always be simplified unless otherwise specified.
(10) Grading complaints must be submitted in writing at the beginning of the class period after the one in which the exam is returned (usually by the Tuesday after the exam).
(11) Time: 50 minutes.

Problems from Midterm 1 from the previous time I taught this course (not including the extra credit):

1. (a) (5 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

(b) (14 points) If $f(x) = 23 - x^2$, compute the derivative $f'(2)$ *directly from the definition.* (You should check your answer using the differentiation formula, but no credit will be given for just using the formula.)

Solution: We find the limit of the difference quotient, using the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{23 - (2+h)^2 - (23 - 2^2)}{h} \\ &= \lim_{h \to 0} \frac{23 - (4 + 4h + h^2) - (23 - 4)}{h} = \lim_{h \to 0} \frac{23 - 4 - 4h - h^2 - 23 + 4}{h} \\ &= \lim_{h \to 0} \frac{-4h - h^2}{h} = \lim_{h \to 0} (-4 - h) = -4. \end{aligned}$$

We can check using the differentiation formulas: $f'(x) = -2x$, so $f'(2) = -4$. (However, you get no credit if this is the only thing you do.)

2. (4 points) Let $T(t)$ be the temperature at time t at the Eugene airport. Assume that t is measured in hours after midnight on 31 Dec. 2000, and that $T(t)$ is measured in °C. Do you expect $T'(11)$ to be positive or negative? Why?

Solution: Positive, because temperature normally increases in the late morning. (The time is 11:00 am.)

3. (8 points/part.) Differentiate the following functions:

(a) $b(t) = \dfrac{q - \sin(t)}{t} + \sqrt{11}$, where q is a constant.

Solution: We use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus,

$$b'(t) = \frac{-\cos(t) \cdot t - (q - \sin(t)) \cdot 1}{t^2} = \frac{-t\cos(t) + \sin(t) - q}{t^2}.$$

Note that $\sqrt{11}$ is a *constant*, so its derivative is zero.

Alternate solution: Rewrite $b(t) = qt^{-1} - t^{-1}\sin(t) - \sqrt{11}$. On the second term, use the product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

This gives

$$b'(t) = -qt^{-2} - \big(-t^{-2}\sin(t) + t^{-1}\cos(t)\big) = -qt^{-2} + t^{-2}\sin(t) - t^{-1}\cos(t).$$

(b) $h(t) = e^{2t^3-t} + \frac{3}{17}$.

Solution: Use the chain rule, remembering that $\frac{3}{17}$ is a constant, so that its derivative is zero, and that $\frac{d}{dx}(e^x) = e^x$:

$$h'(y) = e^{2t^3-t} \cdot (3 \cdot 2t^3 - 1) = (6t^3 - 1)e^{2t^3-t}.$$

4. (8 points/part) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give reasons in all cases.

(a) $\displaystyle\lim_{x\to 3} \frac{x-3}{x^2+x-12}$.

Solution: This has the indeterminate form "$\frac{0}{0}$", so work is needed. We factor the denominator, and then cancel common factors in the fraction:

$$\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \lim_{x\to 3} \frac{x-3}{(x-3)(x+4)} = \lim_{x\to 3} \frac{1}{x+4} = \frac{1}{3+4} = \frac{1}{7}.$$

(b) $\displaystyle\lim_{x\to -\infty} \frac{3x^2-2}{7x^2-2\sin(x^3)}$.

Solution: The limit has the indeterminate form "$\frac{\infty}{\infty}$". Therefore more work is needed. We factor out x^2 from both the numerator and denominator, and then use the limit laws:

$$\lim_{x\to -\infty} \frac{3x^2-2}{7x^2-2\sin(x^3)} = \lim_{x\to -\infty} \frac{3-\frac{2}{x^2}}{7+\frac{2\sin(x^3)}{x^2}} = \frac{3-\lim_{x\to -\infty}\frac{2}{x^2}}{7+\lim_{x\to -\infty}\frac{2\sin(x^3)}{x^2}}.$$

Certainly $\lim_{x\to -\infty}\frac{2}{x^2} = 0$. Since $-2 \leq 2\sin(x^3) \leq 2$ for all x, and $x^2 \to \infty$ as $x \to -\infty$, we get $\lim_{x\to -\infty}\frac{2\sin(x^3)}{x^2} = 0$. Therefore

$$\lim_{x\to -\infty} \frac{3x^2-2}{7x^2-2\sin(x^3)} = \frac{3}{7}.$$

Here is a different way to arrange essentially the same calculation:

$$\lim_{x\to -\infty} \frac{3x^2-2}{7x^2-2\sin(x^3)} = \lim_{x\to -\infty} \frac{\left(\frac{1}{x^2}\right)(3x^2-2)}{\left(\frac{1}{x^2}\right)(7x^2-2\sin(x^3))} = \lim_{x\to -\infty} \frac{3-\frac{2}{x^2}}{7+\frac{2\sin(x^3)}{x^2}}$$

$$= \frac{3-\lim_{x\to -\infty}\frac{2}{x^2}}{7+\lim_{x\to -\infty}\frac{2\sin(x^3)}{x^2}} = \frac{3+0}{7+0} = \frac{3}{7}.$$

(c) $\lim_{x\to 4} \frac{\sqrt{f(x)}}{x-6}$, given that $\lim_{x\to 4} f(x) = 9$.

Solution:

$$\lim_{x\to 4} \frac{\sqrt{f(x)}}{x-6} = \frac{\lim_{x\to 4} \sqrt{f(x)}}{\lim_{x\to 4} x-6} = \frac{\sqrt{\lim_{x\to 4} f(x)}}{-2} = -\frac{\sqrt{9}}{2} = -\frac{3}{2}.$$

(The simplification is necessary.)

The following calculation is not correct, and is an example of getting the right answer through incorrect intermediate steps:

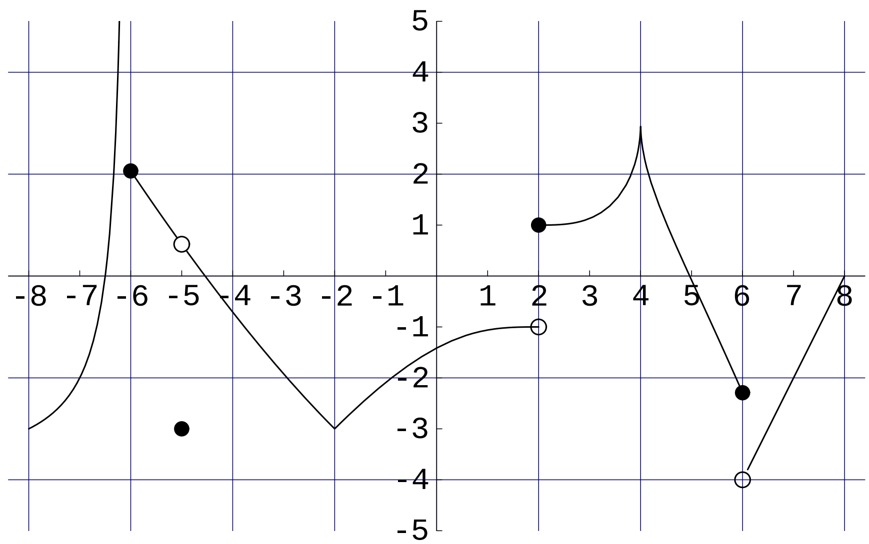
$$\lim_{x\to 4} \frac{\sqrt{f(x)}}{x-6} = \lim_{x\to 4} \frac{\sqrt{9}}{x-6} = \frac{\sqrt{9}}{-2} = -\frac{3}{2}.$$

It is not correct to take the limit of only part of an expression. If you could do that, since $\lim_{x\to 0} \sin(7x) = 0$ and $\lim_{x\to 0} 7x^2 = 0$, you would get

$$\lim_{x\to 0} \frac{\sin(7x)}{3x} = \lim_{x\to 0} \frac{0}{3x} = \lim_{x\to 0} 0 = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{7x^2}{3x^2} = \lim_{x\to 0} \frac{0}{3x^2} = \lim_{x\to 0} 0 = 0.$$

Both limits are in fact $\frac{7}{3}$, not zero.

5. (4 points/part.) For the function $y = h(x)$ graphed below, answer the following questions:



(a) List all numbers a in $(-8, 0)$ such that $\lim_{x\to a} h(x)$ does not exist. Give brief reasons. (This interval is only **part** of what is shown in the graph.)

Solution: The answer is only $a = -6$. It is clear from the graph that $\lim_{x\to -6^-} h(x) = \infty$ and $\lim_{x\to -2^+} h(x)$ is finite (and is close to 2). So the one sided limits disagree.

Note: $\lim_{x\to -5} h(x)$ *does* exist: it is a bit less than 1. Also, $\lim_{x\to 2} h(x)$ and $\lim_{x\to 6} h(x)$ do not exist, but 2 and 6 are not in the interval $(-8, 0)$.

(b) Which of the following best describes $h'(5)$?

(1) $h'(5)$ does not exist.

(2) $h'(5)$ is close to 0.

(3) $h'(5)$ is positive and not close to 0.
(4) $h'(5)$ is negative and not close to 0.

Solution: $h'(5)$ is the slope of the tangent line to the graph of $y = h(x)$ at $x = 5$. You can tell from inspection that this slope is negative and not close to 0 (choice (3) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere around -2. (In fact, it is clear that $h'(x) < 0$ for $4 < x < 6$.)

6. (10 points) Let c be a constant. Find the equation of the tangent line to the graph of $q(x) = 8/x^2 + c$ at $x = 2$. You need not calculate the derivative directly from the definition.

Solution: We need the slope, which is $q'(2)$, and a point on the line, such as $(2, q(2)) = \left(1, \frac{8}{4} + c\right) = (1, 2 + c)$. Now $q(x) = 8x^{-2} + c$, so $q'(x) = -16x^{-3}$ (because c is a constant), and $q'(2) = -\frac{16}{8} = -2$. So the equation is $y - (2 + c) = -2(x - 2)$, which can be simplified to give $y = -2x + 6 + c$. (The simplification is necessary.)

Note that we want the slope at the *particular* value $x = 2$. Therefore we must substitute $x = 2$ in the formula for the derivative $q'(x)$ *before* using it as the slope of a line. The equation

$$y - (2 + c) = -16x^{-3}(x - 2)$$

is wrong—it is not even the equation of a line.

7. (4 points/part) A drone equipped with a camera is hovering over somebody's backyard, spying on it. Its height above the ground varies. During the period from 3:00 pm to 3:05 pm, its height $y(t)$ above the ground, measured in meters (m), at time t, measured in minutes (min) after 3:00 pm, is given by $y(t) = t^3 - 4t^2 + 20$.

(a) Is the drone falling or rising 2 minutes after 3:00 pm? How fast?

Solution: The vertical velocity at time t is $y'(t) = 3t^2 - 8t$, and the vertical velocity at time 2 is $y'(2) = 3 \cdot 2^2 - 8 \cdot 2 = 12 - 16 = -4$. Therefore the drone is falling at 4 m/min.

Note: You *must* include the units in this kind of problem.

Note: It is not correct to say that the drone is falling at -4 m/min. This statement means that it is *rising* at 4 m/min.

(b) What is the average upwards velocity of the drone between 3:00 pm and 3:03 pm?

Solution: The average velocity is the how far it went divided by how long it took to go that far, which here is

$$\frac{y(3) - y(0)}{3 - 0} = \frac{3^3 - (4)(3^2) + 20 - [20]}{3} = \frac{27 - 36 + 20 - 20}{3} = \frac{-9}{3} = -3.$$

So the average upwards velocity is -3 m/min.

Note: You *must* include the units in this kind of problem.

It is not correct to average the velocities at the endpoints of the interval. That is, do not use

$$\frac{y'(3) + y'(0)}{2} = \frac{3}{2}.$$

8. (11 points) If $x\sin(y) + \arctan(19) = x^7 + y$, find $\frac{dy}{dx}$ by implicit differentiation. (You must solve for $\frac{dy}{dx}$.)

Solution: Use the product rule, followed by the chain rule, to differentiate the left hand side. Use the sum rule to differentiate the right hand side. This gives:

$$\frac{d}{dx}\left(x\sin(y)\right) = \frac{d}{dx}\left(x^7 + y\right)$$
$$1 \cdot \sin(y) + x\cos(y)\frac{dy}{dx} = 7x^6 + \frac{dy}{dx}.$$

(The derivative of $\arctan(19)$ is immediately seen to be zero because $\arctan(19)$ is a constant.) Now solve for $\frac{dy}{dx}$:

$$\sin(y) - 7x^6 = \frac{dy}{dx} - x\cos(y)\frac{dy}{dx} = [1 - x\cos(y)]\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{\sin(y) - 7x^6}{1 - x\cos(y)}.$$

Note that this expression can't be simplified.

For those who prefer the other notation, here it is written with y as an explicit function $y(x)$ of x. Start with

$$x\sin(y(x)) + \arctan(19) = x^7 + y(x).$$

Then differentiate with respect to x, just as before:

$$1 \cdot \sin(y(x)) + x\cos(y(x))y'(x) = 7x^6 + y'(x).$$

Now solve for $y'(x)$:

$$\sin(y(x)) - 7x^6 = y'(x) - x\cos(y(x))y'(x) = [1 - x\cos(y(x))]y'(x)$$
$$y'(x) = \frac{\sin(y(x)) - 7x^6}{1 - x\cos(y(x))}.$$

As before, this fraction can't be further simplified.

Additional sample problems:

9. (5 points/part) The following questions refer to the function whose graph is shown in problem 5.

(a) List all numbers a in $(-8, 8)$ such that h is not differentiable at a. Give reasons.

Solution: The answer is $a = -6$, $a = -5$, $a = -2$, $a = 2$, $a = 4$, and $a = 6$. The function h is not differentiable at -6, at -5, at 2, and at 6, because h is not continuous at these places. It is not differentiable at -2 because there is a corner in the graph, and at 4 because there is a cusp; thus, there is no tangent line at either of those places.

(b) List all numbers a in $(-8, 8)$ such that h is continuous at a but not differentiable at a. Give brief reasons.

Solution: The answer is $a = -2$ and $a = 4$. The function h is not differentiable at -2 because there is a corner in the graph, and at 4 because there is a cusp; thus, there is no tangent line at either of those places.

(c) List all numbers a in $(-8, 8)$ such that h is differentiable at a but not continuous at a. Give brief reasons.

Solution: There are no such numbers, because, for any function f and any a, if f is differentiable at a then f is continuous at a.

(d) Which of the following best describes $h'(7)$?
(1) $h'(7)$ does not exist.
(2) $h'(7)$ is close to 0.
(3) $h'(7)$ is positive and not close to 0.
(4) $h'(7)$ is negative and not close to 0.

Solution: $h'(7)$ is the slope of the tangent line to the graph of $y = h(x)$ at $x = 7$. You can tell from inspection that this slope is positive and not close to 0 (choice (3) above). If you actually draw a tangent line on the graph, you should get a slope of somewhere around 2. (In fact, it is clear that $h'(x) > 0$ for $6 < x < 8$.)

(e) Find the largest interval containing 5 on which h is continuous.

Solution: The largest interval containing 5 on which h is continuous is $(2, 6)$. (Note that h is not continuous at either 2 or 6, because $\lim_{x\to 2} h(x)$ and $\lim_{x\to 6} h(x)$ do not exist.)

10. (11 points) Find $\dfrac{dy}{dx}$ by implicit differentiation. (You must solve for $\dfrac{dy}{dx}$.)

(a) $x\sin(y) = (xy + 1)^7$.

Solution: Use the product rule, followed by the chain rule, to differentiate the left hand side. Use the chain rule, followed by the product rule, to differentiate the right hand side. This gives:

$$\frac{d}{dx}\left(x\sin(y)\right) = \frac{d}{dx}\left((xy+1)^7\right)$$

$$\sin(y) + x\cos(y)\frac{dy}{dx} = 7(xy+1)^6\frac{d}{dx}(xy+1) = 7(xy+1)^6\left(y + x\frac{dy}{dx}\right).$$

Now solve for $\dfrac{dy}{dx}$:

$$\sin(y) + x\cos(y)\frac{dy}{dx} = 7(xy+1)^6\cdot y + 7(xy+1)^6\cdot x\frac{dy}{dx}$$

$$\left(x\cos(y) - 7x(xy+1)^6\right)\frac{dy}{dx} = 7y(xy+1)^6 - \sin(y)$$

$$\frac{dy}{dx} = \frac{7y(xy+1)^6 - \sin(y)}{x\cos(y) - 7x(xy+1)^6}.$$

Note that this expression can't be simplified (although it is possible to factor out x in the denominator).

(b) $x^7 y = x + \cos(ky) + \pi^3$, where k is a constant.

Solution: Write the equation as

$$x^7 y(x) = x + \cos(ky(x)) + \pi^3.$$

Use the product rule on the left hand side and the chain rule on the second term on the right hand side. The derivative of the third term on the right hand side is zero because π^3 is a constant. Thus:

$$7x^6 y(x) + x^7 y'(x) = 1 - \sin(ky(x)) \cdot ky'(x).$$

Now solve for $y'(x)$:

$$\begin{aligned} 7x^6 y(x) - 1 &= -x^7 y'(x) - \sin(ky(x)) \cdot ky'(x) = -\left(x^7 + k\sin(ky(x))\right) y'(x) \\ y'(x) &= -\frac{7x^6 y(x) - 1}{x^7 + k\sin(ky(x))}. \end{aligned}$$

This result can't be further simplified.

For those who prefer the other notation, here it is written that way. We use the product rule on the left hand side and the chain rule on the second term on the right hand side. The derivative of the third term on the right hand side is zero because π^3 is a constant. Thus:

$$7x^6 y + x^7 \frac{dy}{dx} = 1 - \sin(ky) \cdot k\frac{dy}{dx}.$$

Now solve for $\frac{dy}{dx}$:

$$\begin{aligned} 7x^6 y - 1 &= -x^7 \frac{dy}{dx} - \sin(ky) \cdot k\frac{dy}{dx} = -\left(x^7 + k\sin(k)\right) \frac{dy}{dx} \\ \frac{dy}{dx} &= -\frac{7x^6 y - 1}{x^7 + k\sin(ky)}. \end{aligned}$$

This result can't be further simplified.

11. (16 points; point values of parts as shown) A small spacecraft takes off from the surface of a planet, reaches a maximum height, and then crashes. Its position at time t is given by $y(t) = 9t^2 - 4t^3$, where $y(t)$ is measured in kilometers (km) above the surface and t is measured in minutes (min). Answer the following questions, being careful to give correct units when called for.

(a) (4 points) Find the speed of the spacecraft at time $t = 1$.

Solution: Calculate $y'(t) = 18t - 12t^2$, so the speed is $y'(1) = 18 - 12 = 6$. Thus, the answer is 6km/min. (The units are *required*.)

(b) (4 points) How long will it take for the spacecraft to reach its maximum height?

Solution: First find the maximum height, which occurs when $y'(t) = 0$. We have $y'(t) = 18t - 12t^2$, so solve:

$$\begin{aligned} 18t - 12t^2 &= 0 \\ 6t(2 - 2t) &= 0 \\ t = 0 \quad \text{or} \quad t &= \frac{3}{2}. \end{aligned}$$

We are told that the spacecraft starts out by rising, reaches a maximum height, and then crashes, to the answer is after $\frac{3}{2}$min. (This can also be checked by applying the methods we have learned to the function $y(t) = 9t^2 - 4t^3$.)

(c) (8 points) At time $t = 2$, what is the acceleration of the spacecraft? Would a person in the craft feel himself speeding up or slowing down at this moment? Explain.

Solution: Calculate $y'(t) = 18t - 12t^2$, so $y''(t) = 18 - 24t$, and $y''(2) - 18 - 48 = -30$. The numerical answer is therefore $-30\text{km}/\text{min}^2$. (The units are *required*.)

The acceleration is negative, and the velocity is $y'(2) = 18 \cdot 2 - 12 \cdot 2^2 = -12$, so the spacecraft is falling fast as time passes. Thus, the person feels himself speeding up towards the ground. Physically, strictly speaking, this is "slowing down", since position is measured in distance above the ground.

12. (10 points/part) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give reasons in all cases.

(a) $\displaystyle\lim_{x \to -2} \frac{x+2}{x^2 - 3x - 10}$.

Solution: This has the indeterminate form $\frac{0}{0}$, so work is needed. We factor the denominator and cancel common factors:

$$\lim_{x \to -2} \frac{x+2}{x^2 - 3x - 10} = \lim_{x \to -2} \frac{x+2}{(x+2)(x-5)} = \lim_{x \to -2} \frac{1}{x-5} = \frac{1}{-2-5} = -\frac{1}{7}.$$

(b) $\displaystyle\lim_{x \to 0^+} \left(\frac{1}{2x} - \frac{1}{2x + \sqrt{x}} \right)$.

Solution: This has the indeterminate form $\infty - \infty$, so work is needed. Subtract the fractions and cancel common factors:

$$\lim_{x \to 0^+} \left(\frac{1}{2x} - \frac{1}{2x + \sqrt{x}} \right) = \lim_{x \to 0^+} \frac{(2x + \sqrt{x}) - 2x}{(2x)(2x + \sqrt{x})} = \lim_{x \to 0^+} \frac{\sqrt{x}}{(2x)(2x + \sqrt{x})} = \lim_{x \to 0^+} \frac{1}{(2\sqrt{x})(2x + \sqrt{x})}.$$

For $x > 0$ and close to 0, the numerator is 1 and the denominator is positive and close to 0. So the function becomes arbitrarily large:

$$\lim_{x \to 0^+} \left(\frac{1}{2x} - \frac{1}{2x + \sqrt{x}} \right) = \infty.$$

(c) $\displaystyle\lim_{x \to -\infty} \frac{x^2 - x + 17}{7x^2 + 9x + 19}$.

Solution: This has the indeterminate form $\frac{\infty}{\infty}$, so work is needed. We factor out x^2 from both the numerator and denominator, and then use the limit laws:

$$\begin{aligned} \lim_{x \to -\infty} \frac{x^2 - x + 17}{7x^2 + 9x + 19} &= \lim_{x \to -\infty} \frac{x^2 \left(1 - \frac{1}{x} + \frac{17}{x^2}\right)}{x^2 \left(7 + \frac{9}{x} + \frac{19}{x^2}\right)} = \lim_{x \to -\infty} \frac{1 - \frac{1}{x} + \frac{17}{x^2}}{7 + \frac{9}{x} + \frac{19}{x^2}} \\ &= \frac{1 - \lim_{x \to -\infty} \frac{1}{x} + \lim_{x \to -\infty} \frac{17}{x^2}}{7 + \lim_{x \to -\infty} \frac{9}{x} + \lim_{x \to -\infty} \frac{19}{x^2}} = \frac{1 - 0 + 0}{7 + 0 + 0} = \frac{1}{7}. \end{aligned}$$

Alternate solution: Here is a different way to arrange essentially the same calculation:

$$\lim_{x\to-\infty}\frac{x^2-x+17}{7x^2+9x+19} = \lim_{x\to-\infty}\frac{\left(\frac{1}{x^2}\right)(x^2-x+17)}{\left(\frac{1}{x^2}\right)(7x^2+9x+19)} = \lim_{x\to-\infty}\frac{1-\frac{1}{x}+\frac{17}{x^2}}{7+\frac{9}{x}+\frac{19}{x^2}}$$

$$= \frac{1-\lim_{x\to-\infty}\frac{1}{x}+\lim_{x\to-\infty}\frac{17}{x^2}}{7+\lim_{x\to-\infty}\frac{9}{x}+\lim_{x\to-\infty}\frac{19}{x^2}} = \frac{1-0+0}{7+0+0} = \frac{1}{7}.$$

(d) $\displaystyle\lim_{x\to-\infty}\frac{x}{\sqrt{5x^2+9}}$.

Solution: This has the indeterminate form $\frac{-\infty}{\infty}$, so work is needed. We multiply the numerator and denominator by $\frac{1}{x}$, since we should consider the highest degree part of the denominator to be $\sqrt{5x^2}=\pm\sqrt{5}\,x$ (the sign depending on whether $x>0$ or $x<0$). When we do the calculation, we are only interested in negative values of x, since what happens for $x>0$ has no effect on a limit as $x\to-\infty$. Therefore when we put $\frac{1}{x}$ under the square root in the denominator, we will have to calculate as follows: $\frac{1}{x}=-\sqrt{1/x^2}$, *not* $\sqrt{1/x^2}$, for $x<0$. Accordingly,

$$\lim_{x\to-\infty}\frac{x}{\sqrt{5x^2+9}} = \lim_{x\to-\infty}\frac{\left(\frac{1}{x}\right)x}{\left(\frac{1}{x}\right)\sqrt{5x^2+9}} = \lim_{x\to-\infty}\frac{1}{-\sqrt{\frac{1}{x^2}}\sqrt{5x^2+9}} = \lim_{x\to-\infty}\frac{1}{-\sqrt{\left(\frac{1}{x^2}\right)(5x^2+9)}}$$

$$= \lim_{x\to-\infty}\frac{1}{-\sqrt{5+\frac{9}{x^2}}} = \frac{1}{-\sqrt{5+\lim_{x\to-\infty}\frac{9}{x^2}}} = \frac{1}{-\sqrt{5+0}} = -\frac{1}{\sqrt{5}}.$$

(e) $\displaystyle\lim_{x\to2^+}\frac{x^2-5x+4}{x-2}$.

Solution: At 2, the denominator is zero but the numerator is not. Moreover, both the denominator and the numerator are continuous at 2. Therefore the function $f(x)=\lim_{x\to2^+}\frac{x^2-5x+4}{x-2}$ has a vertical asymptote at $x=2$.

To understand its behavior, we factor:

$$f(x)=\frac{(x-1)(x-4)}{x-2}.$$

For $x>2$ but close to 2, the denominator is positive. In the numerator, the factor $x-1$ is positive and the factor $x-4$ is negative, so the numerator is negative. Therefore

$$\lim_{x\to2^+}\frac{x^2-5x+4}{x-2}=-\infty.$$

(f) $\displaystyle\lim_{x\to-\infty}\frac{3x-2}{37x^2-6\sin(x)}$.

Solution: The limit has the indeterminate form "$\frac{\infty}{\infty}$". Therefore more work is needed. We factor out x^2 from both the numerator and denominator, and then use the limit laws:

$$\lim_{x\to-\infty}\frac{3x-2}{37x^2-6\sin(x)} = \lim_{x\to-\infty}\frac{\frac{3}{x}-\frac{2}{x^2}}{37+\frac{2\sin(x)}{x^2}} = \frac{\lim_{x\to-\infty}\frac{3}{x}-\lim_{x\to-\infty}\frac{2}{x^2}}{37+\lim_{x\to-\infty}\frac{2\sin(x)}{x^2}}.$$

Certainly $\lim_{x\to-\infty} \frac{3}{x} = 0$ and $\lim_{x\to-\infty} \frac{2}{x^2} = 0$. Since $-6 \le 6\sin(x) \le 6$ for all x, and $x^2 \to \infty$ as $x \to -\infty$, we get $\lim_{x\to-\infty} \frac{6\sin(x)}{x^2} = 0$. Therefore

$$\lim_{x\to-\infty} \frac{3x-2}{37x^2 - 6\sin(x)} = \frac{0+0}{37+0} = 0.$$

Here is a different way to arrange essentially the same calculation:

$$\lim_{x\to-\infty} \frac{3x-2}{37x^2 - 6\sin(x)} = \lim_{x\to-\infty} \frac{\left(\frac{1}{x^2}\right)(3x-6)}{\left(\frac{1}{x^2}\right)(37x^2 - 6\sin(x))} = \lim_{x\to-\infty} \frac{\frac{3}{x} - \frac{2}{x^2}}{37 + \frac{2\sin(x)}{x^2}}$$
$$= \frac{\lim_{x\to-\infty} \frac{3}{x} - \lim_{x\to-\infty} \frac{2}{x^2}}{37 + \lim_{x\to-\infty} \frac{2\sin(x)}{x^2}} = \frac{0+0}{37+0} = 0.$$

13. (9 points/part) Differentiate the following functions:

(a) $f(y) = \left(7y^3 + \frac{1}{2}\right)\left(\frac{1}{8}y^7 - 16\sqrt{y} + \frac{2}{3}\right)$.

Solution: We use the product rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Before doing so, we rewrite the factors to make then easy to differentiate:

$$f(y) = \left(7y^3 + \frac{1}{2}\right)\left(\frac{1}{8}y^7 - 16y^{1/2} + \frac{2}{3}\right).$$

Therefore

$$f'(y) = 21y^2\left(\frac{1}{8}y^7 - 16y^{1/2} + \frac{2}{3}\right) + \left(7y^3 + \frac{1}{2}\right)\left(\frac{7}{8}y^6 - 8y^{-1/2}\right).$$

(b) $h(t) = \dfrac{\sqrt[3]{t} - 2\pi}{\sqrt[3]{t} + 2\pi}$.

Solution: Use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Before doing so, we rewrite the numerator and denominator to make them easy to differentiate:

$$h(t) = \frac{t^{1/3} - 2\pi}{t^{1/3} + 2\pi}.$$

Therefore

$$h'(t) = \frac{\frac{1}{3}t^{-2/3}(t^{1/3} + 2\pi) - (t^{1/3} - 2\pi)\frac{1}{3}t^{-2/3}}{(t^{1/3} + 2\pi)^2} = \frac{\frac{1}{3}t^{-2/3} \cdot 4\pi}{(t^{1/3} + 2\pi)^2} = \frac{4\pi}{3t^{2/3}(t^{1/3} + 2\pi)^2}.$$

(The simplification is necessary.) Note that 2π is a *constant*, so its derivative is zero.

(c) Given that $h'(x) = 3h(x)$, find $\dfrac{d}{dx}\left(\dfrac{x}{h(x)}\right)$. (Your answer might involve the function h.)

Solution: We use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus,

$$\frac{d}{dx}\left(\frac{x}{h(x)}\right) = \frac{1 \cdot h(x) - xh'(x)}{h(x)^2} = \frac{h(x) - x \cdot 3h(x)}{h(x)^2} = \frac{1-3x}{h(x)}.$$

(The simplification is necessary.)

(d) $g(t) = \dfrac{t^2 - 27}{k + e^t} - \sqrt{17}$, where k is a constant.

Solution: We use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Thus,

$$g'(t) = \frac{2t(k+e^t) - (t^2 - 27)e^t}{\left(k+e^t\right)^2} = \frac{2kt + 2te^t - t^2e^t + 27e^t}{\left(k+e^t\right)^2}.$$

The "simplification" at the second step isn't really any simpler, so is not required. Note that $\sqrt{17}$ is a *constant*, so its derivative is zero.

(e) $q(x) = \sin(cxe^x + \pi^2)$, where c is a constant.

Solution: Use the chain rule. The product rule is needed to differentiate xe^x.

$$q'(x) = \cos(cxe^x + \pi^2) \cdot c(e^x + xe^x) = c(1+x)e^x \cos(cxe^x + \pi^2).$$

(The rearrangement is not really necessary.) The derivative of π^2 is zero because π^2 is a constant.

(f) $f(x) = \sin\left((x^2 - k)^{17}\right)$, where k is a constant.

Solution:

$$f'(x) = \cos\left((x^2-k)^{17}\right) \cdot 17(x^2-k)^{16} \cdot 2x = 34x(x^2-k)^{16}\cos\left((x^2-k)^{17}\right).$$

14. (6 points/part) The function $P(t) = (6t+1)e^{k(t-1)}$ models the population of a colony of bacteria at time t, where P is measured in hundreds of bacteria and t is measured in hours. Observations indicate that after one hour there are 700 bacteria, and at that time the colony is growing at a rate of 200 bacteria per hour.

(a) Find k.

Solution: We are told that $P(1) = 7$ and $P'(1) = 2$. (**Caution! Read the problem!** $P(t)$ is measured in **hundreds** of bacteria!) So

$$7 = P(1) = (6 \cdot 1 + 1)e^{k(1-1)} = 7e^0 = 7.$$

This equation doesn't tell us anything: the statement that at $t = 1$ there are 700 bacteria is redundant.

The statement about $P'(1)$ is more useful. Using the product and chain rules, we get

$$P'(t) = 6e^{k(t-1)} + (6t+1)e^{k(t-1)}\frac{d}{dt}\big(k(t-1)\big) = 6e^{k(t-1)} + (6t+1)e^{k(t-1)} \cdot k.$$

Substituting $t = 1$ gives

$$P'(1) = 6e^{k(1-1)} + (6 \cdot 1 + 1)e^{k(1-1)}k = 6 + 7k.$$

This is supposed to be 2, so $2 = 6 + 7k$, that is, $k = -\frac{4}{7}$.

(b) What happens to the population of bacteria in the long run, as $t \to \infty$? Explain.

Solution: Using the information from part (a), we can rewrite the formula as

$$P(t) = (6t + 1)e^{(-4/7)(t-1)}.$$

We are asked about $\lim_{t\to\infty} P(t)$. Write this as

$$\lim_{t\to\infty} \frac{6t + 1}{e^{(4/7)(t-1)}}.$$

In this form, it has the indeterminate form $\dfrac{\infty}{\infty}$. Therefore we may use L'Hospital's Rule. Thus

$$\lim_{t\to\infty} \frac{6t + 1}{e^{(4/7)(t-1)}} = \lim_{t\to\infty} \frac{6}{\frac{4}{7}e^{(4/7)(t-1)}},$$

if the second limit exists. But this limit is zero, since $\lim_{t\to\infty} e^{(4/7)(t-1)} = \infty$. So

$$\lim_{t\to\infty} \frac{6t + 1}{e^{(4/7)(t-1)}} = 0.$$

The population of bacteria goes to zero, that is, the bacteria die out.

14. Let f and g be functions such that:

$$f(-3) = -5, \quad f'(-3) = 12, \quad g(-3) = 2, \quad \text{and} \quad g'(-3) = -3$$

and

$$f(2) = 7, \quad f'(2) = 3, \quad g(2) = -3, \quad \text{and} \quad g'(2) = 2.$$

Let $h(x) = f(g(x))$.

(a) (2 points) Find $h(2)$. (You will not need to use all the information provided.)

Solution: $h(2) = f(g(2)) = f(-3) = -5$.

(b) (8 points) Find $h'(2)$. (You will not need to use all the information provided.)

Solution: Using the chain rule,

$$h'(2) = f'(g(2))g'(2) = f'(-3) \cdot 2 = 12 \cdot 2 = 24.$$

16. (6 points) Express the following statement in terms of calculus. Be sure to define everything that appears in your formulas.

"The population of fire-breathing monsters on the planet Yuggxth was increasing throughout the period 1900–2000."

Solution: Let $Y(t)$ be the population of fire-breathing monsters on the planet Yuggxth at time t, in years AD. (Thus, $t = 1900$ corresponds to 1 Jan. 1900.) Then the statement says $Y'(t) > 0$ for $1900 \leq t \leq 2000$. (If the original statement was intended to mean from the beginning of 1900 through the end of 2000, then the mathematical form would be $Y'(t) > 0$ for $1900 \le t \le 2001$. I would also accept this answer.)

It you hope to safely take a tour of Yuggxth, you had better go soon, before the population of fire-breathing monsters increases much more!

(Note that the problem didn't say anything about *exponential* growth or decay.)

17. (4 points/part) Let $T(h)$ be the temperature at height h above the Eugene airport at noon on 17 July 2001. Assume that h is measured in kilometers, and that $T(h)$ is measured in °C.

(a) What are the units of $T'(h)$?

Solution: °C per kilometer, or °C/km.

(b) For $0 \le h \le 10$, do you expect $T'(h)$ to be positive or negative? Why?

Solution: Negative, because temperature normally decreases with elevation: it gets colder as you get higher. (This is no longer necessarily true when one gets very high.)

(c) Explain the practical significance of the statement $T'(30) = 5$.

Solution: The temperature was increasing with height at the height 30 kilometers above the Eugene airport at noon on 17 July 2001, at 5 °C per kilometer. In particular, if the trend continued over the next kilometer of height, then the temperature would have been 5 °C warmer 31 kilometers above the Eugene airport.

18. (10 points/part) Let f be a function such that $\lim_{x\to 2} f(x) = 7$. Calculate the following expressions, or explain why there is not enough information to do so:

(a) $\displaystyle\lim_{x\to 2} \frac{f(x)}{\sqrt{x}}$.

Solution:

$$\lim_{x\to 2} \frac{f(x)}{\sqrt{x}} = \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2} \sqrt{x}} = \frac{7}{\sqrt{2}}.$$

The following calculation is not correct, and is an example of getting the right answer through incorrect intermediate steps:

$$\lim_{x\to 2} \frac{f(x)}{\sqrt{x}} = \lim_{x\to 2} \frac{7}{\sqrt{x}} = \frac{7}{\sqrt{2}}.$$

It is not correct to take the limit of only part of an expression. If you could do that, since $\lim_{x\to 0} \sin(7x) = 0$ and $\lim_{x\to 0} 7x^2 = 0$, you would get

$$\lim_{x\to 0} \frac{\sin(7x)}{3x} = \lim_{x\to 0} \frac{0}{3x} = \lim_{x\to 0} 0 = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{7x^2}{3x^2} = \lim_{x\to 0} \frac{0}{3x^2} = \lim_{x\to 0} 0 = 0.$$

Both limits are in fact $\frac{7}{3}$, not zero.

(b) $\displaystyle\lim_{x\to 2} \frac{f(x)}{x-2}$.

Solution: This limit does not exist. The solution to part (c) below shows that $\lim_{x\to 2^-} \frac{f(x)}{x-2} = -\infty$. Similar reasoning shows that $\lim_{x\to 2^+} \frac{f(x)}{x-2} = +\infty$. Therefore one can't even correctly say that $\lim_{x\to 2} \frac{f(x)}{x-2}$ is $-\infty$ or that it is $+\infty$.

(c) $\lim_{x\to 2^-} \frac{f(x)}{x-2}$.

Solution: For $x < 2$ but very close to 2, we have $f(x)$ close to 7 and $x-2$ negative and very close to zero. Therefore $\frac{f(x)}{x-2}$ is negative and very far from zero. So $\lim_{x\to 2^-} \frac{f(x)}{x-2} = -\infty$.

(d) $\lim_{x\to 7} f(x)$.

Solution: Nothing can be said about $\lim_{x\to 7} f(x)$. The statement $\lim_{x\to 2} f(x) = 7$ says something about the behavior of $f(x)$ for x close to 2, but nothing about the behavior of $f(x)$ for x close to 7.

(e) $f(2)$.

Solution: Nothing can be said about $f(2)$. The statement $\lim_{x\to 2} f(x) = 7$ says nothing about $f(2)$. It only tells about the values of $f(x)$ for x close to, but not equal to, 2.
(Note: If we also knew that f is continuous at 2, then we could say that $f(2) = 7$.)

(f) $\lim_{x\to 2} \frac{f(x)-7}{x-2}$.

Solution: Nothing can be said about $\lim_{x\to 2} \frac{f(x)-7}{x-2}$. It has the indeterminate form $\frac{0}{0}$, so requires more work. But without knowing anything else about f, there is no way to do any more work.
Examples:
If $f(x) = x+5$, then $\lim_{x\to 2} \frac{f(x)-7}{x-2} = 1$.
If $f(x) = 2x+3$, then $\lim_{x\to 2} \frac{f(x)-7}{x-2} = 2$.
If $f(x) = (x-2)^2+7$, then $\lim_{x\to 2} \frac{f(x)-7}{x-2} = 0$.
If $f(x) = (x-2)^{1/3}+7$, then $\lim_{x\to 2} \frac{f(x)-7}{x-2} = \infty$.
All the given choices of f satisfy $\lim_{x\to 2} f(x) = 7$.

(g) $\lim_{x\to 2} \sqrt{f(x)}$.

Solution:

$$\lim_{x\to 2} \sqrt{f(x)} = \sqrt{\lim_{x\to 2} f(x)} = \sqrt{7}.$$

19. (10 points) The population of the Kingdom of Oggchobb grows at a rate proportional to the existing population. If the population was 9.73 million in 1960 and 11.31 million in 1990, help the Royal Statistical Office determine the population in 2000.

Solution: Let $P(t)$ be the population in millions at time t in years since 1960. With these choices, we have $P(0) = 9.73$ and $P(30) = 11.31$, and we are supposed to determine $P(40)$ under the assumption that the growth rate is always proportional to the population.

Since the growth rate is proportional to the population, the function $P(t)$ must have the form $P(t) = Ce^{kt}$ for some constants C and k. We have two pieces of information to use to determine C and k. First, using $P(0) = 9.73$, we get

$$9.73 = P(0) = Ce^{k \cdot 0} = C,$$

that is, $C = 9.73$. Second, using $P(30) = 11.31$, we get

$$11.31 = P(30) = Ce^{k \cdot 30} = 9.73e^{30k},$$

so

$$e^{30k} = \frac{11.31}{9.73} \quad \text{and} \quad k = \tfrac{1}{30} \ln\left(\frac{11.31}{9.73}\right).$$

Therefore

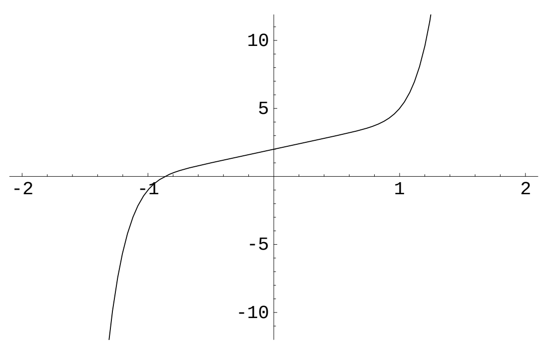
$$P(40) = Ce^{40k} = 9.73 \cdot e^{\frac{40}{30} \ln\left(\frac{11.31}{9.73}\right)}.$$

So the population in 2000 is $e^{4/3} \ln\left(\frac{11.31}{9.73}\right)$ million people.

20. (15 points) Prove that there exists a real solution to the equation $x^9 + 2x + 2 = 0$. Give a complete justification for any theorems that you use, in particular being sure to check that the hypotheses hold.

Solution: We use the Intermediate Value Theorem. Set $f(x) = x^9 + 2x + 2$. Then f is a polynomial function, so it is continuous at all real numbers. By substitution, you can check that $f(0) = 2$ and $f(-1) = -1$. (These numbers were found by trying small integers at random. Note that $x = 0$ is a good choice because $f(0)$ is particularly easy to evaluate. Using $f(-1)$ and $f(1)$, for example, would also work fine.) Since $f(-1) < 0 < f(0)$, the Intermediate Value Theorem therefore tells us that there is some c in the interval $(-1, 0)$ such that $f(c) = 0$.

Here is the graph of f for x near zero (not required as part of the solution):



Note that the quadratic formula can't be used, since the original equation is not a quadratic equation.